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RESOLVING ACYCLIC IMAGES OF HIGHER DIMENSIONAL MANIFOLDS

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Abstract. We prove that every generalized n-manifold ($n \geqslant 4$) which is an acyclic image of a topological n-manifold, has a cell-like resolution. This extends an analogous result for $n=3$, due to J. L. Bryant and R. C. Lacher, to higher dimensions.

1. Introduction

One of the fundamental problems of modern geometric topology is the characterization of topological manifolds. One seeks relatively simple properties which detect manifolds inside a given class of spaces – usually these are already known to be at least ENR homology manifolds (=*generalized manifolds*). It was the idea of J. W. Cannon [2] that one could solve this problem for higher dimensions ($n \geqslant 5$) in essentially two steps:

(1) First, show that every generalized n-manifold X has a *resolution*, $f: M \to X$.

(2) Second, show that if X has *the disjoint disks property* then the induced cell-like decomposition $G(f) = \{f^{-1}(x) | x \in X\}$ of the n-manifold M is shrinkable – hence X is homeomorphic to M and thus nonsingular.

After the announcement by F. S. Quinn [11] in 1978 of a proof of (1) it seemed for a while that Cannon's conjecture was confirmed, since a year before that R. D. Edwards [7] had verified (2). However, in 1985 S. Cappel found a serious error in Quinn's argument [14]. Therefore it again became an interesting problem to find out which (if not all) higher dimensional generalized manifolds admit resolutions. (See [17] for a survey of the situation in dimension 3.)

In this paper we give a partial contribution towards the solution of (1) – we prove that every generalized n-manifold ($n \geqslant 4$) has a resolution, provided it is an acyclic image of some topological n-manifold. This result generalizes an analogous statement for $n=3$ by J. L. Bryant

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and R. C. Lacher [1] (refined by D. Repovš and R. C. Lacher [18]), to higher dimensions.

We remark that, in general, an acyclic map from an n-manifold onto a generalized n-manifold need not have *any* cell-like point-inverses – e. g. R. J. Daverman and J. J. Walsh [4] have constructed examples of proper maps $f: S^n \to X^n$ for every $n > 3$, of the n-sphere onto a generalized n-manifold, such that $f^{-1}(x)$ is an acyclic but *not* even 1-UV (hence not cell-like) compactum in S^n, for every $x \in X$.

This paper was inspired by conversations with J. L. Bryant and F. S. Quinn at the 1985 Georgia Topology Conference in Athens. It was written during my subsequent visit to the Mathematical Sciences Research Institute in Berkeley. I wish to acknowledge the financial support for this trip from the National Academy of Sciences U. S. A.. I also want to thank the referee for pointing out that my argument actually yields a stronger result – an invariance theorem for Quinn's local index (Theorem (3.3)).

2. Preliminaries

We shall use (co)homologies with Z coefficients throughout the paper. A compactum K in an ANR X is *acyclic* if for each neighborhood $U \subset X$ of K there is a neighborhood $V \subset U$ of K such that $H_k(V) \to H_k(U)$ is trivial for all $k \geq 0$ (unless specified otherwise all homomorphisms as above are assumed to be induced by inclusion $i: V \to U$). A compactum K is *cell-like* if there exist a manifold N and an embedding $f: K \to N$ such that $f(K)$ is cellular in N. A map defined on a space (resp. an ANR) X is *monotone* (resp. *acyclic*, *cell-like*) if its point-inverses are continua (resp. acyclic compacta, cell-like compacta) in X. A closed map is *proper* if its point-inverses are compact. A map $f: X \to Y$ is *one-to-one over* $Z \subset Y$ if for every $z \in Z$, $f^{-1}(z)$ is a point. A space X is *locally contractible* if for each $x \in X$ and each neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that V is contractible in U.

A locally compact Hausdorff space X is a *generalized n-manifold with boundary* ($n \in N$) if:

(*i*) X is an *euclidean neighborhood retract* (ENR), i. e. for some integer m, X embeds in $\mathbb{R}^m$ as a retract of an open subset of $\mathbb{R}^m$;

(*ii*) X is a *homology n-manifold with boundary*, i. e. for every $x \in X$ either $\check{H}^*(X, X - \{x\}) \cong \check{H}^{n-*}(\{x\})$ or $\check{H}^*(X, X - \{x\}) \cong 0$. The subset $\dot{X} = \{x \in X \mid \check{H}^*(X, X - \{x\}) \cong 0\}$ of X is called the *boundary* of X and $\mathring{X} = X - \dot{X}$ is the *interior* of X. If $\dot{X} = \emptyset$ we call X a *homology n-manifold*. The set $S(X) = \{x \in X \mid x$ *has no neighborhood in* X *homeomorphic to an open set in* $B^n\}$ is called the *singular set* of X and its complement $M(X) = X - S(X)$ is the *manifold set* of X.

A generalized n-manifold with boundary X is said to have a *resolution* if there exists a pair (M,f), where M is a topological n-manifold with boundary and $f: M \to X$ is a proper cell-like onto map such that $f^{-1}(\dot{X}) = \partial M$. A resolution (M,f) of X is called *conservative* if f is one-to-one over $M(X)$.

F. S. Quinn has isolated an obstruction for generalized manifolds of dimension $\geqslant 4$ to admit a resolution:

THEOREM 2.1. (F. S. Quinn [15]) *Let X be a generalized n-manifold with boundary. Then there exists a "local index" $i(X) \in (1+8\mathbf{Z})$ which has the following properties:*

(i) $i(U) = i(X)$ for every open subset $U \subset X$;

(ii) $i(X \times Y) = i(X) \cdot i(Y)$ for every generalized manifold with boundary Y; and

(iii) If $n \geqslant 5$, or $n=4$ and $S(\dot{X})) = \emptyset$, then X admits a resolution if and only if $i(X) = 1$.

3. The results

Our main result is the following resolution theorem for the class of those generalized manifolds which are acyclic images of genuine manifolds of the same dimension:

THEOREM 3.1. *A generalized n-manifold X has a conservative resolution if and only if there is a topological n-manifold M and a proper acyclic onto map $f: M \to X$.*

A metric space (X,d) has the *disjoint disks property* if for every $\varepsilon > 0$ and every pair of maps $f, g: B^2 \to X$ there exist maps $f', g': B^2 \to X$ such that $d(f',f) < \varepsilon > d(g',g)$ and $f'(B^2) \cap g'(B^2) = \emptyset$.

As an immediate corollary we get a recognition theorem for higher dimensional topological manifolds à la Cannon [2]:

COROLLARY 3.2. *A space X is a manifold of dimension $n \geqslant 5$ if and only if X has the following properties:*

(i) X is a generalized n-manifold;

(ii) X is a proper acyclic image of an n-manifold; and

(iii) X has the disjoint disks property.

Proof. Follows by Theorem (3.1) and [7].

We shall first prove the following invariance theorem for F. S. Quinn's local index (cf. Theorem (2.1)):

THEOREM 3.3. *Let $f: X_1 \to X_2$ be a proper acyclic onto map between generalized n-manifolds X_1 and X_2. Then $i(X_1) = i(X_2)$.*

Proof. Let $N=X_1\times[-1, 1)$ and define Y to be the space obtained from the disjoint union of $X_1\times[-1, 0]$ and $X_2\times[0, 1)$, by identifying $(u, 0)$ with $(f(u), 0)$ for all $u\in X_1$ and put the standard quotient topology on Y. Let $F:N\to Y$ be the obvious quotient map $F((u, t))=[(u, t)]$, for all $(u, t)\in X_1\times[-1, 1)$, where $[w]$ denotes the equivalence class of the image of the point (u, t). Since f is proper, acyclic and onto it follows that the map F possesses the same properties, too.

Assertion 1. *Y is an ENR.*

Proof. Since X_1 is an ENR so is N. Hence N is a locally compact, separable metrizable finite-dimensional ANR [5; (IV. 8.13.1.)]. Since F is proper, Y is thus locally compact [6; (XI. 6.6.)], separable [6; (VIII. 7.2.)], and metrizable [3; (10.C.7.)], therefore by [9; (III.2.B.)], Y is finite-dimensional since it can be expressed as the union of finite dimensional subsets $F(X_1\times[-1, 0))$ and $F(X_2\times[0, 1))$.

Next, we shall verify that the closed subspace $F(X_1\times[-1, 0])=Z_F$ (the mapping cylinder of F [20; p. 365]) is locally contractible. This is clearly true for all $w\in F(X_1\times[-1, 0))$ so let $w\in F(X_1\times\{0\})=X_2$. Choose an open neighborhood $U\subset Z_F$ of w and let $U_0=U\cap X_2$. Since X_2 is an ENR, it is locally contractible [8; (V.7.1.)] so there is an open neighborhood $V_0\subset U_0$ of x in X_2 and a homotopy $H:V_0\times I\to U_0$ such that $H_0=\mathrm{id}_{V_0}$ and $H_1=a$ point in U_0.

Inside the open set $F^{-1}(V_0)$ (i. e. open in $X_1\times\{0\}$) there is an open neighborhood $W_0\subset F^{-1}(V_0)$ of the compactum $F^{-1}(w)$ in $X_1\times\{0\}$, such that for some $\delta<0$, then open set $W=W_0\times(\delta, 0]$ lies entirely inside $F^{-1}(U)$ and $F(W)$ is open in Z_F (recall that the map F is proper). The homotopy $H^*:F(W)\times I\to U$ given by

$$H^*([(u,t)],s)=\begin{cases}[(u,(1-2s)t)]; & 0\leqslant s\leqslant\frac{1}{2}\\ H(f(u),2s-1); & \frac{1}{2}\leqslant s\leqslant 1\end{cases}$$

now shrinks $F(W)$ to a point inside of U. This establishes that Z_F is an ANR [8; (V.7.1.)].

Since X_2 and $X_2\times[0, 1)$ are both ANR's so is therefore their union with Z_F [8; (II. 4.1.), (II.10.1.), (III.3.2.)]. The assertion now follows by [5; (IV.8.13.1.)].

Assertion 2. *Y is a generalized $(n+1)$-manifold with boundary.*

Proof. By Assertion 1 and [16; (1.1.)] we only must check that for every $w\in Y$ and every $q\in\mathbf{Z}_+$ the following holds:

$$(1)\quad H_q(Y, Y-\{w\})\cong\begin{cases}\mathbf{Z}; & q=n+1 \text{ and } w\in\mathring{Y}\\ 0; & \text{otherwise.}\end{cases}$$

By excision [20; (IV.6.5.)], it suffices to check (1) for the open subset $A=F\left(X_1\times\left(-\frac{1}{2},1\right)\right)$. Let $B=F^{-1}(A)$, choose any $w\in A$ and define $W=F^{-1}(w)$, $A'=A-\{w\}$, $B'=B-W$. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
\ldots\to H_q(B') & \xrightarrow{i_*} & H_q(B) & \xrightarrow{j_*} & H_q(B,B') & \xrightarrow{\Delta_*} & H_{q-1}(B') & \xrightarrow{i_*} & H_{q-1}(B) & \to\ldots \\
\downarrow F_*| & & \downarrow F_* & & \downarrow \Phi & & \downarrow F_*| & & \downarrow F_* & \\
\ldots\to H_q(A') & \xrightarrow{i_*} & H_q(A) & \xrightarrow{j_*} & H_q(A,A') & \xrightarrow{\Delta_*} & H_{q-1}(A') & \xrightarrow{i_*} & H_{q-1}(A) & \to\ldots
\end{array}$$

Since $F:(B,B')\to(A,A')$ is proper and acyclic map between paracompact spaces [6; (IX.5.3.)], the Vietoris-Begle mapping theorem [20; (VI.9.15.)] implies that F_* and $F_*|$ are isomorphisms, hence by the five lemma [20; (IV.5.11.)], Φ is an isomorphism. Therefore by the Borel-Moore duality theorem [21; (II.2.2.)] and the excision [20; (IV.6.5.)]:

$$H_q(Y, Y-\{w\})\cong H_q(A,A')\cong H_q(B,B')\cong \check{H}^{n+1-q}(W).$$

The formula (1) is now verified by invoking [10; (2.2.)], and the assertion follows.

We now complete the proof of Theorem (3.3): Let $i(X_k)$ be Quinn's local index of X_k, $k=1$, 2 (cf. Theorem (2.1)) and define $T=\{(u,t)|u\in X_1, 0<t<1\}$. Then

$$i(X_1)=i(Y)=i(T)=i(X_2\times\mathbb{R})=i(X_2)$$

since $\dot{Y}=X_1$, T is open in Y and homeomorphic to $X_2\times\mathbb{R}$, and since Theorem (2.1) applies.

Proof of Theorem (3.1). First, we note that it suffices to show that there is a resolution, for it can always be made conservative by [1; p. 312] if $n=3$, [13; (2.6.2.)] if $n=4$, or [19; p. 271] if $n\geqslant 5$. Also, by [10] ($n\leqslant 2$) and [1] ($n=3$) we may restrict to $n\geqslant 4$. There the conclusion follows immediately from Theorem (3.3) since M has a vanishing Quinn's obstruction, hence $i(X)=i(M)=1$ thus X resolves, too.

Remark. By same techniques one can prove analogous results for generalized manifolds with boundary.

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RAZREŠEVANJE ACIKLIČNIH SLIK MNOGOTEROSTI VIŠJIH DIMENZIJ

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Povzetek

V članku je pokazano, da ima vsaka posplošena n-mnogoterost ($n \geqslant 4$), ki je aciklična slika neke topološke n-mnogoterosti, celičasto razrešitev. To je pospolšitev analognega izreka J. L. Bryanta in R. C. Lacherja za $n=3$, na višje dimenzije.