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C^∞-HOMOGENEOUS CURVES ON ORIENTABLE CLOSED SURFACES

DONČO DIMOVSKI
UNIVERSITY OF SKOPJE, SKOPJE, YUGOSLAVIA

DUŠAN REPOVŠ
UNIVERSITY OF LJUBLJANA, LJUBLJANA, YUGOSLAVIA

EVGENIJ V. ŠČEPIN
STEKLOV MATHEMATICAL INSTITUTE, MOSCOW, USSR

Let M^2 be an oriented smooth closed surface. A <u>Jordan curve on</u> M^2 is a continuous map $K:[0,1] \longrightarrow M^2$ (or $K: R \longrightarrow M^2$) such that for each $t \in (0,1)$ (or $t \in R$) there is an $\varepsilon > 0$ such that K is one-to-one on the open ε-neighborhood of t. The orientation on a Jordan curve K on M^2 will always be the induced orientation by the chosen one on the surface M^2.

A Jordan curve K on M^2 is said to be C^∞-<u>homogeneous in</u> M^2 if for every two points $x,y \in K$ there exists a diffeomorphism $h_{x,y}: M^2 \longrightarrow M^2$ such that (i) $h_{x,y}(x)=y$; (ii) $h_{x,y}(K) \subset K$; (iii) $h_{x,y}$ is orientation preserving for all x and y (or it is orientation reversing for all x and y); and (iv) $h_{x,y}$ preserves the orientation of K for all x and y (or it reverses the orientation of K for all x and y).

PROPOSITION 1. *Let K be a C^∞-homogeneous Jordan curve in the plane R^2. Then for every point $x \in K$ there exist a closed neighborhood $U \subset R^2$ of x and smooth curves $C_1, C_2 \subset U$ such that (i) $K \cap U$ separates U; (ii) C_1 and C_2 lie on the opposite sides of $K \cap U$; and (iii) $C_1 \cap K = \{x\} = C_2 \cap K$ (i.e. x is wedged between C_1 and C_2).*

<u>Proof</u>. Let $x \in K$. Since K is C^∞-homogeneous Jordan curve in R^2 the Jordan Curve Theorem implies that there is a closed neighborhood $U \subset R^2$ of x in R^2 such that $K \cap U$ separates U. Let

$p_i \in U-K$, $i=1,2$, be any two points on the opposite sides of $K \cap U$. Let $p_i^* \in K$ be a point on K nearest to p_i, $i=1,2$ (there can be more than one, in general). Let C_i^* be a circle in R^2 centered at p_i^* with radius $d(p_i, p_i^*)$, $i=1,2$. Modify the circles C_i^* so that eventually $C_i^* \cap K = \{p_i^*\}$, $i=1,2$, while keeping them smooth at all points. Use the C^∞-homogeneity of the curve K to produce two diffeomorphisms $h_i: R^2 \longrightarrow R^2$ such that $h_i(p_i^*)=x$. Define C_i to be the component of $h_i(C_i^*)$ which contains the point x, $i=1,2$. It's is now easy to verify that the curves C_1 and C_2 satisfy the required properties.

A 1-_parameter group of diffeomorphisms_ of an oriented closed smooth surface M^2 is a continuous map $G: R\times M^2 \longrightarrow M^2$ such that (i) $G(t,x)=g^t(x)$ where g^t is a diffeomorphism of M^2; (ii) $g^{t+s}=g^t g^s$; and (iii) $g^0=id_{M^2}$. In other words, a 1-parameter group of diffeomorphisms is a homomorphism $G:(R,+) \longrightarrow (Diff(M^2),o)$ such that $G: R\times M^2 \longrightarrow M^2$ is continuous.

PROPOSITION 2. _Let_ $G: R\times M^2 \longrightarrow M^2$ _be a 1-parameter group of diffeomorphisms of_ M^2 _and let_ $x \in M^2$ _be an arbitrary point of_ M^2. _Then the orbit_ $0_x = \{g^t(x) \mid t \in R\}$ _is a_ C^∞_-homogeneous curve in_ M^2 _unless_ x _is a fixed point_.

Proof. Any two diffeomorphisms g^t and g^s are homotopic via the map $G: [t,s] \times M^2 \longrightarrow M^2$, $t<s$, which implies that either both of them preserve the orientation of M^2 or they both reverse it. The same holds for the orientation of the orbits. Pick $x \in M^2$ and let $y \in 0_x$. Then $H_y = \{t \mid g^t(y)=y\}$ is a subgroup of $(R,+)$ so there are 3 possibilities: H_y is either discrete or dense in R or trivial (i.e. 0). If H_y is dense in R it follows that $H_y=R$ hence $0_x=0_y=\{x\}=\{y\}$. If $0_x \neq \{x\}$ and $H_x=\{0\}$ then 0_x is the image of an embedding of R in M^2. If H_x is a discrete group, i.e. $H_x=rZ$ for some $r>0$, then 0_x is the image of an embedding of $[0,r)$ in M^2, i.e. of a map $f:[0,r] \longrightarrow R$, $f(0)=f(r)$, and f one-to-one on the interval $[0,r)$.

For every two points $y,z \in 0_x$ there exist $t,s \in R$ such that $y=g^t(x)$ and $z=g^s(x)$. Consequently, $g^{s-t}(y)=g^{s-t}(g^t(x))=g^s(x)=z$ hence g^{s-t} is a required diffeomorphism of M^2 since we also have that

$g^{s-t}(O_x)=O_x$. This proves that the orbit O_x is indeed C^∞-homogeneous in M^2 as asserted.

PROPOSITION 3. Let K be a Jordan curve in the plane R^2 and suppose that at a point $x \in K$, K is wedged between two smooth curves, i.e. that there are two smooth curves $C_1, C_2 \subset R^2$ such that $C_1 \cap K = C_2 \cap K = \{x\}$. Then K has a tangent at x.

Proof. Consider the secants $L_n \subset R^2$ of K based at x. For every $n \in N$, pick $q_n \in L_n \cap K$. We may choose the sequence (q_n) so that it converges to x. There are two possible cases to consider.

Case 1. For all but finitely many n, $q_n \notin C_1 \cup C_2$. Then for some subsequence $(q_{s(n)})$, $L_{s(n)}$ have the same slope as the tangent to C_1 (and hence to C_2) at x and therefore so does the limit.

Case 2. For some subsequence $(q_{s(n)})$, $q_{s(n)} \in C_1$ (resp. C_2). Then $L_{s(n)}$ is also a secant for C_1 (resp. C_2) at x hence the slopes of $L_{s(n)}$ must converge - to the derivative of C_1 (resp. C_2) at x. This implies that K is differentiable at x.

EXAMPLE. The following example shows that in Proposition 3 one cannot, in general, also prove that the curve K has a continuous derivative at x (hence much less that K is smooth at x). Let $P_1 = \{(x,x^2) \mid x \in R\} \subset R^2$ and $P_2 = \{(x,-x^2) \mid x \in R\} \subset R^2$. For every $n \in N$, let $A_n, A_n^* \in P_1$ and $B_n, B_n^* \in P_2$ be given by:

$$A_n = \{((2n)^{-1},(4n^2)^{-1})\}, \qquad A_n^* = \{(-(2n)^{-1},(4n^2)^{-1})\},$$

$$B_n = \{((2n-1)^{-1},-(2n-1)^{-2})\}, \quad B_n^* = \{(-(2n-1)^{-1},-(2n-1)^{-2})\},$$

and let $K^* = \bigcup_{n \in N}(\overline{A_nB_n} \cup \overline{A_nB_{n+1}} \cup \overline{A_n^*B_n^*} \cup \overline{A_n^*B_{n+1}^*}) \cup \{(0,0)\}$. Let K be the curve in the plane, obtained from K* by smoothing its corners A_n, A_n^*, B_n, B_n^* (without changing K* near the x-axis). Then the point $T=(0,0) \in K$ is wedged between the smooth curves P_1 and P_2 but the first derivative of K at T isn't continuous (hence, in particular, K isn't smooth at T). To see this, let $f\colon [-1,1] \longrightarrow R$ be the map whose graph is K and define the map $F=(f_1,f_2)\colon [-1,1] \longrightarrow R^2$ by $f_1(t)=t$ and $f_2(t)=f(t)$. Then $df_1/dt\,|(0)=1$ and $df_2/dt\,|(0)=0$. Let $(t_n) \subset R$ be the sequence of points on the x-axis, defined by $F(t_n) = \overline{A_nB_n} \cap$ (x-axis). Then $\lim t_n = 0$ whereas $\lim df_2/dt\,|(t_n) = -2 \neq 0 = df_2/dt\,|(0)$.

REMARK. The curve K constructed above isn't C^∞-homogeneous in R^2. For suppose this were the case and pick any point $T^*\neq T$ on K. We would then have a diffeomorphism $h:R^2\longrightarrow R^2$ such that $h(T)=T^*$. Now, K is clearly smooth at T^* hence it should also be smooth at the image of T^*, $h^{-1}(T^*)=T$. Contradiction.

QUESTION. Under the additional assumption in Proposition 3, that the curve K is C^∞-homogeneous in R^2, can one prove that K is then necessarily smooth at x, i.e. does "C^∞-homogeneous" imply "smooth"? (Note that the converse is true, i.e. every smooth curve in R^2 is locally flat at every point hence one can build diffeomorphisms of R^2 which interchange arbitrary pairs of points on K.)

After this paper was written, W.J.R.Mitchell brought to our attention the work of L.D.Loveland[2)] where he used a similar idea of wedging the curve or a sphere between balls: using entirely different methods from ours he proved e.g. that a curve $K\subset S^3$ is tame in S^3 (i.e. there is an ambient homeomorphism of S^3 which takes K onto a polygonal arc) if for some $t>0$ at each $x\in K$ there are 3-dimensional balls $B_1,B_2\subset S^3$, each with radius t, such that $B_1\cap B_2=(B_1\cup B_2)\cap K=\{x\}$.

Our work, on the other hand, was inspired by a remark in V.I. Arnol'd's textbook[1)] (cf.Problem 1 on p.24). Note that our argument yields a very simple geometric proof of a special case of Theorem (5.2.3) in D.Montgomery-L.Zippin's monograph[3)].

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R E F E R E N C E S

(1) Arnol'd, V.I., "Ordinary Differential Equations"(Russian), Nauka, Moscow 1971.

(2) Loveland, L.D.,"Double tangent ball embeddings of curves in E^3", Pacif. J. Math. 104 (1983), 391-399.

(3) Montgomery, D. and Zippin, L., "Topological Transformation Groups", Interscience Publ. Inc., New York 1955, Interscience Tracts in Pure and Appl. Math. No.1.