HOUSTON JOURNAL OF MATHEMATICS

Volume 26, No. 3, 2000

ON COMPACTA OF COHOMOLOGICAL DIMENSION ONE OVER NONABELIAN GROUPS

MATIJA CENCELJ AND DUŠAN REPOVŠ
Communicated by Jun-iti Nagata

ABSTRACT. We construct a 2-dimensional homogeneous Cannon-Štan'ko compactum which fails to be nonabelian. We also introduce a new class of compact metric spaces, called Daverman compacta and we investigate their applications in the theory of cohomological dimension over nonabelian groups.

1. INTRODUCTION

Cohomological dimension $c-\dim_G X$ of a compact metric space X is defined only for abelian groups, since for $n > 1$ the Eilenberg-MacLane complex $K(G, n)$ is well-defined only in such cases. However, for $n = 1$ a study of compacta of cohomological dimension one with respect to nonabelian groups seems to be a worthwhile project.

First such study was done by Dranishnikov and Repovš [5] in their search of new directions for an attack at the celebrated 4-dimensional *cell-like mapping problem* which asks whether the (Lebesgue) dimension $\dim X$ of the image of a cell-like map $f : M^4 \to X$, defined on an arbitrary topological 4-manifold M^4, is always finite.

In [5] several classes of compacta of cohomological dimension one were introduced – *Cannon-Štan'ko*, *Cainian* and *nonabelian compacta* – depending on which

1991 *Mathematics Subject Classification.* 55M10,57Q55; 54F45,54C25,57Q65.

Key words and phrases. Cohomological dimension, Daverman compactum, Cannon-Štan'ko compactum, nonabelian compactum, weakly Cainian compactum, perfect group, grope, Eilenberg-MacLane complex, cell-like map.

We were supported in part by the Ministry of Science and Technology of the Republic of Slovenia, Research Grant No. J1-0885-0101-98. We also wish to acknowledge R. J. Daverman and A. N. Dranishnikov who originated some of the ideas on which this paper is based.

classes of nonabelian groups one allows as the cohomology coefficients. We should also mention the work of Dydak and Yokoi [6].

The main purpose of our paper is to construct an example of a 2-dimensional homogeneous Cannon-Štan'ko compactum which fails to be nonabelian. Recall that it was proved in [5] that every nonabelian compactum is also a Cannon-Štan'ko compactum. Whether the converse statement is false, was an open problem.

We also introduce a new class of compact metric spaces, called *Daverman compacta* and we investigate their applications in the theory of cohomological dimension over nonabelian groups.

2. Preliminaries

Recall the Kuratowski [8] notation $X\tau Y$: it means that for every closed subset X_0 of X and any map $f : X_0 \to Y$ there exists an extension $\bar{f} : X \to Y$ of f over all of X.

Recall the construction of a *grope* M (see [1] for more about gropes). One defines M as the direct limit $M = \lim_{\to}\{L_i, j^i_{i+1}\}_{i\geq 0}$ of a direct system of compact 2-dimensional polyhedra L_i and injective bonding maps $j^i_{i+1} : L_i \to L_{i+1}$. The polyhedron L_n is called the n-th *stage* of the grope construction. Here, L_0 is an oriented compact connected surface S_g of genus $g > 0$ with an open disk deleted. Let $A_0 \subset S_g$ be a set of $2g$ circles which generate the 1-dimensional (integral) homology of the surface S_g. The complex L_{n+1} is then obtained from L_n for every $n \geq 0$, by attaching for every circle $a \in A_n$, an oriented compact connected surface S_{g_a} of genus g_a, with an open disk deleted, by identifying the boundary ∂S_{g_a} of the surface S_{g_a} with the circle $a \in A_n$. The generators of $H_1(S_{g_a}, \mathbb{Z})$ then determine the set of $2g_a$ circles $A_{n+1} \subset S_{g_a}$ which also generate the 1-dimensional homology of the surface S_{g_a}.

In particular, we shall use the so-called *minimal* grope $M^* = \lim_{\to}\{L^*_i, j^i_{i+1}\}_{i\geq 0}$ which is distinguished by the fact that the genus of L_0 is one and that for every $i \geq 1$, we attach only two orientable 1-handles to each 1-handle pair of generators of the 1-dimensional homology of the complex L^*_i.

Definition 1. [5] A compactum X is said to be a *Cannon-Štan'ko compactum* provided that for the minimal grope M^*, $X\tau M^*$. Equivalently, for the minimal grope M^*, $X\tau K(\pi_1(M^*), 1)$, i.e. $\mathrm{c} - \dim_{\pi_1(M^*)} X \leq 1$.

Every compactum of dimension ≤ 1 is clearly also a Cannon-Štan'ko compactum. The *Pontryagin disk* $\mathbb{D}^2$ (cf. [9]) is an example of a 2-dimensional

Cannon-Štan'ko compactum, the so-called *Riemann surface of infinite local genus* $\mathbb{S}^2$ (cf. [2]) is an example of a homogeneous 2-dimensional Cannon-Štan'ko compactum. It was proved in [5] that for every integer $n \geq 1$ there exists an n-dimensional Cannon-Štan'ko compactum.

Definition 2. [5] Let $T = (S^1 \times S^1) \setminus \mathrm{Int} B$ be a torus with a hole (obtained by removing an open disk B) and denote its boundary by ∂T (hence $\partial T = S^1$). A compactum X is said to be *nonabelian* if for every closed subset $A \subset X$ of X and every continuous map $f : A \to \partial T$ there exists a continuous map $\tilde{f} : X \to T$ such that $\tilde{f}|_A = f$. We shall denote this extension property by $X\tau(T, \partial T)$.

Every compactum of dimension ≤ 1 is clearly nonabelian. An example of a 2-dimensional nonabelian compactum is the classical Pontryagin mod 2 'surface' [10] i.e. the inverse limit of an inverse system of modifications of the 2-sphere where disks are replaced by Möbius bands [10]. Every n-dimensional nonabelian compactum is also a Cannon-Štan'ko compactum. In fact, every n-dimensional nonabelian compactum X has the property $X\tau M$, for every grope M. Also, there exists an n-dimensional nonabelian compactum for every integer $n \geq 0$ (cf. [5]).

Definition 3. [5] A compactum X is said to be *Cainian* provided that for every perfect group Π, $X\tau K(\Pi, 1)$. Equivalently, $\mathrm{c} - \dim_{\Pi} X \leq 1$.

Every compactum of dimension ≤ 1 is Cainian. The Pontryagin disk $\mathbb{D}^2$ and the Riemannian surface of infinite local genus $\mathbb{S}^2$ are examples of 2-dimensional Cainian compacta. Every Cainian compactum is at most 2-dimensional and every 2-dimensional nonabelian compactum is Cainian (cf. [5]). The following remains an interesting open question:

Problem 1. [5] *Let X be a Cannon-Štan'ko compactum. Does X have the property $X\tau M$ for* every *grope M?*

If X is also nonabelian then, as we have already observed above, the answer to this problem is *affirmative*. In Section 4 we shall present an example of a 2-dimensional (homogeneous) Cannon-Štan'ko compactum which fails to be nonabelian. It is unknown if such examples exist in higher dimensions.

3. Grope modifications of polyhedra

Let Γ be any group and define its n-th derived $\Gamma^{(n)}$, as follows: $\Gamma^{(1)} = [\Gamma, \Gamma]$ and for every $n \geq 2$, $\Gamma^{(n)} = [\Gamma^{(n-1)}, \Gamma^{(n-1)}]$.

Lemma 3.1. *Let $\Gamma = \pi_1(T) \cong \mathbb{Z} * \mathbb{Z}$ be the fundamental group of a torus T with one hole. Then*

$$S^1 = \partial T \in \Gamma^{(1)} \qquad , \textit{ but } \qquad \partial T \notin \Gamma^{(2)} .$$

PROOF. The first claim is obvious. For the second claim define the groups Γ_n, as follows: $\Gamma_1 = \Gamma$ and for every $n \geq 2$, $\Gamma_n = [\Gamma_{n-1}, \Gamma]$. The series

$$\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \geq \cdots ,$$

is called the *lower central series* of Γ. For groups Γ_n the following relation holds (cf. [11], Ex. (5.50), p.118) for every i, j:

$$[\Gamma_i, \Gamma_j] \leq \Gamma_{i+j} .$$

Therefore, in particular:

$$\Gamma^{(2)} = [\Gamma^{(1)}, \Gamma^{(1)}] = [\Gamma_2, \Gamma_2] \leq \Gamma_4 .$$

The element $\partial T \in \Gamma$ is a basic commutator of weight 2 (cf. [7]). Therefore by the Basis theorem ([7], p.175) for the lower central series of the free group $\Gamma = \mathbb{Z} * \mathbb{Z}$, the uniqueness of the representation implies that

$$\partial T \notin \Gamma_3 \geq \Gamma_4 \geq \Gamma^{(2)} .$$

□

In the sequel we shall need the following property of the derived groups. If $\varphi : \Pi \to \Gamma$ is a group homomorphism, then

$$\varphi(\Pi^{(2)}) \subset \Gamma^{(2)} .$$

Definition 4. Let σ be a 2-dimensional simplex. Remove intσ and replace it by the n-th stage L_n of the grope construction ($n \geq 1$),

$$\hat{\sigma} = (\sigma \setminus \mathrm{int}\sigma) \cup_{\partial\sigma} L_n ,$$

where ∂L_n is identified with $\partial\sigma$. Call this new compact 2-dimensional polyhedron $\hat{\sigma}$ the *n-stage grope modification* of the 2-simplex σ.

Lemma 3.2. *Let $\hat{\sigma}$ be the 2-stage grope modification of the 2-simplex σ. Then $\partial\hat{\sigma} \in \Pi^{(2)}$, where $\Pi = \pi_1(\hat{\sigma})$.*

PROOF. Left as an exercise. □

Definition 5. Let P be a compact 2-dimensional polyhedron with triangulation T, hence $P = |T|$. The polyhedron $\hat{P}$, obtained from P by replacing every 2-simplex $\sigma \in T^{(2)}$ by the n-stage grope modification $\hat{\sigma}$ of σ, $n \geq 1$,

$$\hat{P} = |\bigcup\{\hat{\sigma}|\sigma \in T^{(2)}\}|$$

is called the *n-stage grope modification* of the polyhedron P (with respect to the triangulation T).

Proposition 3.3. *Let $\hat{L}$ be the 2-stage grope modification of a 2-dimensional compact polyhedron L. Let $\gamma \in \pi_1(L)$, let $g : S^1 \to L^1$ be a representative of γ, where L^1 is the 1-skeleton of L and let $\hat{\gamma} = [g] \in \pi_1(\hat{L})$. If $\gamma \in [\pi_1(L)]^{(2)}$, then also $\hat{\gamma} \in [\pi_1(\hat{L})]^{(2)}$.*

Proof. Let us start with the polyhedron L and first perform the 2-stage modification of only one simplex σ of L to obtain $\tilde{L}$. In finitely many steps of this kind we obtain $\hat{L}$. Therefore it suffices to prove the claim of the proposition for the case of $\hat{L} = \tilde{L}$. Let $i : L \setminus \operatorname{Int}\sigma \hookrightarrow L$, $j : L \setminus \operatorname{Int}\sigma \hookrightarrow \hat{L}$, $p : \hat{L} \to L$ be the obvious maps. Then j induces a morphism of exact sequences

$$\begin{array}{ccccccccc}
1 & \to & \operatorname{Ker} i & \to & \pi_1(L \setminus \operatorname{Int}\sigma) & \xrightarrow{i} & \pi_1(L) & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \operatorname{Ker} p & \to & \pi_1(\hat{L}) & \xrightarrow{p} & \pi_1(L) & \to & 1.
\end{array}$$

If $[g] \notin [\pi_1(L \setminus \operatorname{Int}\sigma)]^{(2)}$, then it differs from an element in $[\pi_1(L \setminus \operatorname{Int}\sigma)]^{(2)}$ by a power of $\partial\sigma$. Since $\partial\sigma \in [\pi_1(\hat{L})]^{(2)}$ the result follows.

□

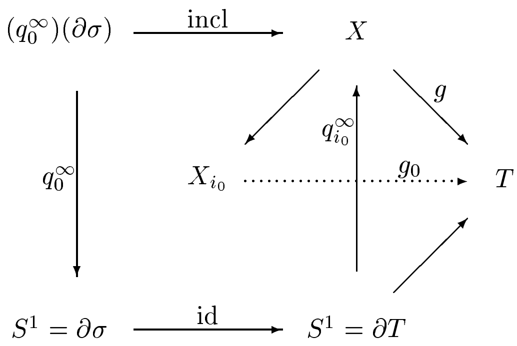
4. The example

As it was pointed out in the introduction, it is already known [5] that every nonabelian compactum must necessarily also be a Cannon-Štan'ko compactum. Whether the converse is true was an important open problem. In this chapter we give a negative answer, by constructing an example of a Cannon-Štan'ko compactum which fails to be nonabelian.

Let $X = \lim_{\leftarrow}\{X_i, q_i^{i+1}\}_{i\geq 0}$, where X_i are compact 2-dimensional polyhedra with triangulations λ_i such that $\operatorname{mesh}\lambda_i \to 0$, and for every $i \geq 0$, the bonding map $q_i^{i+1} : X_{i+1} \to X_i$ is a 2-stage grope modification of the 2-skeleton. The initial polyhedron X_0 is taken to be the boundary of the standard 3-simplex.

Theorem 4.1. *The compactum X fails to be nonabelian.*

Proof. Indeed, suppose not, i.e. suppose that $X\tau(T,\partial T)$. Consider any 2-simplex $\sigma \in X_0^{(2)}$ and observe that $S^1 = \partial\sigma$ is embedded into X by the restriction $(q_0^\infty)^{-1}|_{\partial\sigma} : \partial\sigma \to X$, so we may identify $\partial\sigma$ and its image in X. Let $f : \partial\sigma \to \partial T$ be the identity map on S^1.



By our hypothesis there exists an extension $g : X \to T$ of f over all X. Since T is an ANR, the map g extends over some open neighbourhood $U \subset Q$ of X in the Hilbert cube Q (we may assume $X = \lim_{\leftarrow}\{X_i, q_i^{i+1}\}_{i\geq 0}$ lies in Q). Thus, there is a large enough integer $i_0 \geq 1$ such that $X_{i_0} \subset U$ and hence g extends to a map $g_{i_0} : X_{i_0} \to T$ up to homotopy, i. e.

$$g_{i_0}|_{(q_0^i)^{-1}} \simeq \mathrm{id}_{S^1} \quad \text{and} \quad g_{i_0} \circ p_{i_0} \simeq g\,.$$

Let $K_{i_0} = (q_0^{i_0})^{-1}(\sigma)$. We have the following commutative diagram.

$$\begin{array}{ccccc} & K_i & \dashrightarrow & T & \\ \text{incl} & \uparrow & & \uparrow & \text{incl} \\ & S^1 & \xrightarrow{\text{id}} & S^1 & \end{array}$$

Assertion. Let $\Pi = \Pi_1(K_i)$. Then $S^1 \in \Pi^{(2)}$.

Proof of Assertion. This is verified by induction on i. For $i = 2$ the assertion is obvious. Assume now that it is true for $i < k$ and consider the case $i = k$. Our element is a product of commutators $s = [a_1, b_1] \ldots [a_j, b_j]$ and each of the elements a_l, b_l $(1 \leq l \leq j)$ is itself a product of commutators. This relation defines a 2-cell so we have a map of $\hat{L}_{k+1}$ into $\hat{L}_k$ and when we make a grope modification we get a map of $\hat{L}_{k+1}$ into $\hat{L}_k$. Hence in L_k we have $s([a_1, b_1] \ldots [a_j, b_j])^{-1} = 1$, but in L_{k+1} we have $s([a_1,b_1]\dots[a_j,b_j])^{-1} \in \Pi^{(2)}$ hence $s \in \Pi^{(2)}$ proving the Assertion.

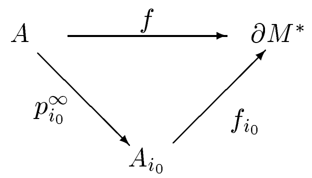
Therefore by Assertion, $\partial T \in \Pi^{(2)} = \Gamma^{(2)}$ so $\partial T \in \Gamma^{(2)}$ which is a contradiction to Lemma (3.1), proving Theorem 4.1. □

Theorem 4.2. *X is a Cannon-Štan'ko compactum.*

Proof. Let $A \subset X$ be an arbitrary closed subset of X and take any map $f : A \to M^*$ of A into the minimal grope $M^* = \lim_{\to}\{L_i, j^i_{i+1}\}_{i\geq 0}$.

By Lemma (4.5) of [5] it suffices to verify the property $(X,A)\tau(M^*,\partial M^*)$, hence we may assume that $f(A) \subset \partial M^* = S^1$. Consider X and M^* embedded in the Hilbert cube and represent $(X,A) = \lim_{\leftarrow}\{(X_i,A_i),(q_i^{i+1}, q_i^{i+1}|_{A_{i+1}})\}_{i\geq 0}$.

Since S^1 is an ANR, there is a large enough $i_0 \geq 1$ such that f homotopically factors through A_{i_0}, i.e. the diagram



commutes up to a homotopy. It is straightforward to extend f_{i_0} over the 1-skeleton $X^{(1)}_{i_0}$ of X_i, so we get a map $\bar{f}_{i_0} : A_{i_0} \cup X^{(1)}_{i_0} \to \partial M^*$, hence a map $g : (q^{\infty}_{i_0})^{-1}(A_{i_0} \cup X^{(1)}_{i_0}) \to \partial M^*$, by taking $g = \bar{f}_{i_0} \circ q^{\infty}_{i_0}$.

It now remains to extend g over to $\tilde{\sigma} = (q^{\infty}_{i_0})^{-1}(\sigma)$, for every 2-simplex $\sigma \in X^{(2)}_{i_0} \setminus A^{(2)}_{i_0}$. Consider all such 2-simplices σ. The map g is already defined on $\tilde{\gamma} = (q^{\infty}_{i_0})^{-1}(\gamma)$, where $\gamma = \partial\sigma$ is the boundary of σ. Now $(q^{i_0+1}_{i_0})^{-1}(\gamma)$ bounds in X_{i_0+1} a 2-stage modification $\hat{\sigma}$ of σ, hence there is a natural map of $\hat{\sigma}$ to M^*. Combining this map with the projection $q^{\infty}_{i_0+1}$ we get the desired extension of H over $\tilde{\sigma}$. We do this for all σ and since there are finitely many, we get a well defined extension of g over X, $\bar{g} : X \to M^*$, such that $\bar{g}|_A \simeq f$. This proves that $(X,A)\tau(M^*,\partial M^*)$, so by Lemma (4.5) of [5], $X\tau M^*$, i.e. X is indeed a Cannon-Štan'ko compactum.

□

5. A new class of compacta

Definition 6. Let $N = N_1 \cup N_2$ be the (boundary connected) sum of two copies $N_1 = M^* = N_2$ of the minimal grope M^* along its boundary circle $\partial M = S^1$.

A compactum X is said to be *weakly Cainian* provided that for the fundamental group $\Pi = \Pi_1(N)$, $X\tau K(\Pi,1)$. Equivalently, $\mathrm{c}-\dim_{\Pi} X \leq 1$.

We define a new class of compacta $\mathcal{K}_n = \{X \mid X\tau(L_{n-1}, \partial L_{n-1})\}$, for every integer $n \geq 1$, where L_n denotes the n-th *stage* of the minimal grope construction. The following properties are easily verified using techniques from [5] and our Section 4.

Theorem 5.1. 1. *For every* $n \geq 1$, $\mathcal{K}_n \subset \mathcal{K}_{n+1}$;
2. *For every* $n \neq m$, $\mathcal{K}_n \neq \mathcal{K}_m$;
3. $\bigcup_{n\in\mathbb{N}} \mathcal{K}_n \subset \mathcal{K}$, *where* $\mathcal{K} = \{X \mid X$ *is weakly Cainian*$\}$;
4. *Every compactum* $X \in \mathcal{K}_1$ *is nonabelian; and*
5. *For every* $n \geq 2$, *no compactum* $X \in \mathcal{K}_n \setminus \mathcal{K}_1$ *is nonabelian.*

We shall call the nested sequence $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}$ the *Daverman series* of compacta and the union

$$\mathcal{D} = \bigcup_{n\geq 1} \mathcal{K}_n$$

the class of *Daverman compacta.*

Theorem 5.2. *The Daverman series is incomplete, i.e.* $\mathcal{D} \neq \mathcal{K}$.

PROOF. Define $K \in \mathcal{K} \setminus \mathcal{D}$ by $K = \lim_{\leftarrow}\{S_i, p_i^{i+1}\}_{i\geq 0}$, where S_0 is 2-sphere and S_{i+1} is obtained from S_i by using the i-th stage grope modifications only. Then clearly $K \notin \mathcal{K}_n$, for any $n \geq 1$ since we exceed the n-th stage grope modifications already in S_{n+1}. On the other hand, K is clearly a Cannon-Štan'ko compactum, so $K \in \mathcal{K}$. □

It follows by Theorem 5.1 above that $\mathcal{D} \subset \mathcal{K}$, i.e. every Daverman compactum is also weakly Cainian. We conclude with the following open problem.

Problem 2. *Does there exist a Cannon-Štan'ko compactum* X *such that:*
1. $\dim X \geq 3$*; and*
2. X *fails to be a nonabelian compactum?*

REFERENCES

[1] J. W. Cannon, *The recognition problem for topological manifolds: What is a topological manifold?* Bull. Amer. Math. Soc. **84** (1978), 832-866.

[2] R. J. Daverman and A. N. Dranishnikov, *Cell-like maps and aspherical compacta*, Illinois J. Math. **40** (1996), 77-90.

[3] A. N. Dranishnikov, *Homological dimension theory*, Uspehi Mat. Nauk **43**(4) (1988), 11-55 (in Russian); English translation in: Russian Math. Surveys **43**(4) (1988), 11-63.

[4] A. N. Dranishnikov, *On a problem of Y. Sternfeld*, Glasnik Mat. **27(47)** (1992), 365–368.
[5] A. N. Dranishnikov and D. Repovš, *Cohomological dimension with respect to perfect groups*, Topol. Appl. **74** (1996), 123-140.
[6] J. Dydak and K. Yokoi, *Hereditarily aspherical compacta*, Proc. Amer. Math. Soc. **124** (1996), 1933-1940.
[7] M. Hall, *The Theory of Groups*, MacMillan, New York, 1959.
[8] K. Kuratowski, *Topology*, Vol. I, PWN, Warsaw, 1968.
[9] W. J. R. Mitchell, D. Repovš and E. V. Ščepin, *On 1-cycles and the finite dimensionality of homology 4-manifolds*, Topology **31** (1992), 605-623.
[10] L. S. Pontryagin, *Sur une hypothèse fondamentale de la théorie de la dimension*, C. R. Acad. Paris, **190** (1930), 1105-1107.
[11] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th Ed., Springer-Verlag, New York, 1995.

Received September 1, 1997
Final version for publication received March 14, 2000

(Matija Cencelj) Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P. O. Box 2964, 1001 Ljubljana, Slovenia
E-mail address: matija.cencelj@imfm.uni-lj.si

(Dušan Repovš) Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P. O. Box 2964, 1001 Ljubljana, Slovenia
E-mail address: dusan.repovs@imfm.uni-lj.si