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A DISJOINT DISKS PROPERTY FOR 3-MANIFOLDS

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We show that the map separation property (MSP), a concept due to H.W. Lambert and R.B. Sher, is an appropriate analogue of J.W. Cannon's disjoint disks property (DDP) for the class $\mathscr{C}$ of compact generalized 3-manifolds with zero-dimensional singular set, modulo the Poincaré conjecture. Our main result is that the Poincaré conjecture (in dimension three) is equivalent to the conjecture that every $X \in \mathscr{C}$ with the MSP is a topological 3-manifold.

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1. Introduction

Cannon's disjoint disks property (DDP) characterizes topological n-manifolds, $n \geq 5$, among generalized n-manifolds [8, 17]. We seek an analogue of DDP for 3-manifolds. We briefly review known results on these topics. Starbird [19] introduced two notions of the disjoint disks property (DDP I and DDP II) for decompositions G of E^3 (rather than for the quotient space E^3/G) and proved that if a cell-like 0-dimensional upper semicontinuous decomposition G satisfies either DDP I or DDP II, then $E^3/G = E^3$. Starbird's result is useful for generalized 3-manifolds X which are already known to be a quotient $X = E^3/G$. A different approach was taken by Bryant and Lacher [5] who showed that if in a compact generalized 3-manifold X the singular set $S(X)$ lies in a compact, tamely embedded 0-dimensional set $Z \subset X$ (i.e., Z is 1-LCC in X), then X is a topological 3-manifold, provided X contains at most finitely many pairwise disjoint fake cubes. (This generalizes previous results of Edwards, Jr. [7] and Wall [23].) However, the condition "$S(X) \subset Z$ where Z is a closed 1-LCC subset of X" is not suitable since

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many potential singular sets may be wildly embedded in X. It is suggested in [13] that one should look for a disjoint disks property for generalized 3-manifolds X with 0-dimensional singular set such that it would imply first, the existence of a resolution $f: M \to X$ and second, the shrinkability of $G = \{f^{-1}(x) | x \in X\}$.

There are few positive results on existence of resolutions of generalized 3-manifolds. Brin and McMillan, Jr. [4] proved that, modulo the Poincaré conjecture, every compact generalized 3-manifold with 0-dimensional singular set has a resolution, provided it satisfies a certain 'torsion-free' hypothesis. This extra condition was inherited from Brin's Loop theorem [2] they used in their proof. Thickstun [20] removed the 'torsion-free' hypothesis from [2] and thus from [4]. He later proved a positive result [21] (obtained independently by R.J. Daverman) to the effect that such generalized 3-manifolds are images of 'tame' generalized 3-manifolds (whose singular set has genus 0 at each point). Another positive result is due to Bryant and Lacher [5] who proved that every locally contractible $\mathbb{Z}_2$-acyclic image of a 3-manifold has a resolution. (For generalizations see [5; Theorem 3] and [18; Theorem 1.1].)

In this paper we show that a concept due to Lambert and Sher [14], called the map separation property (MSP), characterizes the 3-manifold property in certain cases (modulo the Poincaré conjecture). Our main result is: the conjecture that every compact generalized 3-manifold X with $\dim S(X) \leq 0$ satisfying the MSP is a topological 3-manifold is equivalent to the 3-dimensional Poincaré conjecture. We also study a similar concept from [14] called the Dehn's lemma property (DLP) and show that it plays the same role as the MSP.

2. Dehn disks in 3-manifolds

Throughout this paper a *mapping* will mean only a continuous, hence not necessarily PL, map and an *n-manifold* will mean an n-manifold without boundary. A mapping f of a disk (resp. disk with holes) D into a space X is called a *Dehn disk* (resp. *Dehn disk with holes*) if $\partial D \cap S_f = \emptyset$, where $S_f = \text{cl}\{x \in D \,|\, f^{-1}f(x) \neq x\}$ is the *singular set* of f. Also, define $\Sigma_f = f(S_f)$. A space X is said to have the *Dehn's lemma property* (DLP) [14] if for every Dehn disk $f: D \to X$ and every neighborhood $U \subset X$ of Σ_f there exists an embedding $F: D \to f(D) \cup U$ such that $F(\partial D) = f(\partial D)$. A space X is said to have the *map separation property* (MSP) [14] if given any collection $f_1, \ldots, f_k : D \to X$ of Dehn disks such that if $i \neq j$, then $f_i(\partial D) \cap f_j(D) = \emptyset$, and given a neighborhood $U \subset X$ of $\bigcup_{i=1}^k f_i(D)$ there exist mappings $F_1, \ldots, F_k : D \to U$ such that for each i, $F_i|\partial D = f_i|\partial D$ and if $i \neq j$, then $F_i(D) \cap F_j(D) = \emptyset$.

Theorem 2.1. *Let $f: D \to M$ be a Dehn disk in a 3-manifold M (possibly with boundary) and $U \subset M$ a neighborhood of Σ_f. Then there exists an embedding $F: D \to f(D) \cup U$ such that*

(i) $F(D)-U=f(D)-U$,
(ii) $F|\partial D=f|\partial D$.

Proof. By adding a collar on ∂M we may always assume that $f(D)\subset \operatorname{int} M$. Let $U'=f^{-1}(U)$. By [6; Theorem (4.8.3)] there exist families $\{A_i^{(j)}|1\leq i\leq t\}$, $1\leq j\leq 4$, of pairwise disjoint PL disks with holes in U' such that

(a1) for each i and j, $A_i^{(j)}\subset \operatorname{int} A_i^{(j+1)}$,
(a2) $S_f\subset \operatorname{int} B_1$,

where $B_j=\bigcup_{i=1}^t A_i^{(j)}$. Let $V_k=U-f(D-\operatorname{int} B_{2k-1})$, $k=1,2$. Then each V_k is open in M and if we let $V'_k=f^{-1}(V_k)$, then we have

(a3) $S_f\subset V'_1\subset \operatorname{int} B_1$,
(a4) $B_2\subset V'_2\subset \operatorname{int} B_3$.

Let $L\subset D$ be a PL annulus such that $L\cap U'=\emptyset$ and $\partial L\cap\partial D=\partial D$. Finally, let $K\subset L$ be a PL annulus such that $\partial L\cap\partial K=\partial D$.

Apply Bing's surface approximation theorem [1] to replace f by a Dehn disk $f_1:D\to M$ with the following properties

(b1) $f_1|D-D_1=f|D-D_1$,
(b2) $f_1|D_1$ is locally PL,
(b3) $S_{f_1}=S_f$,

where $D_1=\operatorname{int}(B_4-B_1)$. Applying [1] again we get a Dehn disk $f_2:D\to M$ such that

(c1) $f_2|D-\operatorname{int} L=f_1|D-\operatorname{int} L$,
(c2) $f_2|\operatorname{int} L$ is locally PL,
(c3) $S_{f_2}=S_{f_1}$.

Another application of [1] yields a Dehn disk $f_3:D\to M$ such that

(d1) $f_3|D_2=f_2|D_2$,
(d2) $f_3|D-D_2$ is locally PL,
(d3) $S_{f_3}=S_{f_2}$,

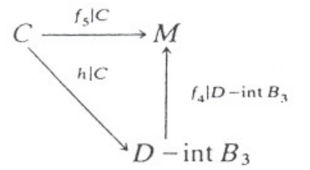
where $D_2=K\cup B_3$. By Zeeman's relative simplicial approximation theorem [24] there is a Dehn disk $f_4:D\to M$ such that

(e1) $f_4|D-\operatorname{int} B_2=f_3|D-\operatorname{int} B_2$,
(e2) $f_4|\operatorname{int} D$ is locally PL,
(e3) $S_{f_4}\subset V'_2$.

By Henderson's extension of Dehn lemma [10; Theorem (IV.3)] there is an embedding $f_5:D\to M$ such that

(f1) $f_5|\operatorname{int} D$ is locally PL,
(f2) $f_5|K=f_4|K$;
(f3) $f_5(D)-V_2=f_4(D)-V_2$.

Note that by (f3), $f_5(D) \subset f_4(D) \cup V_2$ and by (a4), (b1), (c1), (d1), (e1), and (f3) we have $f_4(D - \operatorname{int} B_3) \subset f_5(D)$. Clearly, f_4 and f_5 need not agree pointwisely even outside V_2. Let $C = f_5^{-1} f_4(D - \operatorname{int} B_3)$. By (a3), (b3), (c3), (d3), and (e3) there is a PL homeomorphism $h: D \to D$ which makes the diagram



commute. We now get the desired embedding $F: D \to f(D) \cup U$ by letting

$$F(x) = \begin{cases} f_1 h(x), & x \in C, \\ f_5(x), & x \in D - C. \end{cases}$$

Corollary 2.2. *Every 3-manifold (possibly with boundary) has the DLP.*

Theorem 2.3. *Let $f_1, \ldots, f_k: D \to M$ be Dehn disks in a 3-manifold M (possibly with boundary) such that if $i \neq j$, then $f_i(\partial D) \cap f_j(D) = \emptyset$. Then for every neighborhood $U \subset M$ of $\bigcup_{i=1}^k f_i(D)$ there exist embeddings $F_1, \ldots, F_k: D \to U$ such that*

(i) *for each i, $F_i|\operatorname{int} D: \operatorname{int} D \to U$ is locally PL,*

(ii) *for each i, $F_i|\partial D = f_i|\partial D$, and*

(iii) *if $i \neq j$, then $F_i(D) \cap F_j(D) = \emptyset$.*

Proof. We use induction on k. For $k = 1$ the assertion follows by Theorem 2.1 and Bing's surface approximation theorem [1]. Assume now the assertion is true for all $k \leqslant n$ and consider the case $k = n+1$. By the inductive hypothesis there are embeddings $F_1, \ldots, F_n: D \to U - f_{n+1}(\partial D)$ satisfying (i)–(iii) and f_{n+1} can be replaced by an embedding

$$f'_{n+1}: D \to U - \left(\bigcup_{i=1}^n F_i(\partial D)\right)$$

such that $f'_{n+1}|\operatorname{int} D$ is locally PL, f'_{n+1} is in general position with respect to the surface $S = \bigcup_{i=1}^n F_i(D)$, and $f'_{n+1}|\partial D = f_{n+1}|\partial D$. Hence $f'_{n+1}(D) \cap S$ is a finite collection of pairwise disjoint PL simple closed curves. Starting off with an innermost (on the surface S) of these curves, we can cut $f'_{n+1}(D)$ off S, inside the neighborhood U, thus obtaining F_{n+1}.

Corollary 2.4. *Every 3-manifold (possibly with boundary) has the MSP.*

3. Recognizing 3-manifolds

A *generalized n-manifold* is an euclidean neighborhood retract (ENR) X that is also a $\mathbb{Z}$-homology n-manifold, i.e., for each $x \in X$,

$$H_*(X, X - \{x\}; \mathbb{Z}) = H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}).$$

A *generalized n-manifold with boundary* is an euclidean neighborhood retract X that is also a $\mathbb{Z}$-homology n-manifold with boundary, i.e., for each $x \in X$ either

$$\check{H}^*(X, X-\{x\}; \mathbb{Z}) = \check{H}^{n-*}(\{x\}; \mathbb{Z}) \quad \text{or} \quad \check{H}^*(X, X-\{x\}; \mathbb{Z}) = 0,$$

and such that ∂X is a generalized $(n-1)$-manifold, where

$$\partial X = \{x \in X \mid \check{H}^*(X, X-\{x\}; \mathbb{Z}) = 0\}.$$

The *singular set* $S(X)$ of a generalized n-manifold (resp. generalized n-manifold with boundary) X consists of the *singularities*, i.e., those points of X that have no neighborhood in X homeomorphic to $\mathbb{R}^n$ (resp. $\mathbb{B}^n$). We use $M(X)$ to denote the *manifold set* $X - S(X)$. An *n-resolution* of a space X is a pair (M, f) where M is a topological n-manifold and $f: M \to X$ is a proper, cell-like onto mapping. It is well known that every finite-dimensional cell-like upper semicontinuous decomposition of an n-manifold yields a generalized n-manifold. (For a partial converse in dimension 3 see [18].) It is also known that a generalized n-manifold X $(n \neq 4)$ with a resolution has a *conservative resolution* $f: M \to X$, i.e., $f^{-1}(x) = \mathrm{pt}$ for all $x \in M(X)$ [5]. A generalized 3-manifold X (possibly with boundary) satisfies *Kneser Finiteness* [13] if every compact set $K \subset X$ contains but finitely many pairwise disjoint fake cubes.

Consider a generalized 3-manifold X with $\dim S(X) \leq 0$ and let $p \in X$. Then p has arbitrarily small orientable generalized 3-manifold with boundary neighborhoods with ∂N a compact orientable 2-manifold and $\partial N \cap S(X) = \emptyset$ (see [4; Lemma 1]). If p has arbitrary small such neighborhoods N with the genus of ∂N less than or equal to n, we say that *X has genus $\leq n$ at p*. If X has genus $\leq n$ at p but does not have genus $\leq n-1$ at p, we say *X has genus n at p*. If X does not have genus $\leq n$ at p for any integer n we say *X has genus ∞ at p* [13].

Let G be an upper semicontinuous decomposition of a space X. We shall use H_G to denote the collection of all nondegenerate elements of G and N_G to denote their union. A set $U \subset X$ is *G-saturated if* $\pi^{-1}\pi(U) = U$, where $\pi: X \to X/G$ is the quotient mapping. We say G is *closed 0-dimensional* if $\dim(\mathrm{cl}\, \pi(N_G)) = 0$.

Theorem 3.1. *Let G be a cell-like closed 0-dimensional upper semicontinuous decomposition of a 3-manifold M (possibly with boundary) with $\mathrm{cl}\, N_G \subset \mathrm{int}\, M$. Then the following statements are equivalent:*

(i) *M/G has the DLP.*
(ii) *M/G has the MSP.*
(iii) *M/G is a 3-manifold.*

Proof. The implications (iii)$\Rightarrow$(i) and (iii)$\Rightarrow$(ii) follow by Corollaries 2.2 and 2.4, respectively. We prove (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iii) simultaneously. So assume M/G has either the DLP or the MSP.

Assertion 1. *If every $g \in G$ has a neighborhood embeddable in $\mathbb{R}^3$ then M/G is homeomorphic to M.*

By [15; Theorem 3] G is definable by cubes with handles. Since G is 0-dimensional it suffices to show that G is weakly shrinkable [22; Lemma (2.5)], i.e., we only must prove that for every $\varepsilon > 0$ and every neighborhood U of N_G there exists a homeomorphism $h: M \to M$ such that $h|M - U = \text{identity}$ and $\operatorname{diam} h(g) < \varepsilon$ for all $g \in G$. The proof of [14; Theorem 4] will work except for one change – instead of [16; Theorem (2.1)] we use [12; Lemma A, p. 506].

Assertion 2. *If $G_0 = \{g \in G | g$ has no neighborhood embeddable in $\mathbb{R}^3\}$, then $\pi(G_0)$ is locally finite in M/G.*

If M is orientable apply [11; Theorem 1] and if it is not, apply [18; Proposition 2.1].

Assertion 3. *For every $g \in G$ and every neighborhood $U \subset M$ of g there is a homotopy 3-cell $H \subset U$ such that $g \subset \operatorname{int} H$.*

We may assume that U is saturated. By [15; Theorem 3] G is definable by homotopy cubes with handles hence there is a homotopy cube with handles $H \subset U$ such that $g \subset \operatorname{int} H$. By going further enough in the defining sequence for G we may assume that on some neighborhood $N \subset U$ of ∂H, $\pi|N: N \to M/G$ is an embedding. The idea of the proof is to use the DLP or the MSP to cut the handles of H along pairwise disjoint compressing disks which miss g. We find such disks as follows.

Assume first that M/G has the DLP. Let C_1 and C_2 be disjoint simple closed curves on ∂H such that they are null-homotopic in H but not on ∂H. By Dehn's lemma [9; p. 39] there exist embeddings $f_1, f_2: (D, \partial D) \to (H, \partial H)$ such that $f_i(\partial D) = C_i$, $i = 1, 2$. By running a ribbon in $U - \operatorname{int} H$ between slightly expanded disks $f_1(D)$ and $f_2(D)$ we get an embedding $f: D \to U$ such that for disjoint subdisks $D_1, D_2 \subset \operatorname{int} D$, $f|D_i = f_i$, $i = 1, 2$ and $f(D - (D_1 \cup D_2)) \subset U - H$. Since $\pi|N: N \to M/G$ is an embedding $\pi f: D \to \pi(U)$ is a Dehn disk and $\Sigma_{\pi f} = \Sigma_{\pi f_1} \cup \Sigma_{\pi f_2}$. Therefore $\Sigma_{\pi f} \subset \pi(\operatorname{int} H)$ so using the DLP we can get an embedding $F: D \to \pi f(D) \cup \pi(\operatorname{int} H)$ such that $F(\partial D) = \pi f(\partial D)$. Let $q_i: D \to \pi(H)$ be the subdisks of $F(D)$ bounded by $\pi f_i(\partial D)$, $i = 1, 2$. Note that $q_1(D) \cap q_2(D) = \emptyset$ so there exist disjoint neighborhoods $W_i \subset \pi(U)$ of $q_i(D)$. Let $V_i = \pi^{-1}(W_i)$. By [12; Lemma A] q_i lifts to a Dehn disk $Q_i: D \to V_i \cap H$, $i = 1, 2$. By Theorem 2.1 and [1] we can assume Q_i is a locally PL embedding. Since $V_1 \cap V_2 = \emptyset$, one of the disks $Q_i(D)$ will miss g hence cutting along it we get a homotopy cube with one handle less, H^*, which contains g in its interior. In continuing this process one must be careful to choose the new pair of simple closed curves C_1^*, C_2^* away from the intersections of N_G with ∂H^*. That is because in doing the compression we may have hit some elements of $H_G - \{g\}$ so now $\partial H^* \cap N_G$ may no longer be empty. Since any possible intersections lie inside the two copies of the compressing disk on ∂H^* we can always push C_i^*'s off $H_G \cap \partial H^*$ if necessary. This way π will remain an embedding on a neighborhood of C_i^*, $i = 1, 2$.

If instead of DLP we have the MSP for M/G the procedure is similar. Instead of introducing f we use the MSP to separate $\pi f_1(D)$ and $\pi f_2(D)$ in $\pi(H)$, while the rest of the argument stays the same.

We now finish off the proof of the theorem, first for the case when $\partial M = \emptyset$. By Assertion 2, $G = G_0 \cup G_1$ where $G_1 = G - G_0$ and $\pi(G_0)$ is locally finite in M/G. Consider $M_0 = M/G_0$ and let $\pi_0 : M \to M_0$ be the corresponding quotient map. Since the elements of G are cell-like M_0 is a generalized 3-manifold. Clearly, $S(M_0) \subset \pi_0(G_0)$ where $S(M_0)$ is the singular set of M_0. Also, M_0 satisfies Kneser Finiteness by [5; p. 313].

Assertion 4. *For every $p \in M_0$, $g(M_0, p) = 0$.*

If $p \notin \pi_0(G_0)$, then $p \notin S(M_0)$, so the assertion is clear. Let $p \in \pi_0(G_0)$. By Assertion 2 there is a neighborhood $U \subset M_0$ of p such that $U \cap \pi_0(G_0) = \{p\}$. Let $V = \pi_0^{-1}(U)$. By Assertion 3 there is a homotopy cube $H \subset V$ such that $\pi_0^{-1}(p) \subset \text{int}\, H$ and $\partial H \cap (\bigcup\{g \in G_0\}) = \emptyset$. Therefore, $\pi_0(\partial H)$ is a 2-sphere so $\pi_0(H)$ is the desired neighborhood of p.

It now follows by Assertion 4 and by [13; Corollary (3.1)] that $S(M_0) = \emptyset$, since $\dim S(M_0) \leq \dim \pi_0(G_0) \leq 0$. Thus M_0 is a 3-manifold. Consider $G_1^* = G_1 \cup \pi_0(G_0)$ as a decomposition of M_0. By Assertions 2 and 3 the decomposition G_1^* is cellular, closed 0-dimensional, and upper semicontinuous. Also, $M_0/G_1^* = (M/G_0)/G_1^* = M/G$ so M_0/G_1^* has the DLP (resp. MSP). By Assertion 1, M_0/G_1^* is homeomorphic to M_0, so M/G is homeomorphic to M_0 thus a 3-manifold. This completes the proof if $\partial M = \emptyset$.

In the case when $\partial M \neq \emptyset$ we consider the double DM of M, i.e. we identify two copies of M along ∂M using the identity map and apply the preceeding arguments to the decomposition DG, the double of G. (Note however, that we are not claiming that if M/G has the DLP (or MSP), then DM/DG has this property, too.)

Theorem 3.2. *Let X be a generalized 3-manifold with 0-dimensional singular set, such that for every $p \in X$, $g(X, p) = 0$. Then X has the DLP and the MSP.*

Proof. We first prove the DLP. Let $f : D \to X$ be a Dehn disk. We first show that one may assume $f(\partial D) \cap S(X) = \emptyset$. By hypothesis there is a neighborhood $N \subset D$ of ∂D such that $S_f \cap N = \emptyset$. Thus $N \cap f^{-1}(S(X))$ is 0-dimensional so there is a simple closed curve $J \subset N - f^{-1}(S(X))$ such that J is isotopic in N to ∂D. Let $A \subset D$ be the subdisk of D bounded by J and consider the Dehn disk $f' = f|A : A \to X$. If we show how to find an embedding $F' : A \to f(A) \cup U$, where $U \subset X$ is a neighborhood of $\Sigma_f = \Sigma_{f'}$, such that $F'(J) = f'(J)$, then by defining $F : D \to X$ to be f on $D - A$ and F' on A we get the desired disk.

So assume that $f(\partial D) \cap S(X) = \emptyset$. Using the hypothesis and [4; Lemma 1] we can find a pairwise disjoint collection $N_1, \ldots, N_k$ of generalized 3-manifolds boundary such that

(i) $S(X)\cap(f(D)-U)\subset\bigcup_{i=1}^{p} N_i$,

(ii) $S(X)\cap f(D)\cap U\subset\bigcup_{i=p+1}^{k} N_i\subset U$,

(iii) for each i, ∂N_i is a locally PL 2-sphere, and

(iv) for each i, $\partial N_i\cap S(X)=\emptyset$.

Let $H=\bigcup_{i=1}^{k}\partial N_i$. Then $f(D)\cap H\subset M(X)$. We want $f(D)$ to meet H 'transversely'. But f may not be PL so we must improve it to be PL near H. We do this as follows: close to $\bigcup_{i=p+1}^{k}\partial N_i$ we use the simplicial approximation theorem while close to $\bigcup_{i=1}^{p}\partial N_i$ we use Bing's surface approximation theorem [1] in order to keep f an embedding in that region. By applying general position in $M(X)$ we can make f meet H transversely and by standard methods we can then cut f off at H (in $M(X)$). Denote the new (Dehn) disk by $f':D\to X$. Since $f'(D)\subset M(X)$ it follows by Theorem 2.1 that there is an embedding $F':D\to f'(D)\cup U$ such that

$$F'|\partial D=f'|\partial D \quad\text{and}\quad F'(D)-U=f'(D)-U.$$

Finally, replace the portions which ∂N_i $(1\leqslant i\leqslant p)$ cut off $F'(D)$ by $f(D)\cap N_i$. This yields the desired embedding $F:D\to f(D)\cup U$. Details are omitted since they are similar to those in the proof of Theorem 2.1.

We now prove X has the MSP. Let $f_1,\ldots,f_k:D\to X$ be Dehn disks, $U\subset X$ a neighborhood of $\bigcup_{i=1}^{k}f_i(D)$, and suppose that if $i\neq j$, then $f_i(\partial D)\cap f_j(D)=\emptyset$. As before we may assume that for each i, $f_i(\partial D)\cap S(X)=\emptyset$. Since X was already shown to have the DLP, we may assume all f_i are embeddings. Cover $S(X)\cap\bigcup_{i=1}^{k}f_i(D)$ by a collection $N_1,\ldots,N_t\subset U$ of pairwise disjoint generalized 3-manifolds with boundary such that for each i, ∂N_i is a locally PL 2-sphere and $\partial N_i\cap S(X)=\emptyset$. Let $H=\bigcup_{i=1}^{t}\partial N_i$. As before, we can apply Bing's surface approximation theorem [1] close to H in order to make H meet each $f_j(D)$ transversely. Cut each $f_j(D)$ off H (in $M(X)$) and get a new Dehn disk $f_j':D\to X$ with $f_j'|\partial D=f_j|\partial D$. Since $f_j'(D)\subset M(X)$ we can apply Corollary 2.4 to get f_j''s disjoint in U keeping their boundaries fixed. Since $f_j'|\partial D=f_j|\partial D$ this completes the proof.

Theorem 3.3. *Let $\mathscr{C}$ be the class of all compact generalized 3-manifolds X with $\dim S(X)\leqslant 0$ and let $\mathscr{C}_0\subset\mathscr{C}$ be the subclass of all $X\in\mathscr{C}$ with $S(X)\subset\{p\}$, and X homotopy equivalent to S^3. Then the following statements are equivalent:*

(i) *The Poincaré conjecture in dimension three is true.*

(ii) *If $X\in\mathscr{C}$ has the DLP or the MSP, then $S(X)=\emptyset$.*

(iii) *If $X\in\mathscr{C}_0$ has the DLP or the MSP, then $S(X)=\emptyset$.*

Proof. (i)⇒(ii). If the Poincaré conjecture is true, then X has a resolution [20; Corollary] (see also the concluding remarks in [3]), so by [5; Theorem 1] a conservative resolution $f: M \to X$. Let $G = \{f^{-1}(x) | x \in X\}$. Then G is a cell-like closed 0-dimensional upper semicontinuous decomposition, so by Theorem 3.1, $S(X) = \emptyset$.

(ii)$\Rightarrow$(iii). Obvious.

(iii)$\Rightarrow$(i). Suppose the Poincaré conjecture is false. Let $B_1, B_2, \ldots \subset S^3$ be a sequence of pairwise disjoint 3-cells converging to $p \in S^3$. Deleting interior of each B_i and sewing a fake cube F_i in its place yields a compact generalized 3-manifold X with $S(X) = \{p\}$ (see [5; p. 312]). The map from X onto S^3 which shrinks out each F_i is a homotopy equivalence by [12; p. 510]. Therefore $X \simeq S^3$, so $X \in \mathscr{C}_0$. On the other hand X has the DLP and the MSP by Theorem 3.2. This contradicts the assertion (iii).

Theorem 3.4. *Let X be a generalized 3-manifold satisfying Kneser Finiteness. Suppose that X has the DLP or that X has the MSP (in fact, it suffices to assume the MSP only for pairs of Dehn disks). Then X has no isolated singularities.*

Proof. By [13; Corollary (3.1)] it suffices to show that every point $p \in X$ which has a neighborhood $U \subset X$ such that $U \cap S(X) \subset \{p\}$, satisfies the condition that $g(X, p) = 0$. This is done using standard disk-trading techniques from 3-manifolds except that instead of the classical Loop theorem [9] we must invoke a version of the Loop theorem proved by Thickstun [20], and the classical Dehn lemma [9] is replaced here by the DLP (or the MSP) and Bing's surface approximation theorem [1]. The latter is done as follows: whenever we want to perform a cut along a compressing disk D which hits p we use DLP (or MSP) on two close copies of D to make one of them miss p so that the cut can be performed in $M(X)$.

Remark. Suppose X is a compact generalized 3-manifold with $\dim S(X) \leq 0$, satisfying Kneser Finiteness and having the DLP or MSP. If $S(X) \neq \emptyset$, then X has the following properties:

(i) X admits no resolution ([5; Theorem 1] and Theorem 3.1).
(ii) $S(X)$ is wildly embedded in X ([5; Theorem 4]).
(iii) $S(X)$ has no isolated points (Theorem 3.4).

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