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Acyclicity in 3-Manifolds

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Let K be a continuum in a 3-manifold M. How nice neighborhoods can K have? For example, if K is cellular in M, then K is the intersection of properly nested 3-cells, while if it is cell-like then K is the intersection of properly nested homotopy 3-cells with 1-handles [3; Theorem 3]. We describe below neighborhoods of almost 1-acyclic (over Z_2) continua K.

Theorem 1. *Let K be a continuum in the interior of a 3-manifold M with (possibly empty) boundary. Suppose that K does not separate its connected neighborhoods and that for every neighborhood $U\subset M$ of K there exists a neighborhood $V\subset U$ of K such that the inclusion-induced homomorphism $H_1(V-K;\ Z_2)\to H_1(U;\ Z_2)$ is trivial. Then $K=\cap_{i=1}^{\infty} N_i$, where each $N_i\subset \operatorname{int} M$ is a compact 3-manifold with boundary satisfying the following properties:*

(*i*) *for each i, $N_{i+1}\subset \operatorname{int} N_i$;*

(*ii*) *N_i is obtained from a compact 3-manifold Q_i with a 2-sphere boundary by adding to ∂Q_i a finite number of orientable (solid) 1-handles;*

(*iii*) *for each i, the inclusion-induced homomorphism*

$$H_1(\partial N_{i+1};\ Z_2)\to H_1(N_i;\ Z_2) \text{ is trivial.}$$

Remark. Theorem 1 was proved for orientable 3-manifolds by D. R. McMillan, Jr. [5; Theorem 2]. A. H. Wright observed [9; Theorem 2] that McMillan's theorem generalizes to nonorientable 3-manifolds, but did not obtain orientable 1-handles. Neither of the papers [5] and [9] gave details.

We have decided to present the details in order to explain the specific situation for non-orientable 3-manifolds. Our proof is modelled after the proof of [5; Theorem 2] as outlined in the lecture notes of D. McMillan [4] from which we also quote the following folklore lemmas we shall need at several points.

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Lemma 2. *Let K be a compact set in the interior of a 3-manifold M, $K \neq M$, and let $N \subset M$ be a neighborhood of K. Then there exists a compact polyhedron $U \subset \text{int } N$ with the following properties:*

(*i*) *each component of U is a 3-manifold with boundary;*

(*ii*) *each closed surface in $U-K$ separates $U-K$:*

(*iii*) $K \subset \text{int } U$.

Let M be a compact 3-manifold with boundary and let $F_1, \ldots, F_m \subset \partial M$ be its boundary components. Then we define the t o t a l g e n u s o f ∂M to be the sum of the genera of $F_i (1 \leqq i \leqq m)$ $: g(\partial M) = \Sigma^m_{i=1} g_i$, $g_i =$ genus of F_i.

Lemma 3. *Let M be a compact orientable 3-manifold with boundary and let $R = Z_p$ or the rationals (p a prime). Let $i_* : H_1(\partial M; R) \to H_1(M; R)$ be the inclusion-induced homomorphism. Then, $\text{rank}_R(\text{im } i_*) = g(\partial M)$.*

P r o o f o f T h e o r e m 1. First, we shall prove that $K = \cap_{i=1}^{\infty} N_i$, where N_i satisfy (*i*) and (*ii*). It will follow by hypotheses that we can find a subsequence of $\{N_i\}$ satisfying (*iii*). We shall supress the Z_2 coefficients from the notation.

To prove (*i*)$-$(*iii*) it therefore suffices to show that given a neighborhood $U \subset M$ of K there is a compact 3-manifold neighborhood $N \subset U$ of K such that N is obtained from a compact 3-manifold Q with ∂Q a 2-sphere, by attaching a finite number of orientable (solid) 1-handles to ∂Q. So let $U \subset M$ be a neighborhood of K. We may assume the following about U:

(1) U is a nonorientable connected compact 3-manifold with boundary;

(2) $K \subset \text{int } U$;

(3) $U-K$ is orientable and connected;

(4) each closed surface in $U-K$ separates $U-K$.

The condition (3) follows by [2; Lemma 4.1] since, for sufficiently small U's, the inclusion induces trivial homomorphisms $H_1(U-K) \to H_1(M)$. The condition (4) is provided by Lemma 2.

Let $n_0 \in N$ be Haken's number of U [1; p. 48]. Using the hypothesis, we can construct an ordered (n_0+2)-tuple $Y = \{V_0, V_1, \ldots, V_{n_0+1}\}$ of compact 3-manifolds with boundary such that:

(5) $V_0 = U$;

(6) $V_{i+1} \subset \text{int } V_i$;

(7) ∂V_i is an orientable (possibly disconnected) two-sided closed 2-manifold;

(8) $H_1(\partial V_{i+1}) \to H_1(V_i)$ is trivial;

(9) $K \subset \text{int } V_{n_0+1}$.

(Note that (7) follows by (3) and (4).)

Define the complexity of Y to be the integer $c(Y) = \Sigma_{i=0}^{n_0+1} \Sigma_{n=0}^{\infty} (n+1)^2 g_i(n)$, where $g_i(n)$ is the number of components of ∂V_i with genus n. We shall show that in a finite number of steps we can improve Y, so that it will still satisfy (5)$-$(8) (but not necessarily also (9)) and that for some $i \geqq 1$, ∂V_i will be a collection of 2-spheres. We shall achieve this by compressing $\partial Y = U_{i=0}^{n_0+1} \partial V_i$ in a careful manner to reduce the complexity $c(Y)$, and then we shall apply Haken's Finiteness theorem [1].

The sequence of compressions that accomplish our goal is a sequence of modifications on Y (D. McMillan [3] calls them "simple moves") of two types: if a compression of ∂V_i takes place along a disk contained in V_i, we say that we remove a 1-handle, while if the compressing disk lies outside V_i, we say that we added a 2-handle. So suppose first that there is a disk $D \subset \text{int } V_0$, such that $D \cap \partial Y = \partial D \subset \partial V_i$ for some $i \in \{1, \ldots, n_0+1\}$, and such that ∂D bounds no disk in ∂V_i. So D either lies outside V_i (in int V_{i-1}) or inside V_i (in $V_i - V_{i+1}$). In the first case we add a 2-handle to V_i while in the second case we remove a $1-$handle from V_i. Denote the new V_i and Y by V_i' and Y', respectively. Note that in both cases we did not change any V_j, $i \neq j$. By [3; Lemma 4], $1 \leq c(Y') < c(Y)$ so by a finite number of compressions we get $Y^* = \{V_0^*, \ldots, V_{n_0+1}^*\}$ which cannot be compressed in such a manner anymore. A routine "trading disks" argument now implies that each component of ∂Y^* which is not a 2-sphere is incompressible.

We want to verify that Y^* satisfies the conditions (5)–(8). We first note that, if F is a boundary of a 3-manifold Z, it still bounds after the compression: if we add a 2-handle, then the new F will bound the manifold Z plus the "half-open" 3-cell attached via the 2-handle, while if we removed a 1-handle from Z, then the new F will bound the manifold Z minus the "half-open" 3-cell removed via the 1-handle. Therefore, Y^* is well-defined.

Next, Y^* satisfies (5) and (6) by our construction. To prove (7) we show that a compression of an orientable boundary of a 3-manifold Z always yields an orientable boundary: suppose first that $Z' = Z + $(2-handle) had nonorientable boundary. Then we could find a simple closed curve $J \subset \partial Z'$ such that J would reverse the orientation in $\partial Z'$. We could isotope J off the cocore of the 2-handle and hence off the entire handle and into ∂Z, thus showing ∂Z to be nonorientable. Since removing a 1-handle from Z has the same effect on ∂Z as adding a 2-handle to the complementary 3-manifold component bounded by ∂Z, the preceding argument also proves that for $Z' = Z - $(1-handle), $\partial Z'$ stays orientable. Finally, the condition (8) follows by [3; Lemma B] because we made the simplifications $V_i \to V_i'$ without disturbing V_j, $i \neq j$.

We now prove that for some $k \in \{1, \ldots, n_0+1\}$, ∂V_k^* is a collection of 2-spheres. If not, then by Haken's Finiteness theorem [1] for some $1 \leq p < q \leq n_0+1$ there exist components $S_1 \subset \partial V_p^*$ and $S_2 \subset \partial V_q^*$ that are topologically parallel and different from S^2. So there is an embedding $f: S_1 \times [0,1] \to U$ such that $f(S_s \times \{s\}) = S_s$ where $s = 0,1$. Let $X = f(S_1 \times [0,1])$. We may assume that no surface in (int X)$\cap \partial Y^*$ is parallel to S_1 in X. By [8; Corollary (3.2)] each incompressible surface in int X is parallel to S_1 in X. Therefore, (int X)$\cap \partial Y^*$ consists entirely of 2-spheres. Also, X must be irreducible, for if there were a 2-sphere in X which would not bound a 3-cell in X, then it would be incompressible, hence parallel to $S_1 \neq S^2$. Therefore, X minus the interiors of a finite disjoint collection of 3-cells lies in V_p^*. Hence, every 1-cycle in S_1 is homologous to a 1-cycle in S_2 thus it bounds in V_p^* by (8). Since by Lemma 3, the image of the inclusion-induced homomorphism $H_1(\partial V_p^*) \to H_1(V_p^*)$ has rank (as a vector space over Z_2) equal to $g(\partial V_p^*)$ it follows by (7) that S_1 is a 2-sphere, a contradiction.

Let V be a 3-manifold among V_i^* all of whose boundary components are 2-spheres. Clearly, (9) may no longer be true, so we now take care of that. During the compressions, when we attached a 2-handle, it may have happened that it passed through the space in U that was previously occupied by a 1-handle, which was removed at an earlier stage. In such cases, we require that the boundary of the 2-handle be in general position with respect to the boundary of the 1-handle. In addition, we shall assume that the annulus removed from $\partial V_i'$ (recall $\partial V_i'$ is orientable so it contains no Möbius bands) in the k-th compression be disjoint from all 1-handles or 2-handles involved in the preceeding $k-1$ compressions. So if we now add to ∂V all 1-handles that were removed from V during the compressions, we get several 1-handles attached to ∂V. Note that adding of an old 1-handle H to ∂V may result in many new smaller 1-handles as H may run through several 2-handles that now occupy portions of its original place. (See Figure 1.)

Every resulting 1-handle is orientable. For suppose, in reattaching the 1-handles sequentially, we have added a nonorientable 1-handle. Then for every subsequent reattachement of the remaining 1-handles we have only one isotopy class of attaching maps [7; Theorem (3.34)] so we end up with a nonorientable surface. But this is impossible by (3) and (4). We may also assume that for every resulting 1-handle H both ends of H are attached to the same boundary component, for otherwise we add H to V thus reducing the number of boundary components of V by one.

The 3-manifold N which we get from V by reattaching all 1-handles may be disconnected so we keep only the component which contains K. Thus N is obtained from a compact 3-manifold Q with ∂Q a collection of 2-spheres by attaching a finite number of orientable 1-handles to ∂Q, so that every 1-handle has both ends on the same component of ∂Q. Let $p_i \in \Sigma_i (i=1;\ 2)$ be arbitrary points on two distinct 2-sphere components Σ_1 and Σ_2 of ∂Q. Since K doesn't separate N, there is a polygonal arc A in $N-K$ joining p_1 and p_2. Suppose that A passes through a 1-handle H. We may assume that $A \cap H$ is just one arc meeting ∂Q in only two points on Σ_2. Then, $A \cap H$ can be replaced by another polygonal arc $B \subset N - \text{int}\, H$ attached to Σ_2. So we may assume that A doesn't pass through any of the 1-handles. Therefore, by drilling tunnels, we can effectively join the components of ∂Q thus obtaining the desired neighborhood N. (See Figure 2.)

We can describe the structure of the neighborhoods N of K as follows: $N=Q$ +(1-handles), where Q captures the "nonorientability" of K, while the handles capture the "pathology" of K. (See Figure 3.)

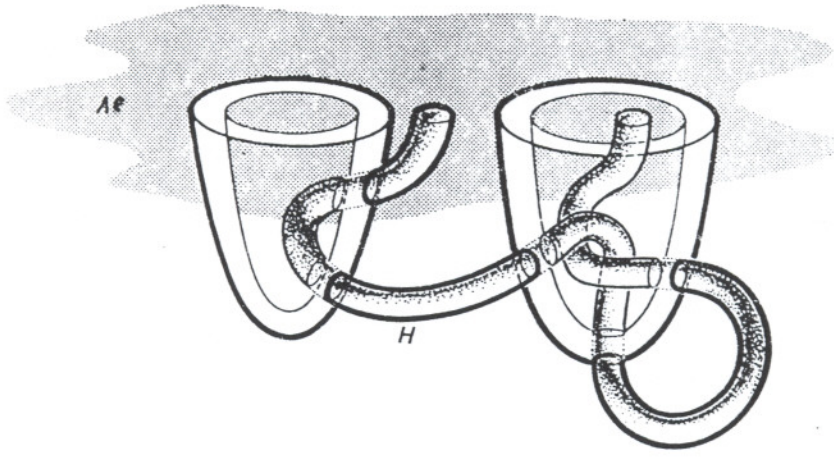

Figure 1

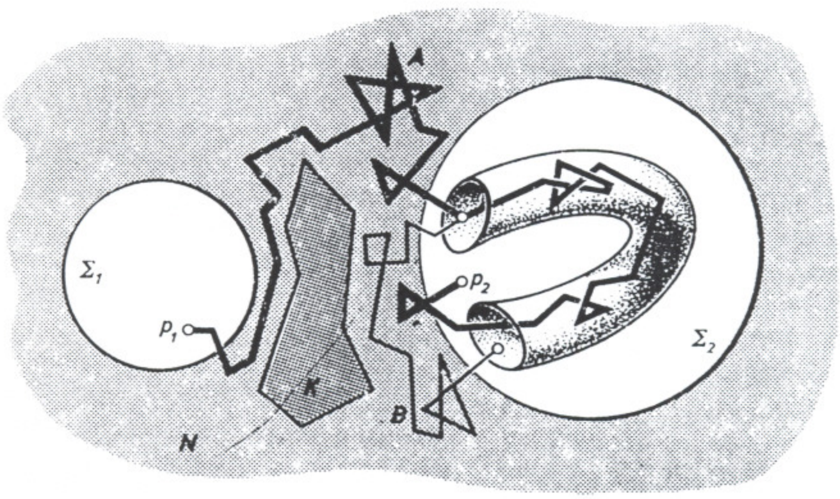

Figure 2

Let K be a compact set in the interior of a 3-manifold M. We say that K can be e n g u l f e d i n M if the interior of some punctured 3-ball in M contains K. A sequence $\{K_i\}$ of compact 3-manifolds with boundary is a W-s e q u e n c e if for every i the following conditions hold:

(i) $K_i \subset \text{int } K_{i+1}$;

(ii) the inclusion-induced homomorphism is trivial:

$$\Pi_1(K_i) \rightarrow \Pi_1(K_{i+1}).$$

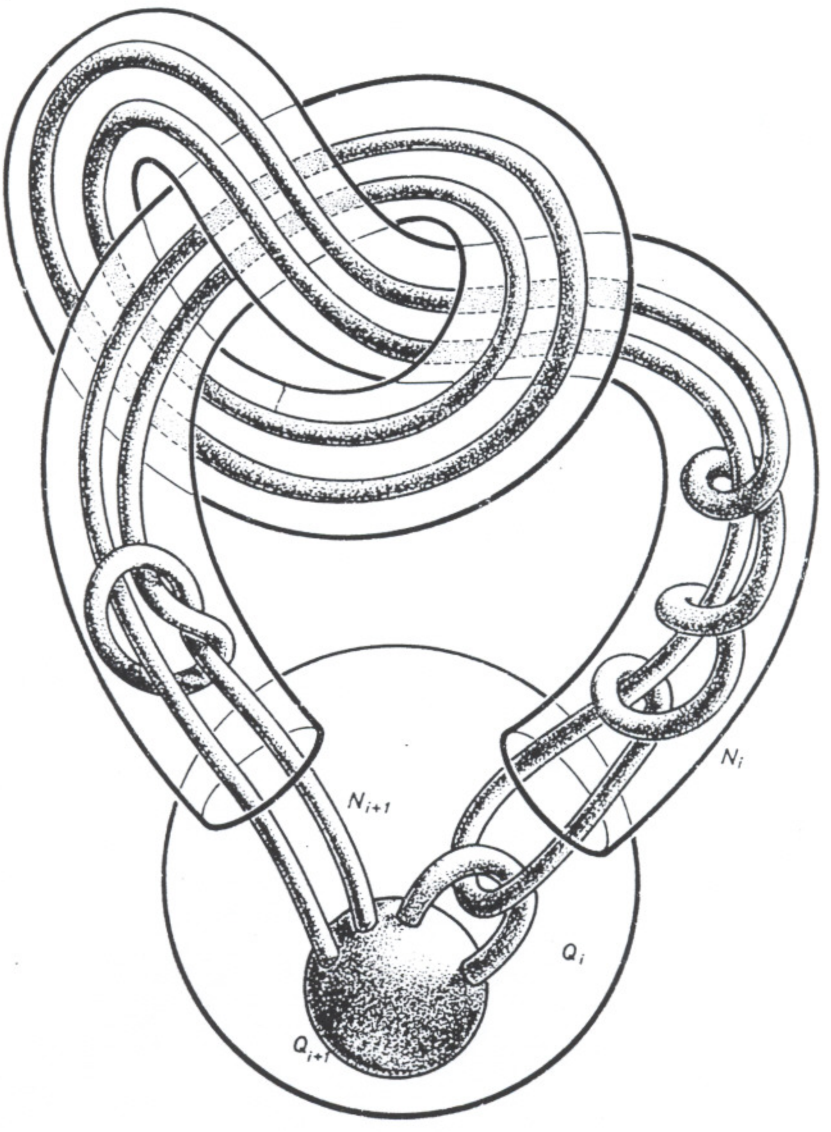

Figure 3

An open 3-manifold M is called a Whitehead manifold if it can be expressed as $M = \cup_{i=0}^{\infty} K_i$ for some W-sequence of handlebodies [6; p. 313].

An examination of the proofs in a recent paper of D. R. McMillan, Jr. and T. L. Thickstun [6] shows that the orientability hypothesis can be removed from all results in [6] if one uses Theorem 1 in the place of [5; Theorem 2]:

Theorem 4. *Let M be a compact 3-manifold (possibly with boundary) and $K \subset \text{int } M$ a compact subset. Then K can be engulfed in M if and only if there is an* *open, connected neighborhood $U \subset M$ of K, such that U embeds in S^3 and $H_1(U; Z)$ vanishes.*

Theorem 5. *Let M be a compact 3-manifold (possibly with boundary). Then M contains no fake 3-cells if and only if each Whitehead manifold that embeds in int M also embeds in S^3.*

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