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A classification of 3-thickenings of 2-polyhedra

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Abstract

We classify 3-thickenings (i.e., 3-dimensional regular neighborhoods) of a given 2-polyhedron P up to a homeomorphism rel P. The partial case of our theorem is that for some class of 2-polyhedra, containing fake surfaces, 3-thickenings of P are classified by the restriction of their first Stiefel–Whitney class to P. The corollary is that for every two homotopic embeddings of a polyhedron P from our class into interior of a 3-manifold M, the regular neighborhoods of their images are homeomorphic.

We also prove that a fake surface is embeddable into some orientable 3-manifold if and only if it does not contain a union of the Möbius band with an annulus (one of the boundary circles of the annulus attached to the middle circle of the Möbius band with a map of degree 1).

Keywords: Thickening; Regular neighborhood; Special 2-polyhedron; Orientation; Embedding of a graph into plane; Faithful embedding; Stiefel–Whitney class; Fake surface

AMS classification: 57M20; 57N10; 57M05; 57M15; 57Q35; 57Q40

1. Introduction

If an (orientable) n-manifold M is a regular neighborhood of a polyhedron $P \subset \operatorname{Int} M$, then the pair (M, P) is called an (orientable) *n-thickening* of P. Note that a 3-thickening of a 2-surface is an I-bundle (possibly, twisted) over this surface. Thickenings of P are equivalent if they are PL homeomorphic, relatively to P. When the polyhedron P is fixed, we shall briefly denote its thickening (M, P) by M. The problems of existence, uniqueness, and classification of n-thickenings of polyhedra were investigated in [2–4,9–17,19,22,24], [6, Theorems 3.2.2, 3.2.3]. The notion of a thickening is analogous and closely related

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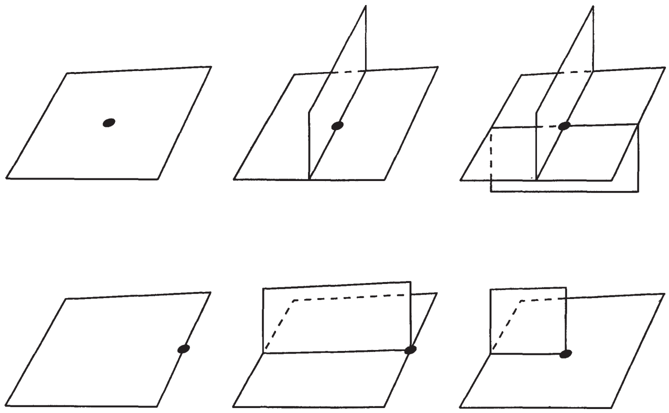

Fig. 1.

to that of a fibre bundle [9], [17, Section 4]. The main result of the present paper is the classification of 3-thickenings of 2-polyhedra. It generalizes [2], [10, p. 222] and the following well-known fact: *Extensions of an* I*-bundle* μ *over a boundary* ∂N *of a compact surface* N *are in* 1–1-*correspondence with the elements* $\nu \in H^1(N)$, *such that* (*if* $\partial N \neq \emptyset$) $\nu|_{\partial N} = w_1(\mu)$.

Let us introduce some notations and definitions. Throughout this paper we shall work in the PL category; by [1] the same results hold in the topological category. In our notations we follow [18]. Denote by $R_Y(X)$ the regular neighborhood of a subpolyhedron X in a polyhedron Y. A *link* of a point of X is its link in some sufficiently small triangulation of X. A vertex of a graph is *hanging* if its degree is one. An edge of a graph is *hanging* if one of its endpoints is hanging. Denote by $\mathcal{T}^n(P)$ the set of all n-thickenings of P. We use (co)homologies with $\mathbb{Z}_2$-coefficients. For a 2-polyhedron P we shall denote by P' its 1-subpolyhedron, which is the set of points of P' having no neighborhood homeomorphic to the 2-disk. By P'' we shall denote the (finite) set of points of P', having no neighborhood homeomorphic to a book with n sheets for some $n \geqslant 1$. For any component of P' containing no points of P'', take a point in it. Denote by F the union of P'' and these points. Then P' is a graph whose vertices are either hanging or they are points of F. Let $H^1(P) \xrightarrow{i} H^1(P') \xrightarrow{\delta} H^2(P, P')$ be a fragment of the exact sequence of the pair (P, P').

Let us begin with a special case and corollaries of our main Theorem 1.3. A 2-polyhedron P is said to be a *fake surface* if each of its points has a neighborhood, homeomorphic to one of those in Fig. 1 [7]. A graph is called 3-*connected* if no two of its points split it into two graphs with more than one edge in each [20].

Corollary 1.1. *Suppose that* P *is a* 2-*polyhedron such that* $\operatorname{lk} A$ *is* 3-*connected for each* $A \in F$ (*in particular, if* P *is fake surface*). *Then*:

(a) (cf. [10, p. 292]) 3-*thickenings of* P *are classified by the restrictions of their first Stiefel–Whitney classes to* P: *either* $\mathcal{T}^3(P) \cong \emptyset$ *or* $\mathcal{T}^3(P) \cong \operatorname{Ker} i$.

(b) (cf. [2], [16, p. 419], [3, Proposition 5]) *For each* 3*-manifold* M *and every two homotopic embeddings* $P \to \operatorname{Int} M$*, the regular neighborhoods of their images are homeomorphic.*

Corollary 1.2 (cf. [24]). *A fake surface is orientably* 3*-thickenable if and only if it does not contain a union of the Möbius band with an annulus* (*one of the boundary circles of the annulus attached to the middle circle of the Möbius band with a map of degree* 1)*.*

An example illustrating Corollary 1.2 is an embedding of the Klein bottle into some *orientable* 3-manifold. Indeed, let

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Then the 3-manifold $S^1 \times [-1,1] \times [0,1]/(z,t,0) \sim (\overline{z},-t,1)$ is orientable and contains the Klein bottle

$$S^1 \times \{0\} \times [0,1]/(z,0,0) \sim (\overline{z},0,1).$$

Another example of an orientable 3-thickening of nonorientable 2-manifold is the regular neighborhood of RP^2, standardly embedded into RP^3.

Now we shall formulate our Main Theorem 1.3. Suppose that $\bigcup_{A\in F} \operatorname{lk} A$ is embeddable into S^2. Take a collection of embeddings $\{g_A : \operatorname{lk} A \to S^2\}_{A\in F}$. Take a nonhanging edge $d \subset P'$ and denote its vertices by A and B (possibly, $A = B$). The edge d meets $\operatorname{lk} A \cup \operatorname{lk} B$ at two points (distinct, even when $A = B$). Regular neighborhoods U and V of these points in $\operatorname{lk} A$ and in $\operatorname{lk} B$ are n-ods, which could be identified with the cone over $\operatorname{lk} d$. If for each such d the maps g_A and g_B give the same or the opposite orders of rotation of the pages of the book at d then the collection $\{g_A\}$ is called *faithful*. This definition differs from that of [13]. What they call 'faithful' we should call 'orientably faithful'. Two faithful collections of embeddings $\{f_A : \operatorname{lk} A \to S^2\}_{A\in F}$ and $\{g_A : \operatorname{lk} A \to S^2\}_{A\in F}$ into (nonoriented) spheres are said to be *isopositioned*, if there is a family of homeomorphisms $\{h_A : S^2 \to S^2\}_{A\in F}$ such that $h_A \circ f_A = g_A$, for each $A \in F$. Evidently, isopositioned collections are faithful or not simultaneously. Denote by $E(P)$ the set of faithful collections up to isoposition.

Let us define e-invariant $e : \mathcal{T}^3(P) \to E(P)$. Suppose that M is a 3-thickening of P. Take any point $A \in F$ and consider its regular neighborhood $R_M(A)$. Since $\partial R_M(A)$ is a sphere, we have a collection of embeddings $\operatorname{lk} A \to \partial R_M(A)$. Since for each edge d of P', $R_P(d)$ is embedded into M, this collection of embeddings is faithful. Let $e(M)$ be its class in $E(P)$. Equivalent thickening yield isopositioned collections of embeddings. Thus $e(M)$ is well-defined.

Let us construct a map $\beta : E(P) \to H^1(P')$. For each $\varepsilon \in E(P)$ take its representative $\{g_A : \operatorname{lk} A \to S^2\}_{A\in F}$. For each nonhanging edge d of P', recall the rotations (the same or the opposite) from the definition of faithful collection of embeddings. Let $\beta(\varepsilon)$ be the class of the cocycle

$$b(d) = \begin{cases} 0, & \text{the rotations are the opposite,} \\ 1, & \text{the rotations are the same.} \end{cases}$$

For collections of embeddings, isopositioned via a family of homeomorphisms $\{h_A : S^2 \to S^2\}_{A\in \operatorname{lk} F}$ the cocycles b differ by a coboundary of a cochain $\varkappa \in C^0(P')$, defined by

$$\varkappa(A) = \begin{cases} 1, & \text{if } h_A \text{ reverses orientation of } S^2, \\ 0, & \text{if } h_A \text{ preserves orientation of } S^2. \end{cases}$$

Thus $\beta(\varepsilon)$ is well-defined.

Theorem 1.3. *Thickenings M_1, M_2 of P are homeomorphic* rel P *if and only if $w_1(M_1)|_P = w_1(M_2)|_P$ and $e(M_1) = e(M_2)$. Moreover, $e \times w_1|_P$ is* 1–1 *correspondence between $\mathcal{T}^3(P)$ and $\{(\varepsilon, \omega) \in E(P) \times H^1(P) \mid \beta(\varepsilon) = \omega|_{P'}\} = \beta^{-1}(\operatorname{Im} i) \times \operatorname{Ker}(i)$.*

The "only if" part in Theorem 1.3 is obvious, the "if" part and the "moreover" part follows from Lemmas 2.1–2.3 of Section 2. Note that $w_1|_P$-invariant is a partial case of invariant $c_n : \mathcal{T}^n(P) \to K(P)$ [9], where K is a *real* K-functor.

The set of embeddings of a given graph into plane up to isoposition was described for 2-connected graphs [23], and there is a simple (folklore) generalization of this description for arbitrary graphs lk F. Notice the similarity between the classification of 3-thickenings of 2-polyhedra and that of graph manifolds [21] and integrable Hamiltonian systems [5].

A polyhedron P is said to be (orientably) *n-thickenable* if it is embeddable into some (orientable) n-manifold. The criteria of (orientable) 3-thickenability [12,19] can be restated as a special case of Theorem 1.3 (cf. [13, Theorem 3.2]): *A* 2-*polyhedron P is* (*orientably*) 3-*thickenable if and only if there exists a faithful embedding $\varepsilon \in E(P)$ such that* ($\beta(\varepsilon) = 0$) $\delta\beta(\varepsilon) = 0$. For partial cases there are simpler criteria of 3-thickenability [10,13–15,24]. Our proof of Corollary 1.2 is based on the above restatement of [12,19]. We also construct a counterexample to the following conjectures, analogous to Corollary 1.2, which arose during a discussion with S.V. Matveev:

Conjecture 1.4.
(a) *A fake surface is* 3-*thickenable if and only if it does not contain the union of the Möbius band and a* 2-*surface with one boundary circle* (*the boundary circle is attached to the middle circle of the Möbius band with a map of degree* 1).
(b) *A special* 2-*polyhedron is* 3-*thickenable if and only if it does not contain the union of the Möbius band with a disk* (*the boundary circle of the disk attached to the middle circle of the Möbius band with a map of degree* 1).

A fake surface P is called a *special* 2-*polyhedron* if $P \setminus P'$ and $P' \setminus P''$ are disjoint unions of open 2- and 1-cells, respectively.

2. Proofs

Lemma 2.1. *If $e(M_1) = e(M_2)$ and $w_1(M_1)|_P = w_1(M_2)|_P$ then $M_1 \cong M_2$* rel P.

Proof. The first two steps are analogous to [2,10], but we present them for completeness.

Construction of a homeomorphism $R_{M_1}(F) \cong R_{M_2}(F)\,\mathrm{rel}\,P$. Choose regular neighborhoods $R_{M_1}(F)$ and $R_{M_2}(F)$ such that $P \cap R_{M_1}(F) = P \cap R_{M_2}(F)$. Take a representative $\{g^i_A\}_{A\in F}$ of $e(M_i)$ described in the construction of e. Take autohomeomorphisms $\{h_A\}_{A\in F}$ rel lk A from the definition of isoposition between $\{g^1_A\}_{A\in F}$ and $\{g^2_A\}_{A\in F}$. Extend h_A canonically to a homeomorphism $h''_A : R_{M_1}(A) \to R_{M_2}(A)$. Since $P \cap R_{M_1}(A)$ is a cone over lk A and h_A is the identity on lk A, h''_A is the identity on $P \cap R_{M_1}(A)$. Let $h'' : P \cup R_{M_1}(F) \to P \cup R_{M_2}(F)$ be the extension of id_P to $P \cup R_{M_1}(F)$ by $\bigsqcup_{A\in F} h''_A$ on the set $R_{M_1}(F)$.

Construction of a homeomorphism $R_{M_1}(P') \cong R_{M_2}(P')\,\mathrm{rel}\,P$. We have that $N_i = \partial R_{M_i}(F)$ is a disjoint union of 2-spheres. For every edge $d \subset P'$ choose a regular neighborhood $D^1_d = R_{N_1}(d \cap N_1)$. This is one or two disks in N_1. We can assume without loss of generality that if d and d' are edges in P', then $D^1_d \cap D^1_{d'} = \emptyset$. Denote $h_0(D^1_d)$ by D^2_d. Choose regular neighborhoods $R_{M_1}(d)$ and $R_{M_2}(d)$ such that $R_{M_1}(d) \cap N_1 = D^1_d$ and $R_{M_2} \cap N_2 = D^2_d$, and $R_{M_1}(d) \cap P = R_P(d) = R_{M_2}(d) \cap P$. Denote by T_d the closure of $R_{M_1}(d) \setminus R_{M_1}(F)$. Then T_d is homeomorphic to a cylinder $D^2 \times I$ with one or two of its bases glued to $R_{M_1}(F)$. Obviously, we may assume that $T_d \cap T_{d'} = \emptyset$ for any edges $d, d' \subset P'$. In T_d we have a cylinder $C_d = P \cap T_d$. For any component V of the set $T_d \setminus P$ the pair $(\mathrm{Cl}(V), \mathrm{Cl}(\partial V \setminus \partial R_{M_1}(P')))$ is homeomorphic to the pair $(I^2 \times I, I^2 \times \{0\})$. Hence we can extend h'' over V independently for each component V. In this way, we obtain a homeomorphism $h' : P \cup R_{M_1}(P') \to P \cup R_{M_2}(P')$ which is the identity on P.

Construction of a homeomorphism $R_{M_1}(P) \cong R_{M_2}(P)\,\mathrm{rel}\,P$. Take a triangulation T of P and a cocycle $a \in Z^1(T)$, representing $w_1(M_1)|_P = w_1(M_2)|_P$. Let T' and T'' be the 1-skeleton and 0-skeleton of T, respectively. Extend h' 'along a' to a homeomorphism

$$R_{M_1}(T') \cong R_{M_2}(T')\,\mathrm{rel}\,R_P(T').$$

Then this new homeomorphism extends to that of $R_{M_1}(P) \cong R_{M_2}(P)\,\mathrm{rel}\,P$. Therefore our lemma follows from the uniqueness of regular neighborhoods.

More precisely, consider $M''_i = R_{M_i}(P' \cup T'')$ such that $M''_1 \cap P = M''_2 \cap P$. Clearly, for $i \in \{1, 2\}$ we can fix orientation in every connected component of M''_i such that (1) h' is orientation-preserving homeomorphism and (2) for any edge $d \subset T' \setminus P'$ going along d in M_i reverses orientation if $a(d) = 1$ and preserves orientation if $a(d) = 0$. Let $h_0 : P \cup R_{M_1}(P' \cup T'') \to P \cup R_{M_2}(P' \cup T'')$ be an orientation-preserving extension of h' to the balls $\{R_{M_1}(A)\}_{A\in T''\setminus P'}$. Since going along the edge d reverses or preserves orientation simultaneously in M_1 and M_2, we can apply the construction from the first and the second step and extend h_0 to a homeomorphism $h_1 : P \cup R_{M_1}(T') \to P \cup R_{M_2}(T')$ which is the identity on P.

Note that $\mathrm{Cl}(P \setminus R_P(T'))$ is a disjoint union of 2-disks. The regular neighborhood of $\mathrm{Cl}(P \setminus R_P(T'))$ in $\mathrm{Cl}(M_i \setminus (R_{M_i}(T'))$ is a disjoint union of 3-balls. These 2-disks and 3-balls are in one-to-one correspondence with the 2-simplices of T. Let D be one of these 2-disks and B_i the corresponding 3-ball. Then $(B_i; D, B_i \cap R_{M_i}(T')) \cong (D^2 \times [-1, 1]; D^2 \times \{0\}, \partial D^2 \times [-1, 1])$. Since the homeomorphism h_1 is already defined on $D^2 \times \{0\}$ and $\partial D^2 \times [-1, 1]$, we can extend it to a homeomorphism $B_1 \to B_2$. By extending

h_1 independently for each disk D of $\mathrm{Cl}(P \setminus R_P(T'))$, we obtain a homeomorphism $h : R_{M_1}(P) \to R_{M_2}(P)$ which is the identity on P. □

Lemma 2.2. *For every* 3*-thickening M of P, $\beta(e(M)) = w_1(M)|_{P'}$.*

Proof. It suffices to prove that for any $\gamma \in Z_1(P')$ carried by a simple closed curve J, $\langle \beta(e(M)), \gamma\rangle = \langle w_1(M), \gamma\rangle$. Indeed, suppose that J is formed by edges $d_1, \ldots, d_n$ of the graph P'. From the definition of the cocycle b it easily follows that if $\sum_{i=1}^n b(d_i) = 1 \bmod 2$ then going around the curve J reverses orientation on M. Similarly, if $\sum_{i=1}^n b(d_i) = 0 \bmod 2$ then going around J does not change the orientation on M. It follows that if $\langle \beta(e(M)), \gamma\rangle = 1$ then going around J reverses the orientation in the bundle $t^{-1}(J) \to J$ (where $t : TM \to M$ is the tangent bundle). Therefore by the definition of $w_1(M)$, $\langle w_1(M), \gamma\rangle = 1$. If, however, $\langle \beta(e(M)), \gamma\rangle = 0$ then going around J does not change the orientation in the bundle $t^{-1}(J) \to J$. In this case $\langle w_1(M), \gamma\rangle = 0$. So $\langle \beta(e(M)), \gamma\rangle = \langle w_1(M), \gamma\rangle$. □

Lemma 2.3. *For any $\varepsilon \in E(P)$, $\omega \in H^1(P)$ such that $\beta(\varepsilon) = \omega|_{P'}$ there exists a thickening $M \in \mathcal{T}^3(P)$ such that $e(M) = \varepsilon$ and $w_1(M)|_P = \omega$.*

Proof. Take a triangulation T of P and a cocycle $a \in Z^1(T)$, representing ω. Let T' and T'' be the 1-skeleton and 0-skeleton of T, respectively. Since $\omega|_{P'} = \beta(\varepsilon)$, using technique from [12] we can construct a 3-manifold M' such that $(M', \partial M')$ is a regular neighborhood of $(R_P(T'), R_P(T') \cap \mathrm{Cl}(P \setminus R_P(T'))$ and $e(M') = \varepsilon$, $w_1(M')|_{T'} = [a] \in H^1(T')$. Since a is a cocycle, $\langle a, \partial\sigma\rangle = \langle \delta a, \sigma\rangle = 0$ for any 2-simplex σ of T. Hence the regular neighborhood of a simple closed curve $\sigma \cap \partial M'$ is an annulus (not Möbius band). Therefore M' extends to a 3-thickening M of P. Clearly, $w_1(M)|_P = \omega$ and $e(M) = e(M') = \varepsilon$. □

Proof of Corollary 1.1. (a) Since $\mathrm{lk}\, A$ is 3-connected, there is at most one embedding $\mathrm{lk}\, A \subset S^2$ [6, Theorem 1.6.6]. Therefore $|E(P)| \leqslant 1$. Thus Conjecture 1.1(a) follows from Corollary 1.2.

(b) Let $h = gf^{-1} : f(P) \to g(P)$ be homeomorphism. Since f, g are homotopic, $h^*(w_1(M)|_{g(P)}) = w_1(M)|_{f(P)}$. Therefore Conjecture 1.1(b) follows from Conjecture 1.1(a). □

Proof of Corollary 1.2. The "only if" part is obvious, so let us prove the "if" part. Since $\mathrm{lk}\, A$ is planar and 3-connected for each $A \in F$, there is a unique embedding $\mathrm{lk}\, A \subset S^2$ [6, Theorem 1.6.6]. Since $|\,\mathrm{lk}\, d| = 3$ for every edge $d \subset P'$, this collection ε of embeddings is faithful. Thus $|E(P)| = 1$. Below we prove that if P does not contain N then $\beta(E(P)) = 0$. Thus Corollary 1.2 follows from the above restatement of [12,19].

Let T be a triod. Since $\mathrm{lk}\, A$ is 3-connected for each $A \in P'$, it follows by Menger's theorem that for each two vertices B, C of $\mathrm{lk}\, A$, whose degrees are more than 2, there are three paths, joining B to C and intersecting only at B, C [20]. Because of this, for every simple closed curve $J \subset P'$, there is a T-fibre bundle over J, embedded in P, where 'zero-section' is identified with J (cf. [24]). There are three types of such bundles. They are obtained from $T \times I$ by identifying $T \times \{0\}$ and $T \times \{1\}$ by autohomeomorphism of T, defined by either identity or 3-cycle or 2-cycle permutation of edges of T, respectively. If P does not contain N then for each J this bundle is of the first or the second type. It is easy to see that then $\beta(E(P)) = 0$. □

Note that these considerations can be applied to prove a criterion for 3-thickenability of a wider class of 2-polyhedra.

Corollary 2.4 (cf. [10, p. 293], [8, Remark 1 on p. 310]). *Suppose that P is a 2-polyhedron such that $\operatorname{lk} A$ is 3-connected for each $A \in P''$ (in particular, if either P is a fake surface or $P'' = \emptyset$). Then P is (orientably) 3-thickenable if and only if the class $\beta(P)$ is defined (see below) and ($\beta(P) = 0$) $\delta\beta(P) = 0$.*

We define $\beta(P)$ independently on connected components of P', containing at least one point of P'', and on those, containing no points of P''. If $\operatorname{lk} A$ is not planar for some $A \in P''$, then $\beta(P)$ is undefined. Otherwise there is a unique collection of embeddings $\{\operatorname{lk} A \subset S^2\}_{A \in P''}$ [6, Theorem 1.6.6]. If it is faithful, then it determines the restriction of $\beta(P)$ to those connected components of P' that contain at least one point of P'' (see the introduction for definition). Otherwise $\beta(P)$ is undefined. Suppose that $J \subset P'$ is a connected component of P', containing no points of P'' (then J is either arc or simple closed curve). Let us define $\langle \beta(P), J \rangle$ in case J is simple closed curve. Clearly, $R_P(J)$ is homeomorphic to a cylinder of a map of finite number of circles onto J. If degrees of the maps of these circles are the same, then put $\langle \beta(P), J \rangle = 0$. If one or two circles have degree 1 and others have degree 2, then put $\langle \beta(P), J \rangle = 1$. If for some J none of these two cases hold, then $\beta(P)$ is undefined.

Construction of the counterexample to Conjectures 1.4(a) *and* 1.4(b). Let P' be a graph with three vertices V_1, V_2, V_3 and six edges: $e_1 = \overrightarrow{V_1V_2}$, $e_2 = \overrightarrow{V_2V_3}$, $e_3 = \overrightarrow{V_3V_1}$ and loops e_4, e_5, e_6 with basepoints V_1, V_2, V_3, respectively. Fix orientation on the loops e_4, e_5 and e_6. Glue three 2-disks to P' along loops $e_1e_5^{-1}e_2e_6^{-1}e_3e_4^{-1}$, $e_1e_2e_6^2e_3e_4^2$ and $e_1e_5^2e_2e_3$. We obtain the polyhedron P. Since none of these disks is embedded in P, P does not contain polyhedra from Conjectures 1.4(a) and 1.4(b). Denote the first disk by D. We have that $\langle \delta\beta(P), D \rangle = \langle \beta(P), \partial D \rangle = 1 \bmod 2$. Then nonthickenability of P follows from Corollary 2.4. □

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