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# On the Relation between the Nonconvexity of a Set and the Nonconvexity of Its $\varepsilon$-Neighborhoods

**D. Repovš and P. V. Semenov**

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**Abstract**—To each closed subset $P$ of a Banach space, a real function $\alpha_P$ characterizing the nonconvexity of this set is associated. Inequalities of the type $\alpha_P(\cdot) < 1$ ensure good topological properties of the set $P$, such as contractibility, the property of being an extensor, etc. In this paper, examples of sets whose nonconvexity functions substantially differ from the nonconvexity functions of arbitrarily small neighborhoods of these sets are constructed. On the other hand, it is shown that, in uniformly convex Banach spaces, conditions of the type "the function of nonconvexity is less than one" are stable with respect to taking $\varepsilon$-neighborhoods of sets.

Key words: *functions of nonconvexity, paraconvex set, paraconvexity with given accuracy, set-valued map, uniformly convex Banach space, Chebyshev center.*

## 1. INTRODUCTION

Let $P$ be a nonempty closed subset of a normed space $B$, and let $D$ be an open ball of radius $r$ intersecting $P$. The number

$$\delta(P, D) = \sup\left\{ \frac{\operatorname{dist}(q, P)}{r} \;\middle|\; q \in \operatorname{conv}(P \cap D) \right\}$$

is a natural measure of the relative nonconvexity of the intersection of $P$ with $D$.

**Definition 1.1.** (a) The *function of nonconvexity* $\alpha_P(\cdot)$ takes each number $r > 0$ to the supremum of the set of all numbers $\delta(P, D)$ over all open balls $D$ of radius $r$.

(b) A set $P$ is called *$\alpha$-paraconvex* if its function of nonconvexity does not exceed (pointwise) a function $\alpha$ on $(0, \infty)$.

(c) A set $P$ is called *$\alpha$-paraconvex with fixed accuracy* $\delta > 0$, if its function of nonconvexity does not exceed (pointwise) a function $\alpha$ on $(\delta, \infty)$.

For a constant function $\alpha$, Definition 1.1(b) coincides with the definition of paraconvexity suggested by Michael in [1]. Clearly, 0-paraconvexity is equivalent to convexity. Functional paraconvexity was considered in [2]; in explicit form, the functions of nonconvexity were introduced in [3]. Definition 1.1(a) admits various modifications useful in particular situations. Replacing open balls in this definitions by closed ones, we obtain the definition of *closed functions of nonconvexity*. Considering only the Chebyshev centers (or the Chebyshev center, if the norm in $B$ is uniformly convex) of the convex bounded set $\operatorname{conv}(P \cap D)$ rather than all points $q \in \operatorname{conv}(P \cap D)$, we obtain the definition of the *central function of nonconvexity*. It is also natural to consider the number

$$\delta_*(P, K) = \sup\left\{ \frac{\operatorname{dist}(q, P)}{r} \;\middle|\; q \in \operatorname{conv} K \right\}$$

for every finite subset $K \subset P$ of Chebyshev radius $r$ and define the *inner function of nonconvexity* of the set $P$, which takes a number $r$ to the supremum of the set of all numbers $\delta_*(P,K)$ over all finite sets $K \subset P$ of Chebyshev radius $r$. The nonconvexity-type functions satisfy certain relations; thus, if $\alpha_P^{\mathrm{in}}$ is the inner function of nonconvexity of a set $P$ and $\alpha_P^{\mathrm{cl}}$ is the closed function of nonconvexity of this set, then

$$\alpha_P(r) \le \alpha_P^{\mathrm{in}}(r) \le \alpha_P^{\mathrm{cl}}(r) = \alpha_P(r+0)$$

for all $r > 0$. In particular, the functions of nonconvexity have right limits (and are in fact left continuous).

Controlling the nonconvexity of a set actually reduces to controlling the behavior of certain iterations of the nonconvexity function of this set. To control this behavior, we associate each function $\alpha\colon (0,\infty) \to [0,\infty)$ to the "geometric progression of ratio $\alpha$"

$$q_\alpha^0(t) = t, \qquad q_\alpha^1(t) = \alpha(t)\cdot t, \qquad q_\alpha^{n+1}(t) = \alpha(q_\alpha^n(t))\cdot q_\alpha^n(t).$$

The function $\alpha$ is extended to zero as $\alpha(0) = 0$.

**Definition 1.2.** (a) A function $\alpha\colon (0,\infty) \to [0,\infty)$ is called *geometrically summable* if the functional series $\sum_{n=0}^{\infty} q_\alpha^n(\,\cdot\,)$ pointwise converges everywhere.

(b) A function $\alpha$ is *less than one from the left* (symbolically, $\alpha < 1_{-0}$) if it has a geometrically summable strict majorant.

It is easy to see [2] that, if all right upper limits of a function $\alpha\colon (0,\infty) \to [0,\infty)$ are less than one on the semiaxis $[0,\infty)$, then $\alpha < 1_{-0}$.

The problem of controllable rejection of convexity in selection theorems for set-valued maps admits the following solution (see [4–8] for basic information about set-valued maps).

**Theorem 1.3** [3, 9]**.** *Let $F\colon X \to 2^Y$ be a lower semicontinuous closed-valued map of a paracompact space $X$ to a Banach space $Y$. If*

$$\sup\{\alpha_{F(x)} \mid x \in X\} < 1_{-0},$$

*then $F$ has a continuous single-valued selection.*

Therefore, if the function of nonconvexity of some set is less than one from the left, then this set is topologically ideal in that it is an absolute extensor (and a local absolute extensor) for the class of paracompact spaces. In particular, it is contractible and locally contractible.

In this paper, we examine the stability of similar properties of sets with respect to taking metric neighborhoods of these sets. To be more precise, we consider the following question. Let the nonconvexity function of some set $P$ be less than one from the left. Are then the nonconvexity functions of the $\varepsilon$-neighborhoods of this set also less than one from the left?

In this paper, we construct an example showing that in general the answer to this question is negative (see Theorem 2.1): paraconvex sets may have arbitrarily small nonparaconvex $\varepsilon$-neighborhoods even in a four-dimensional Banach space. However, in Banach spaces with sufficiently smooth unit spheres, the answer is positive (see Theorem 2.3). On the other hand (see Proposition 2.4), even in the Euclidean plane, there exists a curve such that the nonconvexity functions of its $\varepsilon$-neighborhoods do not converge to the nonconvexity function of the curve itself as $\varepsilon \to 0$; thus the degree of nonconvexity of arbitrarily small neighborhoods of such a curve can *essentially* differ from the degree of nonconvexity of the curve (and still be strictly less than one).

## 2. STATEMENTS OF THE RESULTS

**Theorem 2.1.** *The four-dimensional Banach space $Y = l_\infty^3 \oplus_2 \mathbb{R}$ with the norm*

$$\|(x,y,z,t)\| = \sqrt{(\max\{|x|,\ |y|,\ |z|\})^2 + t^2}$$

*has a one-dimensional $q$-paraconvex subset $P$ with $0 \le q < 1$ such that, for some sequence of positive numbers $\varepsilon_n \to 0$, the nonconvexity functions $\alpha_{P_n}$ of the closed $\varepsilon_n$-neighborhoods $\overline{D}(P,\varepsilon_n) = P_n$ of $P$ are identically equal to one on some intervals $(0,\tau_n) \subset (0,\infty)$.*

We stress that the equality to one is the "worst" possible case for functions of nonconvexity. The proof of Theorem 2.1 is, roughly speaking, based on the observation that the Banach space $Y = l_\infty^3 \oplus_2 \mathbb{R}$ contains many triples of noncollinear points for which the triangle *equality* is valid. The main technical details are collected in the following proposition.

**Proposition 2.2.** *In the Banach space $Y = l_\infty^3 \oplus_2 \mathbb{R}$, there exists a straight line $l \subset l_\infty^3$ with the following property: for any positive numbers $R$ and $\varepsilon$, the plane $\Pi = l \oplus_2 \mathbb{R}$ contains a $q$-paraconvex $(0 \le q < 1)$ graph $A \subset \Pi$ of a continuous function $f\colon l \to \mathbb{R}$ such that the nonconvexity function $\alpha_{A_\varepsilon}$ of the closed $\varepsilon$-neighborhood $A_\varepsilon$ of this graph equals one at the point $R$. Moreover, a monotonically increasing function of the ratio $R/\varepsilon$ can be taken for $q$ (see Fig. 1).*

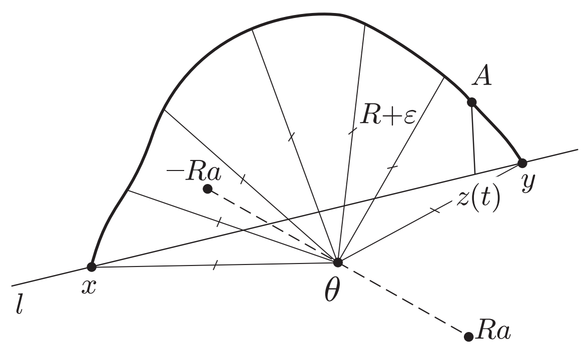


**Fig. 1**

In Banach spaces with "good" norms, there are no examples with such peculiar geometric properties.

**Theorem 2.3.** *For any uniformly convex Banach space $Y$, any function $\alpha(\,\cdot\,) < 1_{-0}$, and any $\varepsilon > 0$, there exists a function $\beta(\,\cdot\,) < 1_{-0}$ such that the closed $\varepsilon$-neighborhood $\overline{D}(P,\varepsilon)$ of an arbitrary $\alpha$-paraconvex set $P$ is $\beta$-paraconvex.*

The following proposition occupies an intermediate position between Theorems 2.1 and 2.3. On the one hand, it lies in the domain of applicability of Theorem 2.3. On the other hand, it shows that the functions $\beta = \beta(\alpha,\varepsilon)$ whose existence is asserted by Theorem 2.3 cannot converge pointwise to the function $\alpha$ as $\varepsilon \to 0$ even on the Euclidean plane. At the same time, the situation described in Theorem 2.1 with $\beta(\,\cdot\,) = 1$ at some points cannot occur in this case.

**Proposition 2.4.** *In the Euclidean plane, for an arbitrary $0 \le q < 1$, there exist a number $p \in (q,1)$, a $q$-paraconvex compact subset $K$, and a sequence of positive numbers $\varepsilon_n \to 0$ such that the neighborhoods $\overline{D}(K,\varepsilon_n)$ are not $p$-paraconvex.*

## 3. PROOF OF THEOREM 2.1

**Derivation of Theorem 2.1 from Proposition 2.2.** Let us take arbitrary numbers $\tau_0 > 0$, $\tau \in (0, \tau_0)$, $R > 0$, and $\varepsilon > 0$ such that $R/\varepsilon = \tau$. By Proposition 2.2, there exists a $q(\tau)$-paraconvex set $A = A(R, \varepsilon) \subset \Pi = l \oplus_2 \mathbb{R}$ which is the graph of some function $f\colon l \to \mathbb{R}$. Since $q(\,\cdot\,)$ is a monotone function of the ratio $R/\varepsilon$, the set $A$ is $q(\tau_0)$-paraconvex for $q(\tau_0) = q_0 < 1$; at the same time,

$$\alpha_{\overline{D}(A,\varepsilon)}(R) = 1.$$

Now, take a sequence $\varepsilon_n \to 0$ and consider *all rational* numbers $R_{n,m} \in (0, \varepsilon_n \cdot \tau_0)$, where $m \in \mathbb{N}$, for each $\varepsilon_n$. As above, we construct the sets $A_{n,m} = A(R_{n,m}, \varepsilon_n)$ and arrange isometric copies (parallel translations) of these sets in the Euclidean plane $\Pi$ along the straight line $l$ in such a way that the endpoints of the arcs $A_{n,m}$ lie on $l$ "very far apart," i.e., the distances between neighboring copies are substantially larger than the sums of their diameters. We join all neighboring copies by segments of $l$. The set $P \subset \Pi$ thus obtained is the graph of some continuous function.

Let us show that this set is $q$-paraconvex for some $q \in (q_0, 1)$. For this purpose, we apply the inner function of nonconvexity. Let $\Delta$ be an arbitrary simplex with vertices from $P$. Since $P$ is the graph of a continuous function, we can assume that $\Delta$ is a straight line segment (see [3]). If its endpoints belong to different copies, then the length of the segment $\Delta$ is very large as compared to the diameters of these copies. Therefore, the relative remoteness of the points of $\Delta$ from the set $P$ is bounded from above by a small number. There remain only two cases: either the endpoints of the segment $\Delta$ lie in the same set $A_{n,m}$ or one of its endpoints lies in $A_{n,m}$ and the other endpoint lies on the segment going away from an endpoint of the arc $A_{n,m}$. In the first case, the distance from the points of $\Delta$ to $P$ is bounded by virtue of the $q_0$-paraconvexity of the arc $A_{n,m}$. In the second case, the relative remoteness does not exceed $\sqrt{2}/2$ (see [3]).

Now, consider the closed $\varepsilon_n$-neighborhood $\overline{D}(P, \varepsilon_n) = P_n$ of the set $P$. By construction, for each rational $R_{n,m} \in (0, \tau_n)$, where $\tau_n = \varepsilon_n \tau_0$, there exists a point $\theta_{n,m}$ whose distance from $P_n$ is $R_{n,m}$ which is the center of some segment of length $2R_{n,m}$ with endpoints from $P_n$. Clearly, the minimal distance from $\theta_{n,m}$ to the straight line $l$ is attained at some point between the endpoints of the arc $A_{n,m}$. Therefore, $\alpha_{P_n}(R_{n,m}) = 1$ for all $m \in \mathbb{N}$. The left continuity of the functions of nonconvexity implies that the function of nonconvexity of the set $P_n$ *identically* equals one on the interval $(0, \tau_n)$. $\square$

**Proof of Proposition 2.2.** Take

$$a = (1,0,1,0), \qquad b = (0,1,1,0), \qquad c = (0,0,1,0),$$
$$x = Ra + \varepsilon b = (R, \varepsilon, R+\varepsilon, 0), \qquad y = -Ra - \varepsilon c = (-R, 0, -R-\varepsilon, 0), \qquad l = \operatorname{span}\{x, y\}.$$

We have $\|x\| = R + \varepsilon = \|y\|$ and $\|x - y\| = \|(2R, \varepsilon, 2R + 2\varepsilon, 0)\| = 2R + 2\varepsilon = \|x\| + \|-y\|$.

The norm of an arbitrary point $z(t) = (1-t)x + ty$ in the segment $[x, y]$ can be evaluated explicitly; it equals

$$\|z(t)\| = \max\{\varepsilon(1-t),\ (R+\varepsilon)|1-2t|\}.$$

The function $\|z(t)\|$ attains its minimum at the point $t_0 = (R + 2\varepsilon)/(2R + 3\varepsilon)$; the minimum value equals

$$\|z_0\| = \|z(t_0)\| = \frac{(R+\varepsilon)\varepsilon}{2R+3\varepsilon}.$$

Consider the Euclidean plane $\Pi = l \oplus_2 \mathbb{R} \subset Y$ and the graph $A$ of the function

$$f(z(t)) = \sqrt{(R+\varepsilon)^2 - \|z(t)\|^2}, \qquad f\colon [x, y] \to \mathbb{R}$$

on the segment $[x,y]$ in this plane. The function $f$ is unimodal; to be more precise, it increases on $[x,z(t_0)]$ and decreases on $[z(t_0),y]$.

Since the endpoints of $[x,y]$ lie in $A$, the points $\pm Ra$ belong to the closed $\varepsilon$-neighborhood $\overline{D}(A,\varepsilon)=A_\varepsilon$ of the set $A$. Therefore, the origin $\theta$ lies in $\operatorname{conv}(A_\varepsilon)$. At the same time, by construction, all points of the set $A_\varepsilon$ are precisely distance $R+\varepsilon$ apart $\theta$. Hence $\alpha_{A_\varepsilon}(R)=1$, and it only remains to verify the paraconvexity of the set $A$, which is the graph of a continuous function on a line segment.

Unfortunately, this function is not Lipschitz, and we only have the estimate

$$\frac{|f(z(t_1))-f(z(t_2))|}{\|z(t_1)-z(t_2)\|}=\frac{\big|\|z(t_2)\|-\|z(t_1)\|\big|}{\|z(t_1)-z(t_2)\|}\cdot\frac{\|z(t_2)\|+\|z(t_1)\|}{f(z(t_1))+f(z(t_2))}\le\frac{2(R+\varepsilon)}{f(z(t_1))+f(z(t_2))}.$$

The last fraction becomes infinite at the endpoints of the segment; so, we cannot directly apply the technique developed in [2, 3].

**Lemma 3.1.** *Suppose that, in the notation introduced above,*

$$t_0=\frac{R+2\varepsilon}{2R+3\varepsilon},\qquad f_0=f(z(t_0)),\qquad p=\frac{f_0}{R+\varepsilon}=\sqrt{1-\left(\frac{\varepsilon}{2R+3\varepsilon}\right)^2},$$

*$\lambda$ is a number in the interval $(1,p^{-1})$, and $\mu$ is a number in the interval $(0,\sqrt{\lambda^2-1}/(p\lambda))$. Then the graph of the function $f$ (i.e., the set $A$) is $q$-paraconvex for*

$$q=\max\{p\lambda,\ \sin(\arctan(L'))\},\qquad \textit{where}\quad L'=\tan\frac{\arctan(L)+\pi/2}{2},\quad L=\frac{1}{p\mu}$$

*(see Fig. 2).*

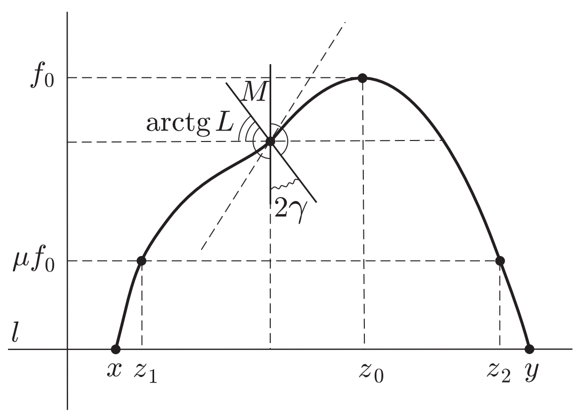


**Fig. 2**

**Proof.** Let us draw a horizontal line at the altitude $\mu f_0$. The function $f$ takes the value $\mu f_0$ at precisely two points $z_1<z_2$ on the segment $[x,y]$. Let us find the difference $\|z_2-z_1\|$. We have

$$f(z_i)=(R+\varepsilon)\sqrt{1-(1-2t_i)^2}=\mu f_0=\mu(R+\varepsilon)p,$$
$$(1-2t_i)^2=1-\mu^2p^2,\qquad |t_2-t_1|=\sqrt{1-\mu^2p^2}.$$

Therefore,

$$\|z_2-z_1\|=\|(1-t_2)x+t_2y-(1-t_1)x-t_1y\|=|t_2-t_1|\cdot\|x-y\|=|t_2-t_1|\cdot 2(R+\varepsilon)=2\sqrt{1-\mu^2p^2}(R+\varepsilon).$$

To estimate the degree of paraconvexity of the graph of an arbitrary continuous function, it is sufficient to consider only *straight line segments* with endpoints on this graph [3]. Moreover, it suffices to find an upper bound of only the distances from the *midpoints* of these segments to the graph of the function. Indeed, for points close to the midpoint of a segment, the upper bound is almost the same, and the points significantly different from the midpoint are close to the endpoints of the segment, which lie on the graph. We also use the following simple geometric observation [3]. If the endpoints of a straight line segment of length $2r$ lie on the graph of a function Lipschitz with constant $K$, then the midpoint of this segment is within distance $\sin(\arctan(K))\cdot r$ from the graph of the function.

Let us return to our function $f\colon [x,y]\to\mathbb{R}$. Take any segment with endpoints on the graph of this function. For the projection of such a segment on the "horizontal" axis $\operatorname{span}\{x,y\}$, the following three cases are possible:

(a) the left endpoint belongs to $[x,z_1]$ and the right endpoint belongs to $[z_2,y]$;
(b) the right endpoint belongs to $[x,z_2]$;
(c) the left endpoint belongs to $[z_1,y]$.

*Case* (a). The length of the segment is in this case no less than $\|z_2-z_1\|$, and the distances from the points of the segment to the graph of the function are no larger than $f_0$. Therefore, in the case under consideration, the relative remoteness of the midpoint of this segment from the graph of the function $f$ is bounder from above by the ratio

$$\frac{2f_0}{\|z_2-z_1\|}=\frac{p}{\sqrt{1-\mu^2p^2}}<\lambda p;$$

the last inequality is implied by the choice of $\mu\in(0,\sqrt{\lambda^2-1}/(p\lambda))$.

*Case* (b). First, note that we have already estimated the Lipschitz constant of the function $f$ restricted to the segment $[z_1,z_2]$; indeed, for any $t_1'$ and $t_2'$ from the segment $[t_1,t_1]$, we have

$$\frac{|f(z(t_1'))-f(z(t_2'))|}{\|z(t_1')-z(t_2')\|}\le\frac{2(R+\varepsilon)}{f(z(t_1'))+f(z(t_2'))}\le\frac{2(R+\varepsilon)}{f(z_1)+f(z_2)}=\frac{R+\varepsilon}{\mu f_0}=\frac{1}{\mu p}=L.$$

Take an arbitrary point $M$ on the graph such that its horizontal projection lies on $[x,z_2]$. First, consider the left angle with vertex at this point formed by a vertical line directed downward and a ray with negative slope equal to $\arctan(L)$ in absolute value. Certainly, this angle is less than the straight angle. We claim that the graph of the function $f$ on the left of $M$ lies in this angle, whose measure is

$$\frac{\pi}{2}+\arctan(L)=2(\arctan(L)+\gamma),\qquad \gamma=\frac{\pi/2-\arctan(L)}{2}.$$

For the points on the graph whose horizontal projections are larger than $z_1$, this assertion follows from (a), and for the points of the interval $[x,z_1]$, it holds because the function $f$ increases on this interval, and the graph on the left of its arbitrary point lies in the third quadrant.

To the right of the point $M$, the graph of $f$ (up to the point $z_2$) lies in the angle of measure $2\arctan(L)$ with vertex at $M$ and horizontal bisector. Therefore, it lies in the right angle formed by a vertical line directed upward and the continuation of the oblique ray constructed above. In other words, we have constructed two angles symmetric about the point $M$, both of size $2(\arctan(L)+\gamma)$, such that their union contains the graph of the function $f$ restricted to the interval $[x,z_2]$.

Thus, in the coordinate system obtained by rotating the initial coordinate system counterclockwise through the angle $\gamma$, the fragment of the graph of $f$ under consideration is the graph of a function Lipschitz with constant

$$L'=\tan(\arctan(L)+\gamma)=\tan\frac{\arctan(L)+\pi/2}{2}.$$

Therefore, the relative remoteness of the midpoint of an arbitrary segment with endpoints on the graph of the function $f$ on the interval $[x, z_2]$ is bounded from above by the number $\sin(\arctan(L'))$.

*Case* (c) is considered similarly.

This completes the proof of the lemma. $\square$

We continue the proof of Proposition 2.2. It remains to explicitly express the paraconvexity index $q$ of the set $A$ in terms of $R$, $\varepsilon$, and $\tau = R/\varepsilon$. By definition, we have

$$p = \sqrt{1 - \left(\frac{\varepsilon}{2R + 3\varepsilon}\right)^2} = \sqrt{1 - \left(\frac{1}{2\tau + 3}\right)^2}.$$

Therefore, $p = p(\tau)$ monotonically increases from $p(0) = 2\sqrt{2}/3$ to $p(\infty) = 1$. Next, according to Lemma 3.1, we can assume that the function $\lambda = \lambda(\tau)$ is equal to $(p+1)/(2p)$ and the function $\mu = \mu(\tau)$, to $\sqrt{\lambda^2 - 1}/(2p\lambda)$. Then both $\lambda$ and $\mu$ monotonically decrease, while

$$L = \frac{1}{\mu p}, \qquad L' = \tan\frac{\arctan(L) + \pi/2}{2}, \qquad \text{and} \qquad \lambda p = \frac{p+1}{2}$$

monotonically increase. Therefore, the upper bound $q(\tau) = q = \max\bigl\{\lambda p,\ \sin(\arctan(L'))\bigr\}$ of the nonconvexity function of the set $A$ can also be assumed monotonically increasing with respect to the variable $\tau = R/\varepsilon$. This completes the proof of Proposition 2.2 and Theorem 2.1. $\square$

## 4. THE PROOFS OF THEOREM 2.3 AND PROPOSITION 2.4

We shall use the following lemma proved in [9] for the case of a Hilbert space. Roughly speaking, this lemma asserts that the "altitudes" of a triangle are small if one of its sides is almost equal to the sum of the two other sides. Recall that a Banach space $Y$ is called uniformly convex if the modulus of convexity of the unit sphere in this space is positive, i.e.,

$$0 < \delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x - y\| \ge \varepsilon\right\}, \qquad \varepsilon \in (0, 2].$$

**Lemma 4.1.** *Let $(Y, \|\cdot\|)$ be a uniformly convex Banach space. Then, for any $\lambda > 0$, there exists a continuous function $\sigma_\lambda\colon (0,\infty)\times(0,\infty) \to (0,1)$ such that the relations $\|x - y\| = a$, $\|y - z\| = b$, and $\|x - z\| \ge \sigma_\lambda(a, b)\cdot a + b$ imply the inequality* $\operatorname{dist}(z, \operatorname{span}\{x, y\}) \le \lambda b$ (*see Fig.* 3).

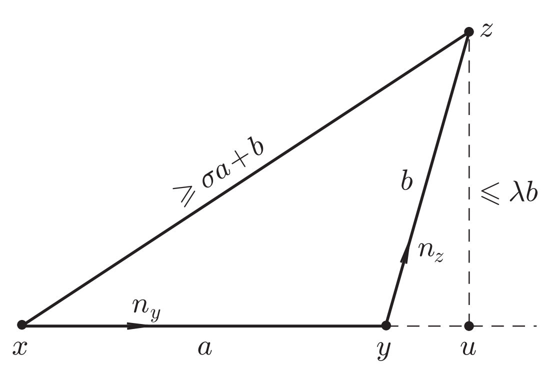


**Fig. 3**

**Proof.** Let $n_y$ and $n_z$ be the unit vectors whose directions coincide with those of $y-x$ and $z-y$, respectively. Then

$$y = x + an_y, \qquad z = y + bn_z = x + an_y + bn_z.$$

Let us take the point $u = tx + (1-t)y$ with $t = -b/a$ on the straight line passing through $x$ and $y$ and find the distance between $z$ and $u$. We have

$$\begin{aligned}\|z-u\| &= \|tz + (1-t)z - tx - (1-t)y\| = \|t(z-x) + (1-t)(z-y)\| \\ &= \|t(an_y + bn_z) + (1-t)bn_z\| = b\|n_y - n_z\|.\end{aligned}$$

Therefore, $\operatorname{dist}(z, \operatorname{span}\{x,y\}) \le b\|n_y - n_z\|$. It remains to gain the inequality $\|n_y - n_z\| \le \lambda$ or $\delta(\|n_y - n_z\|) \le \delta(\lambda)$, where $\delta(\cdot)$ is the modulus of convexity of the unit sphere in the space $Y$.

First, consider the case $a \ge b$. The crucial technical point (adding and subtracting $bn_y$) was suggested to us by V. M. Kadets, to whom we express our thanks. We have

$$\begin{aligned}\sigma_\lambda(a,b)a + b \le \|z-x\| = \|an_y + bn_z + bn_y - bn_y\| &\le (a-b) + 2\left\|\frac{n_y+n_z}{2}\right\| \\ &\le (a-b) + 2b\bigl(1 - \delta(\|n_y - n_z\|)\bigr) = a + b - 2b\delta(\|n_y - n_z\|).\end{aligned}$$

Therefore,

$$\delta(\|n_y - n_z\|) \le \frac{a}{2b}\bigl(1 - \sigma_\lambda(a,b)\bigr).$$

Thus, for the inequality $\delta(\|n_y - n_z\|) \le \delta(\lambda)$ to hold, it suffices to ensure that

$$\frac{a}{2b}\bigl(1 - \sigma_\lambda(a,b)\bigr) \le \delta(\lambda).$$

In other words, the formula

$$\sigma_\lambda(a,b) = \max\left\{0,\ 1 - \frac{2b}{a}\delta(\lambda)\right\}$$

gives the required result.

Now, let us show that we can put $\sigma(a,b) = \sigma(a,a)$ for $a < b$. Consider the triangle with vertices $z$, $x$, and $y + an_z$. The length of its side $[x, y + an_z]$ is no less than the difference between the lengths of the two other sides; hence

$$\|y + an_z - x\| \ge \|z - x\| - (b-a)\|n_z\| \ge \sigma(a,b)a + b - b + a = \sigma(a,a)a + a.$$

Therefore, the argument used in the preceding case applies to the triangle with vertices $x$, $y$, and $y + an_z$; thus the distance from the point $y + an_z$ to the straight line $\operatorname{span}\{x,y\}$ does not exceed $\lambda \cdot a$. Considering the homothety with factor $b/a$ and center $y$, we see that the distance from the point $z$ to the straight line $\operatorname{span}\{x,y\}$ does not exceed $\lambda \cdot b$. $\square$

Note that, by virtue of the continuity of the modulus of convexity, the function $\sigma$ constructed above is also continuous with respect to $\lambda$.

Uniformly convex Banach spaces retain many geometric properties of Hilbert spaces. For instance, the distance from a point to a convex closed set is attained at a unique point of this set. In addition, a bounded subset of a uniformly convex Banach space has a unique Chebyshev center [10]. We use yet another property of this kind: If the convex hull of an $(n+1)$-element set has dimension $n$, and if this hull contains its Chebyshev center, then all distances from the Chebyshev center to points of this set are equal to the Chebyshev radius of this set.

In the proof of Theorem 2.3, we estimate the inner function of nonconvexity of the set under consideration rather than the nonconvexity function proper. In other words, we take finite subsets of the given set, consider their Chebyshev centers, and estimate the remoteness of these centers from the set itself in terms of a suitable real-valued function of the Chebyshev radii of these finite sets.

**Proof of Theorem 2.3.** Let $P$ be a set whose nonconvexity function $\alpha_P$ is majorized by a function $\alpha < 1_{-0}$ on the half-line $(0, \infty)$. Let us estimate the function of nonconvexity of the set $\overline{D}(P, \varepsilon) = Q$ from above on this half-line. Take an arbitrary open ball $D_r = D$ of radius $r$ intersecting the set $Q$.

Suppose that $K = \{y_1, \dots, y_n\} \subset D \cap Q$ and $y \in \operatorname{conv}\{y_1, \dots, y_n\}$. We must estimate the distance $\operatorname{dist}(y, Q)$ from above. To this end, we take points $z_i \in P$ $\varepsilon$-close to the corresponding points $y_i$. If some distance $\|y_i - z_i\|$ is less than $\varepsilon$, then we take a point $u_i$ such that $\|y_i - u_i\| = \varepsilon$ on the interval $[y_i, z_i)$.

Consider the situation where $y$ is the Chebyshev center of the polyhedron $\operatorname{conv}\{y_1, \dots\dots, y_n\}$ (it has a unique Chebyshev center, because the unit sphere in the space $Y$ is uniformly convex). If the dimension of the polyhedron $\operatorname{conv}\{y_1, \dots, y_n\}$ is less than $n-1$ or $y$ lies on the boundary of this polyhedron, then we can pass to a smaller number of points and argue by induction. Thus, it is sufficient to consider the case where all the distances $\|y - y_i\|$ are equal to the same number, namely, to the Chebyshev radius $r$ of the polyhedron.

So we are given a function $\alpha < 1_{-0}$ and positive numbers $r$ and $\varepsilon$. We put $\lambda = 1 - \alpha(r+\varepsilon) \in (0, 1]$ and consider the function $\sigma_\lambda(\,\cdot\,,\,\cdot\,)$ whose existence is asserted by Lemma 4.1. Three are three possible cases:

(a) $\sigma(r, \varepsilon)r + \varepsilon < r$ and $\|y - u_i\| < \sigma(r, \varepsilon)r + \varepsilon$ for some $i$;
(b) $r \le \sigma(r, \varepsilon)r + \varepsilon$ and $\|y - u_i\| < \sigma(r, \varepsilon)r + \varepsilon$ for some $i$;
(c) $\|y - u_i\| \ge \sigma(r, \varepsilon)r + \varepsilon$ for all $u_i$.

In case (a), one of the sides incident to the vertex $y$ in the triangle $\Delta y y_i u_i$ has length $r$ and the other is shorter. By uniform convexity, we have $\operatorname{dist}(y, P) \le \|y - z_i\| < r$, and hence $\operatorname{dist}(y, Q)$ vanishes if $r \le \varepsilon$ and is less than $r - \varepsilon$ if $r > \varepsilon$. As a result, we have

$$\frac{\operatorname{dist}(y, Q)}{r} \le \max\left\{0,\ 1 - \frac{\varepsilon}{r}\right\}$$

in this case.

In case (b), the consideration of the same triangle gives the estimate

$$\operatorname{dist}(y, P) \le \|y - z_i\| < \sigma(r, \varepsilon)r + \varepsilon,$$

whence

$$\frac{\operatorname{dist}(y, Q)}{r} < \sigma(r, \varepsilon).$$

In case (c), Lemma 4.1 applies to each triangle $\Delta y y_i u_i$ with $\lambda = 1 - \alpha(r+\varepsilon) \in (0, 1]$. The points $u_i$ and $z_i$ are then $(\lambda\varepsilon)$-close to the straight lines passing through $y$ and $y_i$, and the points $y_i'$ at which the distance $\lambda\varepsilon$ is attained are the endpoints of the segments $[y, y_i']$, which contain the points $y_i$. Therefore, $y$ lies in the convex hull $\operatorname{conv}\{y_1', y_2', \dots, y_n'\}$ of points which are $(\lambda\varepsilon)$-close to the set $P$.

Therefore, the point $y$ is $(\lambda\varepsilon)$-close to some point $z$ of the polyhedron $\operatorname{conv}\{z_1, \dots, z_n\}$. By assumption, the distance of such a point from the set $P$ obeys the estimate

$$\operatorname{dist}(z, P) \le \alpha_P(r+\varepsilon)\cdot(r+\varepsilon) < \alpha(r+\varepsilon)\cdot(r+\varepsilon).$$

Therefore,

$$\operatorname{dist}(y, P) < \lambda\varepsilon + \alpha(r+\varepsilon)\cdot(r+\varepsilon) = (1-\alpha)(r+\varepsilon)\cdot\varepsilon + \alpha(r+\varepsilon)\cdot(r+\varepsilon) = \varepsilon + \alpha(r+\varepsilon)\cdot r$$

by the choice of the parameter $\lambda$. Hence $\operatorname{dist}(y, Q)/r < \alpha(r+\varepsilon)$. As a result, we have

$$\frac{\operatorname{dist}(y, Q)}{r} < \max\left\{\alpha(r+\varepsilon),\ \sigma_\lambda(r, \varepsilon),\ 1 - \frac{\varepsilon}{r},\ 0\right\}.$$

The right-hand side of the last inequality gives the required upper bound $\beta(\cdot) < 1_{-0}$ for the *central* function of nonconvexity of the set $Q$. By Lemma 14 from [9], the usual function of nonconvexity of this set also has a geometrically summable majorant depending only on the majorant $\beta(\cdot)$ found above and on the modulus of uniform convexity of the space $Y$. This concludes the proof of Theorem 2.3. $\square$

**Proof of Proposition 2.4.** For a given number $0 \le q < 1$, we define the number $p \in (q, 1)$ as a solution to the equation

$$\frac{\sqrt{1+q^2}-1}{p-q} = 2 \cdot \frac{p - q/\sqrt{1+q^2}}{1-p^2}.$$

Both sides of this equation are continuous with respect to the variable $p \in (q, 1)$. At $p = 1$, the left-hand side is finite and the right-hand side is infinite, and at $p = q$, *vice versa*. Therefore, the interval $(q, 1)$ indeed contains a solution of this equation.

For an arbitrary $\varepsilon > 0$, consider the points $C_{\pm} = (\pm R, 0)$, where $R$ is the product of $\varepsilon$ by the left-hand (= right-hand) side of the equation written above, in the Euclidean plane $\mathbb{R}^2$. Let $A_{\pm}$ be the points obtained from $C_{\pm}$ by the translation along the $OX$-axis by $\pm\varepsilon q/\sqrt{1+q^2}$ and the lift along the $OY$-axis by $1/\sqrt{1+q^2}$, and let $B_{\pm}$ be the points obtained by lifting the points $A_{\pm}$ along the $OY$-axis by $q \cdot x(A_+)$.

Consider the set $K_\varepsilon$ defined to be the polygonal line $A_-B_-B_+A_+$. This set is $q$-paraconvex by construction. In addition, the points $C_{\pm}$ are $\varepsilon$-close to the points $A_{\pm}$; therefore, the origin $\theta$ lies in $\operatorname{conv}(\overline{D}(K_\varepsilon, \varepsilon))$. Clearly, the minimum distance between $\theta$ and the points of the set $K_\varepsilon$ is attained at the endpoints $A_{\pm}$ of the set $K_\varepsilon$ and at the point $B = (B_- + B_+)/2$. But the numbers $R > 0$ and $p \in (q, 1)$ are chosen so that

$$\operatorname{dist}(\theta, A_{\pm}) = \operatorname{dist}(\theta, B) = pR + \varepsilon.$$

Therefore, $\operatorname{dist}(\theta, \overline{D}(K_\varepsilon, \varepsilon)) = pR$, and the set $\overline{D}(K_\varepsilon, \varepsilon)$ is not $p$-paraconvex, because the value of its (closed) function of nonconvexity at the point $R$ is precisely $p$.

Note that a similar effect occurs for the value of the nonconvexity function at the point $\lambda R$ of the closed $(\lambda\varepsilon)$-neighborhood of a set homothetic to $K_\varepsilon$ with factor $\lambda > 0$. It remains to take a convergent series of positive numbers $\lambda_n$, arrange $\lambda_n$-homothetic copies of $K_\varepsilon$ at suitable distances apart on a fixed straight line, join them by segments of this line, and add one limit point. The compact set $K$ thus obtained is $q$-paraconvex, but, for each $n \in \mathbb{N}$, the nonconvexity function of its closed $(\lambda_n\varepsilon)$-neighborhood takes a value $p > q$ at the point $\lambda_n R$. This completes the proof of Proposition 2.4. $\square$

## 5. CONCLUSION

Examination of upper semicontinuous set-valued maps is usually reduced to the case of single-valued maps by means of (graphic) approximations rather than selections. A single-valued map $f\colon X \to Y$ of metric spaces is called an *$\varepsilon$-approximation* of a set-valued map $F\colon X \to 2^Y$ if the graph of $f$ lies in the $(\varepsilon \times \varepsilon)$-neighborhood of the graph of $F$. The approximability (i.e., the existence of $\varepsilon$-approximations for all $\varepsilon > 0$) of an upper semicontinuous closed-valued map of a compact absolute extensor (= $AE$) $X$ to itself ensures the existence of fixed points for this map [4–7]. For an $n$-dimensional (possibly noncompact) space $X$, the approximability of an upper semicontinuous compact-valued map $F\colon X \to Y$ is a corollary of the following purely topological property of the values $F(x)$ known as the *$UV^n$-property*: for any neighborhood $U$ of the set $F(x)$, there exists a smaller neighborhood $V$ such that the identity embedding $V \hookrightarrow U$ is homotopically trivial in dimension $n$. In full generality, this result has recently been obtained by Shchepin and Brodskii [11], who used the ideas of [12, 13]. For an infinite-dimensional *compact* absolute neighborhood extensor (=$ANE$) $X$, the approximability of a map $F\colon X \to X$ is ensured by the

$UV(\infty)$-property of the values of $F$. This result is due to Granas, Gorniewicz, and Kryszewsky [14]; Kryszewsky [15] generalized it to locally finite-dimensionally polyhedral spaces $X$. In the case of an arbitrary infinite-dimensional domain $X$, the approximability problem has no purely topological solution even for *compact* spaces $X$. The point is that the approximable cell-like surjections of compact spaces do not increase the Lebesgue dimension [16], while general cell-like surjections may increase it [17]. At the same time, adding the assumption that the map $F$ is *convex-valued*, we arrive immediately to a solution of this problem, even if we do not assume this map to be compact-valued; this is the classical von Neumann–Cellina approximation theorem [6, 7].

Thus, in the search for a nonconvex-valued analog of the von Neumann–Cellina approximation theorem for infinite-dimensional domains, the question about the correlation between the nonconvexity properties of a set and those of its $\varepsilon$-neighborhoods naturally arises. Theorem 2.3 shows that the neighborhoods of paraconvex subsets of uniformly convex Banach spaces are topologically trivial. In particular, such paraconvex sets are $UV(\infty)$-subsets of the ambient space. As a consequence, any upper semicontinuous $\alpha$-paraconvex-valued map of a metric space to a uniformly convex Banach space is approximable for $\alpha < 1_{-0}$. We believe that this approximation result is valid for *any* normed space.

To conclude, we mention several points about paraconvexity with given accuracy (see Definition 1.1(c) ). The property of being paraconvex is not stable with respect to the Hausdorff metric in the exponent of a Banach space. Even small (with respect to the Hausdorff metric) deformations of a paraconvex set may yield nonparaconvex and even topologically nontrivial sets. On the contrary, paraconvexity with given accuracy *is* stable in this sense. The inequality $\mathrm{Hausd}(P,Q) < \lambda$ in the statement of the following proposition means that the sets $P$ and $Q$ lie in each other's $\lambda$-neighborhoods.

**Proposition 5.1.** *For every normed space $Y$ and any numbers $q \in [0,1)$, $\delta > 0$, and $p \in (q,1)$, there exists a number $\lambda \in (0,\delta)$ such that, if $P \subset Y$ is $q$-paraconvex with accuracy $\delta$ and* $\mathrm{Hausd}(P,Q) < \lambda$, *then $Q$ is $p$-paraconvex with the same accuracy $\delta$.*

Finally, we state an analog of Theorem 2.3 for paraconvexity with given accuracy.

**Theorem 5.2.** *For any uniformly convex Banach space $Y$, any function $\alpha(\,\cdot\,) < 1_{-0}$, and any positive numbers $\varepsilon$ and $\delta$, there exists a function $\beta(\,\cdot\,) < 1_{-0}$ such that the closed $\varepsilon$-neighborhood $\overline{D}(P,\varepsilon)$ of an arbitrary set $P$ $\alpha$-paraconvex with accuracy $\delta$ is $\beta$-paraconvex with accuracy $\delta$.*

The proof of Theorem 5.2 is a mere repetition of the proof of Theorem 2.3 with the only alteration that all finite subsets under consideration must have Chebyshev radii larger than the given accuracy $\delta$.

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(D. Repovš) University of Ljubljana, Slovenia
*E-mail*: dusan.repovs@fmf.uni-lj.si
(P. V. Semenov) Moscow State Pedagogical University
*E-mail*: pavels@orc.ru