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On generalized 3-manifolds which are not homologically locally connected

Umed H. Karimov [a], Dušan Repovš [b,c,*]

[a] Institute of Mathematics, Academy of Sciences of Tajikistan, Ul. Ainy 299[A], Dushanbe 734063, Tajikistan
[b] Faculty of Education, University of Ljubljana, Kardeljeva pl. 16, Ljubljana 1000, Slovenia
[c] Faculty of Mathematics and Physics, University of Ljubljana, P.O. Box 2964, Ljubljana 1001, Slovenia

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ABSTRACT

We show that the classical example X of a 3-dimensional generalized manifold constructed by van Kampen is not homologically locally connected (i.e. not HLC) space. This space X is not locally homeomorphic to any of the compact metrizable 3-dimensional manifolds constructed in our earlier paper which are not HLC spaces either.

1. Introduction

In our earlier paper [10] we constructed for every natural number $n > 2$, examples of n-dimensional compact metrizable cohomology n-manifolds which are not homologically locally connected with respect to the singular homology (i.e. they are not *HLC* spaces). In the present paper we shall call them *singular quotient n-manifolds.*

Subsequently, we have discovered that van Kampen constructed a compact metrizable generalized 3-manifold which "is not locally connected in dimension 1 in the *homotopy sense*" [2, p. 573]. The description of van Kampen's construction can be found in [2, p. 573] (see also [16, p. 245]).

The obvious modification of van Kampen's construction gives an infinite class of examples – we shall call them *van Kampen generalized* 3*-manifolds*. The main purpose of the present paper is to prove the following theorem:

Theorem 1.1. *No van Kampen generalized* 3*-manifold is homologically locally connected in dimension* 1 *with respect to the singular homology. They are neither locally homeomorphic to any singular quotient* 3*-manifold. Furthermore, no singular quotient* 3*-manifold is locally homeomorphic to any van Kampen generalized* 3*-manifold.*

2. Preliminaries

We shall denote the singular homology (resp. Čech cohomology) groups with integer coefficients by H_* (resp. $\check{H}^*$). All spaces considered in this paper will be assumed to be metrizable, locally compact and finite-dimensional. Under these

* Corresponding author at: Faculty of Education, University of Ljubljana, Kardeljeva pl. 16, Ljubljana 1000, Slovenia.
E-mail addresses: umedkarimov@gmail.com (U.H. Karimov), dusan.repovs@guest.arnes.si (D. Repovš).

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circumstances the classes of *generalized manifolds*, *homology manifolds*, and *cohomology manifolds* in the classical sense [16] coincide (cf. [3,9,11]). For the history and importance of generalized manifolds see [13].

Definition 2.1. (Cf. [3, p. 377, Corollary 16.9].) A locally compact, cohomologically locally connected with respect to Čech cohomology (*clc*), and cohomologically finite-dimensional space X is called a *generalized* n*-manifold*, $n \in \mathbb{N}$, if

$$\check{H}^p\big(X, X \setminus \{x\}\big) \cong \check{H}^p\big(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}\big)$$

for all $x \in X$ and all $p \in \mathbb{Z}_+$.

Definition 2.2. A closed 3-manifold is called a *homology* 3*-sphere* if its homology groups are isomorphic to the homology groups of the standard 3-sphere (cf. e.g. [8]). The complement of the interior of any 3-simplex in any triangulation of any homology 3-sphere is called a *homology* 3*-ball*.

In 1904 Poincaré constructed the first example of a homology 3-sphere with a nontrivial fundamental group [12,14]. The following theorem is due to Brown [4]:

Theorem 2.3 *(Generalized Schönflies theorem). Let $h : S^{n-1} \times [-1, 1] \to S^n$ be an embedding. Then the closures of both complementary domains of $h(S^{n-1} \times 0)$ in S^n are topological n-cells.*

Definition 2.4. A space X is said to be *locally homeomorphic* to the space Y if for every point $x \in X$ there exists an open neighborhood $U_x \subset X$ which is homeomorphic to some open subspace of Y.

Definition 2.5. (Cf. e.g. [1,5,10].) Let G be any group and $g \in G$ any element. The *commutator length* $cl(g)$ of the element g is defined as the minimal number of the commutators of the group G the product of which is g. If such a number does not exist then we set $cl(g) = \infty$. Moreover, $cl(g) = 0$ if and only if $g = e$, where e denotes the neutral element.

Theorem 2.6. *(Cf. [6,7].) Let G be a free product of groups $\{G_i\}_{i=1}^k$, $G = G_1 * G_2 * \cdots * G_k$. Let $g \in G$, $g_i \in G_i$ for $i = \overline{1,k}$ and let $g = g_1 g_2 \cdots g_k$. Then $cl(g) = \sum_{i=1}^k cl(g_i)$.*

Assertion 2.7. *The commutator length function has the following properties*:

(1) *if $\varphi : G \to H$ is a homomorphism of groups and $g \in G$ then $cl(\varphi(g)) \leqslant cl(g)$; and*
(2) *for every element $g \in G$, $cl(g) = \infty \Leftrightarrow g \notin [G, G]$.*

All undefined terms can be found in [3], [14] or [15].

3. The construction of the van Kampen example and the proof of the main theorem

To prove the main theorem we shall need a more detailed description of the original van Kampen construction [2, p. 573]. For nonnegative numbers i, p and q let the *ball layer* $A_{i,p,q}$ be the following subspace of $\mathbb{R}^3$:

$$A_{i,p,q} = \big\{\bar{a} \in \mathbb{R}^3 \;\big|\; p \leqslant \big|\bar{a} - (0, i, 0)\big| \leqslant p + q,\ (0, i, 0) \in \mathbb{R}^3\big\}.$$

In particular, $A_{i,0,q}$ is a 3-ball with the center at the point $(0, i, 0) \in \mathbb{R}^3$ and of radius q. For $i \in \mathbb{N}$, let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of a homology 3-balls with a nontrivial fundamental group. All spaces B_i are compact 3-manifolds the boundaries of which are homeomorphic to S^2.

Let for $n, k \in \mathbb{N}$

$$X_{n,k} = \overline{A_{0,(n-\frac{1}{2}),k} \setminus \bigcup_{j=n}^{n+k-1} A_{j,0,\frac{1}{4}}} \cup \bigcup_{i=n}^{n+k-1} B_i$$

with the topology of identification of the boundaries of $A_{i,0,\frac{1}{4}}$ and B_i (the bar over the space denotes the closure of that space). Let

$$X_k = X_{1,k} \cup A_{0,0,\frac{1}{2}}.$$

Obviously, $X_{n,k}$ and X_k are compact 3-manifolds with boundaries. Since the homology groups of B_i are the same as the homology groups of the 3-ball $A_{i,0,\frac{1}{4}}$ it follows from the exactness of the Mayer–Vietoris homology sequence that the homology groups of X_k are the same as the homology groups of $A_{0,0,(k+\frac{1}{2})}$, i.e. they are trivial and X_k is a homology 3-ball.

Consider $X = \varinjlim(X_k \subset X_{k+1})$ with the topology of the direct limit. Then

$$H_*(X) \cong H_*(*) \quad \text{and} \quad \check{H}^*(X) \cong \check{H}^*(*), \quad \text{i.e.} \quad X \text{ is an acyclic 3-manifold.} \tag{1}$$

The one-point compactification of this space, by the point $*$, is a 3-dimensional compact metrizable space X^*. We have

$$X^* \cong \varprojlim\left(X_k/\partial A_{0,0,(k+\frac{1}{2})}\right), \tag{2}$$

where $X_k/\partial A_{0,0,(k+\frac{1}{2})}$ is the quotient of the manifold X_k via the identification of all of its boundary points to one point and where the projections

$$X_k/\partial A_{0,0,(k+\frac{1}{2})} \leftarrow X_{k+1}/\partial A_{0,0,(k+1+\frac{1}{2})}$$

map the subspaces $\overline{(X_{k+1}/\partial A_{0,0,(k+1+\frac{1}{2})}) \setminus X_k}$ of the spaces $X_{k+1}/\partial A_{0,0,(k+1+\frac{1}{2})}$ to the point of identification of $X_k/\partial A_{0,0,(k+\frac{1}{2})}$.

Obviously, the spaces $X_k/\partial A_{0,0,(k+\frac{1}{2})}$ are homology 3-spheres and all projections in the inverse sequence generate isomorphisms of homology and cohomology groups. Therefore

$$\check{H}^*(X^*) \cong \check{H}^*(S^3). \tag{3}$$

It follows from the exact sequence

$$\cdots \to \check{H}^{p-1}(X^* \setminus *) \to \check{H}^p(X^*, X^* \setminus *) \to \check{H}^p(X^*) \to \check{H}^p(X^* \setminus *) \to \cdots,$$

from the facts that $X^* \setminus * = X$, $\check{H}^p(X^*) = 0$ for $p \neq 3$, and from (1) and (3), that

$$\check{H}^p(X^*, X^* \setminus *) \cong H^p(\mathbb{R}^3, \mathbb{R}^3 \setminus \{0\}). \tag{4}$$

Let us show that X^* is a *clc* space. Since X is a 3-manifold, the space X^* is a *clc* space at every point of $X^* \setminus *$. Since $X = \varinjlim(X_k \subset X_{k+1})$ and X_k are compact spaces, the system of the sets $\{X^* \setminus X_k\}$ is a basis of neighborhoods of the point $*$. According to (2), we have

$$\overline{X^* \setminus X_k} = \varprojlim \overline{(X_n/\partial A_{0,0,(n+\frac{1}{2})}) \setminus X_k}, \quad \text{for } n > k.$$

The spaces $\overline{(X_n/\partial A_{0,0,(n+\frac{1}{2})}) \setminus X_k}$ are homology 3-balls. It follows that $\overline{X^* \setminus X_k}$ is an acyclic space with respect to the Čech cohomology. Therefore X^* is a *clc* space. Hence it follows from (4) that X^* is a generalized 3-manifold (cf. Definition 2.1).

The topological type of the space X^* depends on the choice of the sequence of homology 3-balls B_i. In the case when all B_i are homeomorphic to the Poincaré homology 3-sphere, the space X^* is the example of van Kampen. In general, since there exist infinitely many distinct homology 3-balls (cf. [8]), there exists an infinite class of van Kampen generalized 3-manifolds.

Let X^* be any van Kampen generalized 3-manifold. We shall show that X^* is not *HLC* in dimension 1. It suffices to prove that for every k the embedding $X^* \setminus X_k \subset X^* \setminus X_1$ is not homologically trivial with respect to singular homology.

The space $\overline{X^* \setminus X_k}$ is a retract of $\overline{X^* \setminus X_1}$. Indeed, the 3-balls $A_{i,0,\frac{1}{4}}$ are AR spaces so there exists for every i, a mapping of B_i on $A_{i,0,\frac{1}{4}}$ which is the identity on the boundary of B_i. Therefore we have a mapping of $X_{1,k}$ onto the ball layer $A_{0,\frac{1}{2},k}$.

However, the sphere $A_{0,k+\frac{1}{2},0}$ is a retract of this ball layer and $A_{0,k+\frac{1}{2},0} \subset \overline{X^* \setminus X_k}$. Therefore $\overline{X^* \setminus X_k}$ is a retract of $X^* \setminus X_1$. So in order to prove that $\overline{X^* \setminus X_k} \subset X^* \setminus X_1$ is not homologically trivial it suffices to prove that $H_1(\overline{X^* \setminus X_k}) \neq 0$.

Since for any path-connected space the 1-dimensional singular homology group is isomorphic to the abelianization of the fundamental group of this space, it suffices to show that $\pi_1(\overline{X^* \setminus X_k})$ is a nonperfect group.

Consider the union of spheres $A_{j,\frac{1}{4},0}$, for $j > k$, with the compactification point as a subspace of the space $X^* \setminus X_k$. Join them by the segments

$$\mathrm{I}_j = \left\{(0, y, 0) \;\middle|\; y \in \left[j + \frac{1}{4}, j + 1 - \frac{1}{4}\right]\right\}, \quad j = k+1, \ldots, \infty.$$

We get the compactum

$$A = \bigcup_{j=k+1}^{\infty} A_{j,\frac{1}{4},0} \cup \mathrm{I}_j \cup \{*\}.$$

The space A also lies in a 3-dimensional cube with a countable number of open balls $\{\operatorname{int} A_{j,0,\frac{1}{4}}\}_{j=k+1}^{\infty}$ removed and it is obviously its retract. Therefore the space

$$B = A \cup \bigcup_{i=k+1}^{\infty} B_i$$

with the natural topology of identification is a retract of $X^* \setminus X_k$.

Consider nontrivial loops $\alpha_i \in B_i$, with their homotopy classes $[\alpha_i]$. Then $0 < l([\alpha_i])$. Since $H_1(B_i) = 0$ it follows that $l([\alpha_i]) < \infty$. Let α be any loop in B such that for the canonical projection of this space to B_i, the image of α generates the loop α_i (obviously such a loop exists). We have a homomorphism

$$\pi_1(B) \to \pi_1(B_1) * \pi_1(B_2) * \cdots$$

such that the image of $[\alpha]$ has projections $[\alpha_i]$ in $\pi_1(B_i)$. Hence according to Theorem 2.6 and Assertion 2.7, the commutator length of $[\alpha] \in \pi_1(B)$ is infinite, $cl([\alpha]) = \infty$, and the group $\pi_1(B)$ is not perfect (Assertion 2.7). Therefore and since B is a retract of $X^* \setminus X_k$ it follows that $\pi_1(\overline{X^* \setminus X_k})$ is nonperfect and hence the van Kampen generalized 3-manifold is not an *HLC*-space.

Let us prove that no van Kampen generalized 3-manifold is locally homeomorphic to any singular quotient 3-manifold. Consider any open neighborhood U of the singular point $*$ of a van Kampen generalized 3-manifold. Suppose that it were homeomorphic to some open subset of some almost 3-manifold. Now, every singular quotient 3-manifold is a quotient space of a topological 3-manifold by some continuum, which generates a singular point [10]. Therefore there should exists an index i such that some neighborhood of B_i in U embeds into the Euclidean 3-space with some ball layer $A_{i,\frac{1}{4},\varepsilon}$.

By Theorem 2.3, the bounded component of $\mathbb{R}^3 \setminus A_{i,\frac{1}{4}+\frac{\varepsilon}{2},0}$ is a 3-cell. So the space $B_i \cup A_{i,\frac{1}{4}+\frac{\varepsilon}{2},0}$ embeds in a 3-cell. This embedding cannot be an onto mapping to the 3-cell because B_i is not simply connected. Therefore the sphere $A_{i,\frac{1}{4}+\frac{\varepsilon}{2},0}$ must be a retract of B_i. However, this is impossible since the space B_i is acyclic whereas the sphere $A_{i,\frac{1}{4},0}$ is not acyclic. So no van Kampen generalized 3-manifold is locally homeomorphic to any almost 3-manifold.

By the argument above and since every local homeomorphism must map singular points to singular points it follows that no singular quotient 3-manifold is locally homeomorphic to any van Kampen generalized 3-manifold. □

4. Epilogue

Cohomological local connectedness does not imply local contractibility even in the category of compact metrizable generalized 3-manifolds (this follows e.g. from Theorem 1.1). However, the following problem remains open:

Problem 4.1. Does there exist a finite-dimensional compact metrizable generalized manifold (or merely finite-dimensional compactum) which is homologically locally connected but not locally contractible?

Mardešić formulated the following interesting problem:

Question 4.2. Is it true that every compact generalized n-manifold which is an ANR can be represented as an inverse limit of compact n-manifolds?

As it was mentioned above (Chapter 3 of reference (2)), every van Kampen generalized 3-manifold is an inverse limit of compact 3-manifolds. In contrast with this we set forth the following conjecture:

Conjecture 4.3. *No singular quotient* 3*-manifold can be represented as an inverse limit of closed* 3*-manifolds.*

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References

[1] C. Bavard, Longueur stable des commutateurs, Enseign. Math. (2) 37 (1991) 109–150.
[2] E.G. Begle, Locally connected spaces and generalized manifolds, Amer. J. Math. 64 (1942) 553–574.
[3] G.E. Bredon, Sheaf Theory, second ed., Grad. Texts in Math., vol. 170, Springer, Berlin, 1997.
[4] M. Brown, A proof of the generalized Schoenflies theorem, Bull. Amer. Math. Soc. 646 (1960) 74–76.
[5] K. Eda, U.H. Karimov, D. Repovš, On (co)homology locally connected spaces, Topology Appl. 120 (2002) 397–401.
[6] R.Z. Goldstein, T.E. Turner, A note on commutators and squares in free products, in: Combinatorial Methods in Topology and Algebraic Geometry, Rochester, NY, 1982, in: Contemp. Math., vol. 44, Amer. Math. Soc., Providence, RI, 1985, pp. 69–72.
[7] H.B. Griffiths, A note on commutators in free products, II, Proc. Cambridge Philos. Soc. 51 (1955) 245–251.
[8] J.L. Gross, An infinite class of irreducible homology 3-spheres, Proc. Amer. Math. Soc. 25 (1970) 173–176.
[9] A.E. Harlap, Local homology and cohomology, homology dimension and generalized manifolds, Mat. Sb. 96 (1975) 347–373.
[10] U.H. Karimov, D. Repovš, Examples of cohomology manifolds which are not homologically locally connected, Topology Appl. 155 (2008) 1169–1174.

[11] W.J.R. Mitchell, Homology manifolds, inverse system and cohomological local connectedness, J. Lond. Math. Soc. (2) 19 (1979) 348–358.
[12] H. Poincaré, Cinquième complèment a l'analysis situs, Rend. Circ. Mat. Palermo 18 (1904) 45–110.
[13] D. Repovš, Detection of higher dimensional topological manifolds among topological spaces, in: M. Ferri (Ed.), Giornate di Topologia e Geometria delle Varietà, Bologna, 1990, Univ. degli studi di Bologna, 1992, pp. 113–143.
[14] H. Seifert, W.A. Threlfall, Textbook of Topology, Academic Press, New York, 1980.
[15] E.H. Spanier, Algebraic Topology, Springer-Verlag, Berlin, 1966.
[16] R.L. Wilder, Topology of Manifolds, Amer. Math. Soc. Colloq. Publ., vol. 32, Amer. Math. Soc., New York, 1949.