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RESOLVING ACYCLIC IMAGES OF NONORIENTABLE THREE-MANIFOLDS

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ABSTRACT. We show that every 1-LC $\mathbf{Z}_2$-homology 3-manifold (without boundary) which is an almost 1-acyclic (over $\mathbf{Z}_2$) proper image of a *nonorientable* 3-manifold M (without boundary) is a resolvable generalized 3-manifold. The analogous result for the case when M is *orientable* was recently proved by J. L. Bryant and R. C. Lacher.

1. Introduction. A space X is said to be *locally simply connected* (1-LC) if for every $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that any loop in V is null-homotopic in U. A compact subset Y of an ANR X is *cell-like* if for every neighborhood $U \subset X$ of Y there is a neighborhood $V \subset U$ of Y such that V is contractible in U. A mapping f of an ANR M onto a space N is *cell-like* (resp. *monotone*) if for every $x \in N$, $f^{-1}(x)$ is a cell-like set (resp. compact and connected). A mapping $f\colon X \to Y$ is *proper* if it is closed and if $f^{-1}(y)$ is compact for all $y \in Y$.

Let R be a principal ideal domain. A metrizable space X is an *R-homology n-manifold* (with respect to singular homology and without boundary) provided $H_*(X, X - \{x\}; R) \cong H_*(\mathbf{R}^n, \mathbf{R}^n - \{0\}; R)$ for each $x \in X$, where $H_*(\ ; R)$ is the singular homology with coefficients in R. A *generalized n-manifold* is a euclidean neighborhood retract (ENR) that is also a $\mathbf{Z}$-homology n-manifold. An *n-dimensional resolution* of a space X is a pair (M, f) where M is an n-manifold without boundary and $f\colon M \to X$ is a proper, cell-like onto mapping.

J. L. Bryant and R. C. Lacher [**2**, Theorem 2] have proved that every locally contractible 1-acyclic over $\mathbf{Z}_2$ image X of a 3-manifold M without boundary admits a resolution. In particular, X is a generalized 3-manifold. A refinement of their proof enabled them to omit the acyclicity hypothesis over a 0-dimensional set, provided that M was orientable [**2**, Theorem 3]. We prove that orientability is not necessary.

THEOREM 1.1. *Let f be a closed, monotone mapping from a 3-manifold M without boundary onto a locally simply connected $\mathbf{Z}_2$-homology 3-manifold X. Suppose there is a 0-dimensional set $Z \subset X$ such that $\check{H}^1(f^{-1}(x); \mathbf{Z}_2) = 0$ for all $x \in X - Z$. Then the*

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set $C = \{x \in X \mid f^{-1}(x)$ *is not cell-like*$\}$ *is locally finite in* X. *Moreover*, X *is a resolvable generalized* 3-*manifold.*

As a corollary we obtain a partial converse in dimension 3 to the well-known fact that a cell-like upper semicontinuous decomposition G of an n-manifold M without boundary always yields a generalized n-manifold (if $n \geqslant 4$ one must assume, in addition, that M/G is finite dimensional) [**5**, **7**].

COROLLARY 1.2. *Let* G *be a* 0-*dimensional upper semicontinuous decomposition of a closed* 3-*manifold* M *such that* M/G *is a* 1-*LC* $\mathbf{Z}_2$-*homology* 3-*manifold. Then the set* $C = \{g \in G \mid g$ *is not cell-like*$\}$ *is finite.*

REMARK 1.3. Let $\pi\colon M \to M/G$ denote the quotient map, H_G the collection of all nondegenerate elements of G, and N_G their union.

(1) The Hopf maps or the Bing map [**1**] show that if $\pi(N_G)$ is a 1-manifold then all nondegenerate elements of G may fail to be cell-like.

(2) Spine maps [**1**] show that $C = \{g \in G \mid g$ is not cell-like$\}$ may have any finite number of elements even when $C = H_G$.

(3) An easy modification of the construction of the Whitehead continuum [**12**] shows that all nondegenerate elements of G may fail to be cellular even when $\pi(N_G)$ is a Cantor set and G is cell-like. (For details see [**11**].)

2. Neighborhoods of compacta in nonorientable 3-manifolds. Under some additional hypotheses, Knoblauch's finiteness theorem [**4**, Theorem 1] extends to nonorientable 3-manifolds.

PROPOSITION 2.1. *For every closed nonorientable* 3-*manifold* M *there exists an integer* K *such that if* $X_1, \ldots, X_{K+1} \subset M$ *are pairwise disjoint compact sets and each* X_i *has a neighborhood* $U_i \subset M$ *such that the inclusion-induced homomorphism* $H_1(U_i - X; \mathbf{Z}_2) \to H_1(M; \mathbf{Z}_2)$ *is trivial, then at least one* X_i *has a neighborhood in* M *which embeds in* $\mathbf{R}^3$.

PROOF. We shall suppress the $\mathbf{Z}_2$ coefficients from the notation. Let $X_1, \ldots, X_n \subset M$ be pairwise disjoint compact sets and suppose each X_i has a neighborhood $U_i \subset M$ such that the inclusion-induced homomorphism $H_1(U_i - X_i) \to H_1(M)$ is trivial, and if $i \neq j$ then $U_i \cap U_j = \varnothing$. Let $X = \bigcup_{i=1}^n X_i$ and $U = \bigcup_{i=1}^n U_i$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & \bigoplus_{i=1}^{n} H_1(U_i - X_i) & \to & \bigoplus_{i=1}^{n} H_1(U_i) & & & \\
 & \downarrow\cong & & \downarrow\cong & & & \\
\cdots \to & H_1(U - X) & \to & H_1(U) & \to & H_1(U, U - X) & \to \cdots \\
 & \downarrow & & \downarrow & & \Psi\downarrow\cong & \\
\cdots \to & H_1(M - X) & \to & H_1(M) & \to & H_1(M, M - X) & \to \cdots
\end{array}$$

where the horizontal sequences are exact and Ψ is the excision isomorphism. It is easy to see that the image of the inclusion-induced homomorphism $H_1(U) \to H_1(M)$

is the direct sum of the images of the inclusion-induced homomorphisms $H_1(U_i) \to H_1(M)$, $1 \leq i \leq n$. So if we let $\beta_1 = \operatorname{rank} H_1(M)$ then $n - \beta_1$ of the homomorphisms $H_1(U_i) \to H_1(M)$ are trivial. It follows by [**8**, Lemma (4.1)] that $n - \beta_1$ of the neighborhoods U_i are orientable. Let $k(\tilde{M})$ be the Knoblauch number of the orientable 3-manifold double cover of M [**4**, Theorem 1]. Since every orientable neighborhood lifts in $\tilde{M}$ to two (homeomorphic) copies, it follows that if $2(n - \beta_1) > k(\tilde{M})$ then some $\tilde{X}_i$ has a neighborhood in M which embeds in $\mathbf{R}^3$. We can now determine the number K from the equation $2(K - \beta_1) = k(\tilde{M})$.

PROPOSITION 2.2. *Let K be a compact connected subset of the interior of a 3-manifold M. Suppose K does not separate its connected neighborhoods and, for every neighborhood $U \subset M$ of K there exists a neighborhood $V \subset U$ of K such that the inclusion-induced homomorphism $H_1(V - K; \mathbf{Z}_2) \to H_1(U; \mathbf{Z}_2)$ is trivial. Then $K = \bigcap_{i=1}^{\infty} N_i$ where each $N_i \subset \operatorname{int} M$ is a compact 3-manifold with boundary satisfying the following properties*:

(i) $N_{i+1} \subset \operatorname{int} N_i$;

(ii) *N_i is obtained from a compact 3-manifold Q_i with a 2-sphere boundary by adding to ∂Q_i a finite number of orientable (solid) 1-handles*;

(iii) *the inclusion-induced homomorphism $H_1(\partial N_{i+1}; \mathbf{Z}_2) \to H_1(N_i; \mathbf{Z}_2)$ is trivial*;

(iv) *there is a homeomorphism h_i: $N_i \to N_i$ such that $h_i | \partial N_i =$ identity and $h_i(Q_i^*) = Q_{i+1}$, where $Q_i^* \subset \operatorname{int} Q_i$ is formed by pushing Q_i into $\operatorname{int} Q_i$ along a collar of ∂Q_i.*

REMARK 2.3. An examination of the proofs in [**10**] shows that the orientability hypothesis can be removed from all results in [**10**] if one uses Proposition 2.2 in place of [**9**, Theorem 2].

PROOF OF PROPOSITION 2.2. By [**13**, Theorem 2], $K = \bigcap_{i=1}^{\infty} N_i$ where each $N_i \subset \operatorname{int} M$ is a compact 3-manifold with boundary satisfying (i) and (ii) above (the orientability of the 1-handles follows by [**8**, Lemma (4.1)]). By choosing an appropriate subsequence of $\{N_i\}$ we can satisfy (iii). We prove (iv). Let $K_i \subset \operatorname{int} Q_i$ be a spine of Q_i. Let $\hat{Q}_i$ be the closed 3-manifold we obtain by attaching a 3-cell to ∂Q_i. For each $i \geq 1$, $N_1 = (N_1/K_i) \# \hat{Q}_i$ (the interior connected sum [**3**]). Since N_1 is nonorientable, it admits a unique normal, prime decomposition $N_1 = M_1 \# \cdots \# M_n$, $M_i \neq S^2 \times S^1$ [**3**, Theorem (3.15) and Lemma (3.17)]. Consider normal, prime decompositions of N_1/K_1 and $\hat{Q}_i$ $(i \geq 1)$. Since N_1/K_i is clearly orientable, its normal, prime decomposition $N_1/K_i = A_1 \# \cdots \# A_p \# B_1 \# \cdots \# B_q$ may contain $p > 0$ summands $A_i = S^2 \times S^1$. On the other hand, $\hat{Q}_1$ is nonorientable (since N_1 is) so its normal prime decomposition $\hat{Q}_i = C_1 \# \cdots \# C_r$ contains no $S^2 \times S^1$ summands. By [**3**, Lemma (3.17)] we may replace each A_i by $P =$ the nonorientable S^2-bundle over S^1 to get a normal, prime decomposition $N_1 = P \# \cdots \# P \# B_1 \# \cdots \# B_q \# C_1 \# \cdots \# C_r$ (p summands P) of N_1. It follows by the uniqueness of normal, prime decompositions that $p + q + r = n$ and that after a suitable permutation of the summands each C_i is homeomorphic to some M_i. We may conclude that among any $n + 1$ $\hat{Q}_i$'s at least two have the same prime summands (up to a homeomorphism). By choosing an appropriate subsequence of $\{Q_i\}$ we may henceforth assume that for each $i \leq j$ there is a homeomorphism s_{ij}: $Q_i \to Q_j$.

We first construct h_1. The identity on ∂N_1 induces a homeomorphism t'_{ij}: $\partial(N_1/K_i) \to \partial(N_1/K_j)$ for each $i \leq j$. Using Dehn's lemma we can extend t'_{ij} to a homeomorphism t_{ij}: $N_1/K_i \to N_1/K_j$. Finally, define h_{ij}: $N_1 \to N_1$ by $h_{ij}(x) = s_{ij}(x)$ if $x \in Q_i$ and $h_{ij}(x) = t_{ij}(x)$ if $x \in N_1 - Q_i$. Clearly, $h_{ij} | \partial N_1 =$ identity and $h_{ij}(Q_i^*) = Q_j^*$. We define h_2 as the composition of h_{12} and a homeomorphism of N_1 that is the identity outside a neighborhood of ∂Q_2 in N_2 and pushes Q_2^* onto Q_2. We can get h_i, $i \geq 2$, in a similar way. For details see [**11**].

3. The proof of Theorem (1.1). We shall suppress the $\mathbf{Z}_2$ coefficients from the notation. Let $A = \{x \in X \mid \check{H}^1(f^{-1}(x)) \neq 0\}$. By [**6**, Theorem (4.1)], A is locally finite in X. Let $B = \{x \in X \mid f^{-1}(x)$ has no neighborhood embeddable in $\mathbf{R}^3\}$. In order to show that B is locally finite in X it suffices, by Proposition 2.1, to prove that for each $x \in X$, $f^{-1}(x)$ possess a neighborhood $U \subset M$ such that

$$H_1(U - f^{-1}(x)) \to H_1(M)$$

is trivial. So let $x \in X$. Since A is locally finite there is a neighborhood $W \subset X$ of x such that $W \cap A \subset \{x\}$. By hypothesis X is LC^1 so there is a connected neighborhood $W' \subset W$ of x such that any loop in W' is null-homotopic in W. Consider the following commutative diagram:

$$\begin{array}{ccc}
H_1(f^{-1}(W') - f^{-1}(x)) & \overset{i'_*}{\to} & H_1(f^{-1}(W) - f^{-1}(x)) \\
\cong\downarrow f|_* & & \cong\downarrow f|_* \\
H_1(W' - \{x\}) & \to & H_1(W - \{x\}) \\
\downarrow j'_* & & \downarrow j_* \\
H_1(W') & \overset{i_*}{\to} & H_1(W)
\end{array}$$

where the horizontal homomorphisms are induced by inclusions, $f|_*$ is the Vietoris-mapping theorem isomorphism [7, 3.4], while j_* and j'_* are the isomorphisms from the homology sequence of the pairs $(W, W - \{x\})$ and $(W', W' - \{x\})$, respectively. By hypothesis, $i_* = 0$, hence $i'_* = 0$. Thus we may apply Proposition 2.1. By Proposition 2.2, $f^{-1}(x)$ is definable by (orientable) cubes with handles for all $x \in X - B$, so by [**9**, Theorem 3], $f^{-1}(x)$ has the 1-UV property. Since cubes with handles have no higher homotopy, each $f^{-1}(x)$ has the UV^∞ property and hence $C \subset B$ (cf. [7]). Therefore, C is locally finite in X. In particular, $X - C$ is finite dimensional by [**5**]. A resolution of X is now obtained by improving f over the points of C. This is done similarly as in [**2**]. For details see [**11**].

References

1. J. L. Bryant and R. C. Lacher, *A Hopf-like invariant for mappings between odd-dimensional manifolds*, Topology Appl. **8** (1978), 47–62.

2. ______, *Resolving acyclic images of three-manifolds*, Math. Proc. Cambridge Philos. Soc. **88** (1980), 311–319.

3. J. Hempel, *3-manifolds*, Ann. of Math. Studies, vol. 86, Princeton Univ. Press, Princeton, N.J., 1976.

4. T. Knoblauch, *Imbedding compact 3-manifolds in E^3*, Proc. Amer. Math. Soc. **48** (1975), 447–453.

5. G. Kozlowski and J. J. Walsh, *The cell-like mapping problem*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 315–316.

6. R. C. Lacher, *Some mapping theorems*, Trans. Amer. Math. Soc. **195** (1974), 291–303.

7. ______, *Cell-like mappings and their generalizations*, Bull. Amer. Math. Soc. **83** (1977), 495–552.

8. R. C. Lacher and D. R. McMillan, Jr., *Partially acyclic mappings between manifolds*, Amer. J. Math. **94** (1972), 246–266.

9. D. R. McMillan, Jr., *Acyclicity in three-manifolds*, Bull. Amer. Math. Soc. **76** (1970), 942–964.

10. D. R. McMillan, Jr. and T. L. Thickstun, *Open three-manifolds and the Poincaré conjecture*, Topology **19** (1980), 313–320.

11. D. Repovš, *Generalized three-manifolds with zero-dimensional singular set*, Ph. D. Thesis, Florida State University, Tallahassee, 1983.

12. J. H. C. Whitehead, *A certain open manifold whose group is a unity*, Quart. J. Math. Oxford Ser. **6** (1935), 268–279.

13. A. Wright, *Mappings from* 3-*manifolds onto* 3-*manifolds*, Trans. Amer. Math. Soc. **167** (1972), 479–495.

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