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MATHEMATIQUES
Topologie

ISOLATED SINGULARITIES IN GENERALIZED 3-MANIFOLDS

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Generalized manifolds have had an important role in modern geometric topology ever since they were introduced in the 1930's. In recent years they were instrumental in the work by M. G. Brin, J. L. Bryant, J. W. Cannon, R. D. Edwards, R. C. Lacher, D. R. McMillan, Jr., F. S. Quinn, T. L. Thickstun, and others — on the recognition problem for topological manifolds [6]. In this paper we prove that generalized 3-manifolds which possess a certain amount of general position cannot have isolated singularities. As an application we give a geometric criterion for the Freudenthal compactification of an open 3-manifold to be nonsingular.

Preliminaries. We shall work in the category of locally compact Hausdorff spaces and continuous mappings. We shall use (co)-homologies with Z coefficients throughout the paper. A general reference for 3-manifolds is [5].

A space X is a generalized n-manifold with boundary ($n\in N$ if: (i) X is an euclidean neighbourhood retract (ENR), i. e. for some integer m, X embedes in R^m as a retract of an open subset of R^m: (ii) X is a homology n-manifold with boundary, i. e. for every $x\in X$ either $\check{H}^*(X, X-\{x\})\cong\check{H}^{n-*}(\{x\})$ or $\check{H}^*(X, X-\{x\})\cong 0$. The subset $\dot{X}=\{x\in X\mid \check{H}^*(X, X-\{x\})\cong 0\}$ of X is called the boundary of X and $\mathring{X}=X-\dot{X}$ is the interior of X. If $\dot{X}=\emptyset$ we call X simply a homology n-manifold (resp. generalized n-manifold, if it is also an ENR). The set $S(X)=\{x\in X\mid x$ has no neighbourhood in X homeomorphic to an open subset of $B^n\}$ is called the singular set of X and its complement $M(X)=X-S(X)$ is the manifold set of X. A singularity $p\in S(X)$ of X is said to be isolated if there is a neighbourhood $U\subset X$ of p such that $(U-\{p\})\cap S(X)=\emptyset$.

A space X is LC^n if it is $k-LC$ for all $k\in\{0, 1, \dots, n\}$, i. e. if for every $x\in X$ and every neighbourhood $U\subset X$ of x there is a neighbourhood $V\subset U$ of x such that $\Pi_k(V)\to\Pi_k(U)$ is trivial. A space X is LC^∞ (=locally contractible) if for every $x\in X$ and every neighbourhood $U\subset X$ of x there is a neighbourhood $V\subset U$ of x such that V is null-homotopic in U.

Let X be a σ-compact space and represent it as $X=\cup_{i=1}^{\infty}K_i$ of compact subsets $K_{i+1}\subset \operatorname{int} K_i\subset X$. An end of X is a sequence $e=\{U_i\}$ of the complements $U_i=X-K_i$. The Freudenthal compactification $\hat{X}$ of X is $X\cup\{e\}$ with $\{U_i\}$ as the basis of topology at the end e [1].

A mapping of a disk $f: B^2\to X$ into a space X is called a Dehn disk if $S_f\cap\partial B^2=\emptyset$, where $S_f=\overline{\{x\in B^2\mid f^{-1}f(x)\neq x\}}$ is the singular set of f. Finally, a fake cube is a compact contractible 3-manifold with boundary not homeomorphic to B^3. The classical Poincaré conjecture asserts that there are no fake cubes.

Theorem. Suppose that X is a generalized 3-manifold with the following two properties:

(A) No compact subset of X contains infinitely many pairwise disjoint fake cubes; and

(B) for every pair $f_1, f_2: B^2 \to X$ of Dehn disks such that $f_1(B_2) \cap f_2(B^2) = f_1$ (int $B^2) \cap f_2$ (int B^2), and for every neighbourhood $U \subset X$ of $f_1(B^2) \cup f_2(B^2)$, there is a pair of mappings $f'_1, f'_2: B^2 \to U$ such that $f'_1(B^2) \cup f'_2(B_2) = \emptyset$ and $f'_i | \partial B^2 = f_i | \partial B^2$ for both i's.

Then X has no isolated singularities.

Remark. Both conditions (A) and (B) are necessary, as the examples in [2, 3, 7] show.

Proof. Let $p \in X$ and let $U \subset X$ be an open neighbourhood of p such that $U \cap S(X) \subset \{p\}$. By [3] (Lemma 1) there is a compact orientable connected generalized 3-manifold $N \subset U$ with boundary ~~a compact orientable connected generalized 3-manifold $N \subset U$ with boundary~~ a compact orientable 2-manifold such that $p \in \text{int}\, N$. Since X is an ENR it is locally contractible so we may assume that N is null-homotopic in U. Let $c = \Sigma_{n=0}^{?} (n+1)^2 g(n)$ where $g(n)$ is the number of components of $\dot{N}$ with genus n. Choose N so that c is minimal. We shall show that $c = 0$. So suppose that $c > 0$. Then there is a boundary component $C \subset \dot{N}$ with positive genus; C is a 2-sphere with $k > 0$ handles since $\dot{N}$ is orientable. Let $L: \partial B_2 \to C$ be an essential simple closed curve. By our choice of N the inclusion-induced homomorphism $\Pi_1(\dot{N}) \to \Pi_1(U)$ is trivial hence there is an extension $f: B^2 \to U$ of L over B^2. Using methods similar to those employed in [7] (pp. 167-168) we can assume that f is locally PL near C and that it is in a general position with respect to C, because $C \subset M(\dot{X})$. Thus we may assume $f^{-1}(C)$ is a finite collection of pairwise disjoint PL simple closed curves in B^2, one of them being ∂B^2. Let $J \subset \text{int}\, B^2$ be an innermost such curve and let $E \subset f(B^2)$ be the (singular) subdisk bounded by $f(J)$. There are three possibilities.

Case 1. $f(J)$ is inessential on C. Then $f(J)$ bounds a (singular) disk $E' \subset C$. Exchanging E with E' we can go to the next innermost curve.

Case 2. $f(J)$ is essential on C and $E \subset U$-int N. Since U-int $N \subset M(X)$ we can use Dehn's lemma to attach a 2-handle to N after we have made E locally PL by [1]. This reduces c which, in its turn, contradicts the minimality of c. Hence this case cannot occur.

Case 3. $F(J)$ is essential on C and $E \subset N$. By [4] (p. 168), $f(J)$ can be replaced by a simple closed curve $J' \subset C$ such that J' is nontrivial on C but bounds a Dehn disk in N. Let $R \subset C$ be a regular neighbourhood of J' in C and let J_1 and J_2 be two simple closed curves boundary components of R. Then J_i bounds a Dehn disk D_i in N for each $i = 1, 2$. Using (B), we can get D_1 and D_2 disjoint in N — denote them by

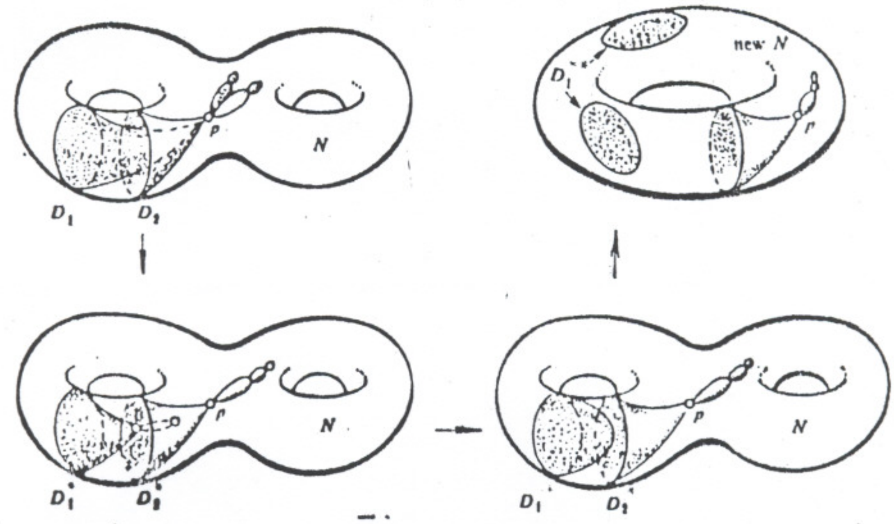

Fig. 1

D_1^* and D_2^*, respectively. Thus one of them will miss p, say $p \notin D_1^*$. By [1] we can make D_1^* locally PL — denote it by D_1^{**}. Then by cutting N along D_1^{**} we reduce the complexity c which, in turn, again contradicts its minimality. Hence this case cannot occur either. (See the Figure).

We conclude that indeed $c=0$ hence $g(N, p)=0$. Since N satisfies $f(N)$ and since $S(N)=\{p\}$ it follows from [6] (Theorem 7.5*) that N is a 3-manifold. In particular, $n \in M(X)$.

Corollary. Suppose that M is an open 3-manifold with finitely many ends and let $\widehat{M}$ be its Freudenthal compactification. Then $\widehat{M}$ is a 3-manifold iff $\widehat{M}$ is an LC^2 homology 3-manifold and it also satisfies the conditions (A) and (B) above.

Proof. ($\Rightarrow$) It follows from [7] (Theorem 2.3) and from Kneser's Finiteness theorem [5].

($\Leftarrow$): $\widehat{M}$ is clearly finite-dimensional hence it is an ENR as soon as it is LC^∞ at the points $p_1, \ldots, p_t$ of compactifictions (assume that M has t ends). Since for each i, $\widehat{M}$ is always 0-LC at p_i and since $\widehat{M}$ deforms onto a Freudenthal compactification of a locally finite 2-dimensional polyhedron with t ends, it suffices to show that $\widehat{M}$ is LC^2 at each p_i. The assertion now follows from our Theorem above.

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ERRATUM

In the D. Repovš' article 'Isolated singularities in generalized 3-manifolds', published in 'Comptes rendus de l'Académie bulgare des Sciences' Vol. 39, 1986, No. 10, pp. 13-15, a whole line is missing. Namely, in page 14 after line 6 from above it has to be added the sentence concluding the formulation of the Theorem: Then X has no isolated singularities.