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GENERALIZED 3-MANIFOLDS WITH BOUNDARY

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Generalized manifolds have held an important position in topology ever since they were introduced in the 1930's [25]. For low dimensions ($\leqslant 2$) their local algebraic properties are strong enough to imply that they are genuine manifolds [7], [25]. In higher dimensions they are interesting for at least two reasons:

(i) they arise in many different classes of spaces (e.g., as quotient spaces of cell-like upper semicontinuous decompositions of manifolds, as manifold factors, as quotients of Lie group actions on manifolds, as suspensions of homology spheres, etc.), and

(ii) they have the same global algebraic properties possessed by topological manifolds (e.g., local orientability, duality, etc.) [14].

Recent success in higher dimensions – a remarkably simple characterization of n-manifolds ($n \geqslant 5$) by R. D. Edwards [8] and F. Quinn [17] – has stimulated an upsurge in interest in the geometric topology of generalized manifolds.

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Particulary under investigation are generalized manifolds of dimensions three and four:

(i) a partial analogue of Edwards–Quinn's result is now known in dimension three, modulo the Poincaré conjecture [20], and

(ii) Quinn has recently extended his resolution theorem [17] to dimension four [18].

However, an analogue of Edwards' shrinking theorem [8] for this dimension is still missing. (For a review of the present situation in dimension three see [15]).

On the other hand, little is known about the topology of generalized manifolds *with boundary* although they occur quite naturally in many different situations, as we shall show in this paper. We propose to derive some basic properties of these spaces, most of which are analogous to those already known for generalized manifolds without boundary. We are mainly interested in dimension three – the main part of the paper is on generalized 3-manifolds with boundary. We also give a list of some interesting open problems in this area.

1. PRELIMINARIES

A space X is an *euclidean neighborhood retract* (ENR) if it is homeomorphic to a retract of an open subset of some $\mathbf{R}^n$. Equivalently, X is a separable, locally compact, finite-dimensional metrizable ANR, [3]. Let R be a principal ideal domain (PID). A Hausdorff space X is an *R-homology n-manifold* ($n \in \mathbf{N}$) if for each $x \in X$,

$$\check{H}^*(X, X - \{x\}; R) \cong \check{H}^{n-*}(\{x\}; R),$$

where $\check{H}^*(\_;R)$ is the Čech cohomology with coefficients in R. A Hausdorff space X is an *R-homology n-manifold with boundary* ($n \in \mathbf{N}$) if for each $x \in X$, either

$$\check{H}^*(X, X - \{x\}; R) \cong \check{H}^{n-*}(\{x\}; R) \quad \text{or} \quad \check{H}^*(X, X - \{x\}; R) \cong 0.$$

The subset

$$\dot{X} = \{x \in X \mid \check{H}^*(X, X - \{x\}; R) \cong 0\}$$

of X is called the *boundary* of X and $\overset{\circ}{X} = X - \dot{X}$ the *interior* of X.

Lemma 1.1. *Let X be an* ANR *and R a* PID. *If X is an R-homology n-manifold $(n \in \mathbf{N})$ then for each $x \in X$ and each $q \in \mathbf{Z}$:*

$$H_q(X, X - \{x\}; R) \cong \begin{cases} R, & q = n \\ 0, & q \neq n \end{cases}$$

while if X is an R-homology n-manifold with boundary then for each $x \in X$ and each $q \in \mathbf{Z}$:

$$H_q(X, X - \{x\}; R) \cong \begin{cases} R, & q = n \text{ and } x \in \overset{\circ}{X} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. On the class of ANR's the Čech cohomology agrees with the singular cohomology [23]. The conclusion now follows by the Universal Coefficients theorem [23].

Let R be a PID and consider an R-homology n-manifold X with nonempty boundary $(n \in \mathbf{N})$. We observe that $\dot{X}$ need not be an R-homology $(n-1)$-manifold (as it would be the case if X was a topological manifold with boundary). A simple example is $X =$ the interior of any n-manifold with boundary $(n > 1)$ together with one point from its boundary. It may also happen that $\dot{X}$ is an R-homology $(n-1)$-manifold with nonempty boundary, e.g., let $X =$ the interior of the standard n-ball B^n plus an $(n-1)$-ball on ∂B^n. The next proposition gives a criterion for determining the boundary points of $\dot{X}$:

Proposition 1.2. *Let X be an* ANR *and an R-homology n-manifold with boundary, R a* PID, *$n \in \mathbf{N}$. Suppose that $p \in \dot{X}$ and that*

$$H_*(\dot{X} - \{p\}; R) \cong H_*(\dot{X}; R).$$

Then $p \in (\dot{X})^{\cdot}$.

Proof. We suppress the coefficients from the notation. Consider the homology sequence of the triple $(X, \dot{X}, \dot{X} - \{p\})$ over R:

$$\ldots \xrightarrow{i_*} H_{q+1}(X, \dot{X} - \{p\}) \xrightarrow{j_*} H_{q+1}(X, \dot{X}) \xrightarrow{\Delta_*}$$

$$\longrightarrow H_q(\dot{X}, \dot{X} - \{p\}) \xrightarrow{i_*} H_q(X, \dot{X} - \{p\}) \xrightarrow{j_*}$$

$$\longrightarrow H_q(X, \dot{X}) \longrightarrow \ldots$$

Since

$$H_*(\dot{X}) \cong H_*(\dot{X} - \{p\})$$

it follows by [23, Lemma 6 on p. 202] that

$$H_*(X, \dot{X} - \{p\}) \cong H_*(X, \dot{X}).$$

Hence $\operatorname{im} \Delta_* = 0 = \ker i_*$ so

$$H_*(\dot{X}, \dot{X} - \{p\}) \cong \ker j_* = 0$$

thus by Lemma 1.1, $p \in (\dot{X})^{\cdot}$.

A *generalized n-manifold* $(n \in \mathbf{N})$ is an ENR that is also a **Z**-homology n-manifold. A *generalized n-manifold with boundary* $(n \in \mathbf{N})$ is an ENR X such that X is a **Z**-homology n-manifold with boundary and $\dot{X}$ is a generalized $(n-1)$-manifold. Let X be a generalized n-manifold (possibly with boundary), $n \in \mathbf{N}$. The set $S(X) = \{x \in X \mid x$ has no neighborhood in X homeomorphic to an open subset of $B^n\}$ is the *singular set* of X, its complement $M(X) = X - S(X)$ is the *manifold set* of X. The points of $S(X)$ (resp. $M(X)$) are called the *singularities* (resp. *manifold points*) of X. If $\dot{X} = \phi$ or if $S(X) \subset \mathring{X}$ then $M(X)$ is a topological n-manifold (possibly with boundary, in the second case).

The next two propositions give an interesting relationship between generalized manifolds and generalized manifolds with boundary.

Proposition 1.3. *Let X be a generalized n-manifold $(n \geqslant 3)$ with $S(X) \subset Z$, where $Z \subset X$ is a compact 0-dimensional set. Then there exists an n-cell $B \subset X$ and a generalized n-manifold with boundary $Y \subset X$, such that $S(Y) \subset Z$, $Y = X - \operatorname{int} B$, and $Z \subset \partial B$.*

Proof. Let $B_0 \subset X - Z$ be any tamely embedded n-cell. We get B from B_0 by pushing out from B_0 wildly embedded (in X) "feelers" towards the points of Z.

Remark 1.4. The restriction $n \geqslant 3$ comes from the fact that generalized n-manifolds (possibly with boundary) are genuine n-manifolds (with boundary) as soon as $n \leqslant 2$ [25].

Example 1.5. Let $X = S^3$ and $B =$ a thickened one half of the Fox–Artin wild arc [9, Example (3.1)]. Then $S(Y) = \{p\} =$ the only wild point of the arc (see Figure 1).

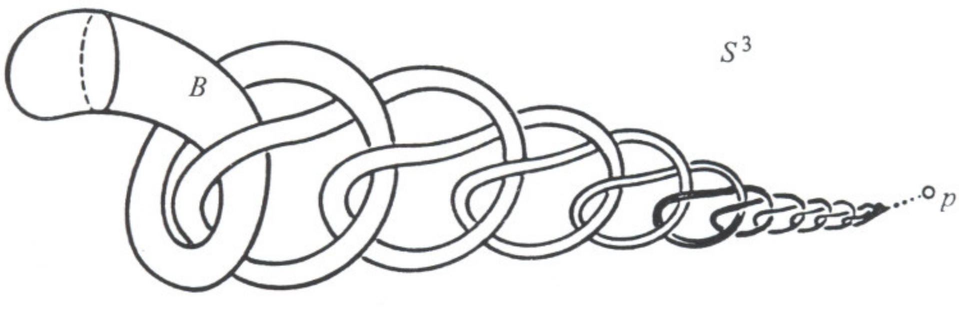

Figure 1

Like manifolds, generalized n-manifolds with homeomorphic boundaries can also be glued together to produce new generalized n-manifolds:

Proposition 1.6. *Let X and Y be generalized n-manifolds with boundary $(n \in \mathbf{N})$ and suppose that there exists a homeomorphism $h \colon \dot{X} \to \dot{Y}$. Then $X \cup_h Y$ is a generalized n-manifold.*

Proof. Since $\dot{X}$ is an ENR so is $X \cup_h Y$, by [3, Theorem (IV.6.1)]. It therefore suffices to show that $X \cup_h Y$ is a **Z**-homology n-manifold. We shall suppress the coefficients from the notation. The argument presented below is clearly valid over any PID. Consider the Mayer–Vietoris sequence for the pairs $(X, X - \{p\})$ and $(Y, Y - \{h(p)\})$. (By the Excision theorem [23] it suffices to consider only the case when $p \in \dot{X}$.)

$$\ldots \to H_q(X, X - \{p\}) \oplus H_q(Y, Y - \{h(p)\}) \to$$
$$\to H_q(X \cup_h Y, (X \cup_h Y) - \{p\}) \to H_{q-1}(\dot{X}, \dot{X} - \{p\}) \to$$
$$\to H_{q-1}(X, X - \{p\}) \oplus H_{q-1}(Y, Y - \{h(p)\}) \to \ldots .$$

Since $p \in \dot{X}$ and $h(p) \in \dot{Y}$ it follows by Lemma 1.1 that

$$H_*(X, X - \{p\}) \cong 0 \cong H_*(Y, Y - \{h(p)\}).$$

Also, $\dot{X}$ is a generalized $(n-1)$-manifold hence

$$H_q(X \cup_h Y, (X \cup_h Y) - \{p\}) \cong H_{q-1}(\dot{X}, \dot{X} - \{p\}) \cong R$$

if $q = n$ and is trivial otherwise. The assertion now follows by Lemma 1.1.

Remark 1.7. Let M be a closed PL n-manifold $(n \geqslant 3)$ and $N \subset M$ a separating $(n-1)$-submanifold. Then we can *split* M *along* N, i.e., we remove one of the components of $M - N$ from M and we obtain a closed subspace X of M. In general, X need not be (even a topological) n-manifold with boundary – this entirely depends on how "wildly" N is embedded in M. For example, if $M = \mathbf{R}^3$ and $N =$ the Alexander's horned sphere [6], then X can be a 3-cell or it can have a Cantor set of singularities in the boundary, depending upon our choice for X between the two components of $M - N$. (The singularities in the second case are precisely the wild points of N.) However, an argument analogous to the one in the proof of Proposition 1.6, shows that X is always a generalized n-manifold with boundary $(\dot{X} = N)$.

A generalized n-manifold (resp. generalized n-manifold with boundary) $(n \in \mathbf{N})$ is said to have a *resolution* if there exists a pair (M, f), where M is an n-manifold (resp. n-manifold with boundary) and f: $M \to X$ is a proper cell-like onto map (resp. a proper cell-like onto map such that $f(\partial M) \subset \dot{X}$). A resolution (M, f) is called *conservative* if $f^{-1}(x) =$ point, for every $x \in M(X)$. It is known that all generalized n-manifolds $(n \geqslant 4)$ have (conservative) resolutions [17], [18], and certain generalized 3-manifolds are also known to be resolvable, modulo the Poincaré conjecture [4], [24]. These results imply the following observation:

Proposition 1.8. *Let X be a generalized n-manifold with boundary. If X has a resolution then $\mathring{X}$ has a conservative resolution.*

Proof. Let (M, f) be a resolution of X. Then the restriction $f | f^{-1}(\mathring{X})$: $f^{-1}(\mathring{X}) \to \mathring{X}$ is a resolution of $\mathring{X}$. The assertion now follows by [5, Theorem 1] if $n = 3$, by [18, Corollary 2.6.2] if $n = 4$, and [22, Approximation Theorem] if $n \geqslant 5$. (The case $n \leqslant 2$ is trivial, as we have already observed above.)

2. DIMENSION THREE

Dimension three is in many respects peculiar, mostly due to the unresolved status of the Poincaré conjecture in that dimension. We first prove the analogues of the two finiteness theorems of J.L. Bryant and R.C. Lacher [5]. A $\mathbf{Z}_2$*-homology* 3*-cell* is a compact 3-manifold with boundary X such that X has the $\mathbf{Z}_2$-homology of the 3-cell.

Theorem 2.1. *For every compact generalized 3-manifold with boundary X there is an integer k_0 such that among any $k_0 + 1$ pairwise disjoint $\mathbf{Z}_2$-homology 3-cells in X at least one is contractible.*

Proof. By Proposition 1.6 the double DX of X is a generalized 3-manifold so there exists the Bryant–Lacher number n_0 for DX [5, p. 312]. Let $k_0 = \left[\frac{1}{2}(n_0 + 1)\right]$, where $[t] = \max\{n \in \mathbf{Z} \mid n \leqslant t\}$.

Theorem 2.2. *Let X be a compact generalized 3-manifold with boundary and assume that X has a resolution. Then there exists an integer k_0 such that among any $k_0 + 1$ pairwise disjoint $\mathbf{Z}_2$-homology 3-cells there is at least one genuine 3-cell.*

Proof. Let (M, f) be a resolution of X, let k_1 be the Kneser's number of M [11, Lemma 3.14], and let k_2 be the number given for X by Theorem 2.1. Put $k_0 = k_1 + k_2$ and consider an arbitrary $(k_0 + 1)$-tuple $F_1, \ldots, F_{k_0+1} \subset X$ of pairwise disjoint $\mathbf{Z}_2$-homology 3-cells. By pushing each F_i into $\operatorname{int} F_i$ along a collar on ∂F_i, we may assume that each F_i lies in $M(X) \cap \mathring{X}$. By Proposition 1.8 we may assume that f is a homeomorphism over $M(X)$. Therefore the F_i's lift to M. By our choice of k_2 at least $k_1 + 1$ among them are contractible. Thus at least one of them is a 3-cell.

There is an appropriate name for the property of X described in the conclusion of the preceding theorem: we say that a space X has *Kneser Finiteness* (KF) if for each compact subset $X_0 \subset X$ there is an integer k_0 such that X_0 contains at most k_0 pairwise disjoint *fake cubes,* i.e.

homotopy 3-cells which are not 3-cells (the Poincaré conjecture asserts there are no fake cubes).

Next, we shall prove an analogue of T.L. Thickstun's resolution theorem [24] (also obtained, independently, by R.J. Daverman (unpublished)) for generalized 3-manifolds with boundary (Theorem 2.4). As a consequence we prove an extension of the main result of [20, Theorem 3.3] to generalized 3-manifolds with boundary (Theorem 2.5).

Let X be a generalized 3-manifold with 0-dimensional singular set. Then by [4, Lemma 1] every $p \in X$ has arbitrary small compact generalized 3-manifold with boundary neighborhoods $N \subset X$ such that $\dot{N}$ is a closed orientable surface in $M(X)$. We say that X has *genus* $\leqslant n$ *at* p if p has arbitrarily small such neighborhoods N with $N =$ surface of genus $\leqslant n$. We say that X has *genus* n *at* p if X has genus $\leqslant n$ at p and doesn't have genus $\leqslant n-1$ at p. If X doesn't have genus $\leqslant n$ at p for any n we say X has *genus* ∞ *at* p. We shall denote the genus of X at p by $g(X,p)$ [15]. A sequence of pairwise disjoint compacta $\{C_i\}$ in a metric space X is a *null-sequence* if for every $\epsilon > 0$ all but finitely many among the C_i's have diameter $< \epsilon$.

Example 2.3. It is not surprising that the Poincaré conjecture enters into the picture as soon as we try to resolve generalized 3-manifolds with boundary, since the same is true with generalized 3-manifolds [4]. We consider an example which will be used later on. Suppose fake cubes exist and consider in $\operatorname{int} B^3$ a null-sequence $\{B_i\}$ of pairwise disjoint 3-cells converging to a point $p \in \partial B^3$. Replace each B_i by a fake cube F_i and choose a metric in

$$W = \left(B^3 - \bigcup_{i=1}^{\infty} \operatorname{int} B_i\right) \cup \left(\bigcup_{i=1}^{\infty} F_i\right)$$

so that the F_i's also converge (in W) to p (see Figure 2). Then W is a compact generalized 3-manifold with boundary $\dot{W} = S^2$. Since there are no fake cubes in $\mathbf{R}^3$, the point p cannot possess an euclidean neighborhood in W hence $S(W) = \{p\}$. We shall call such singularities *soft*, or *Wilder type* singularities [15].

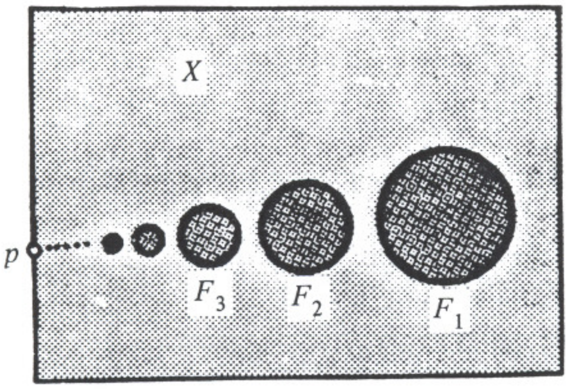

Figure 2

Consider the map $f\colon W \to B^3$ which is obtained by shrinking out all F_i's. Then f is clearly cell-like hence by [13, Theorem 4.2] a homotopy equivalence. In particular, W is contractible.

Theorem 2.4. *Let X be a compact generalized 3-manifold with boundary and suppose that $\dim S(X) \leq 0$. Then there exist:*

(i) *a compact generalized 3-manifold with boundary Y such that $\dim S(Y) \leq 0$, all singularities are soft and lie in $\mathring{Y}$, and $g(Y, y) = 0$ for every $y \in Y$;*

(ii) *a proper cell-like onto map $f\colon Y \to X$, with $f(\dot{Y}) \subset \dot{X}$.*

Hence if Y has KF *then (Y, f) is a resolution of X.*

Proof. Let $Z = X + C$, where $C = \dot{X} \times I$ is a collar on $\dot{X}$. By the arguments employed in the proof of Proposition 1.6, Z is a compact generalized 3-manifold with boundary and $S(Z) \subset S(X)$. By [24] there exist a compact generalized 3-manifold with boundary Y satisfying the requirements (i) above, and a proper cell-like onto map $h\colon X \to Z$. Let $g\colon Z \to X$ be the map induced by the contraction of C onto $\dot{X}$ along the fibers of C. Clearly, g is cell-like so it follows by [13, Corollary 4.2.2] that $f = g \circ h\colon Y \to X$ is cell-like, as desired.

Theorem 2.5. *Let $\mathscr{C}$ be the class of all compact generalized 3-manifolds with boundary X such that $\dim S(X) \leq 0$ and let $\mathscr{C}_0 \subset \mathscr{C}$ be the* *subclass of all contractible* $X \in \mathscr{C}$ *with at most one singularity. Then the following statements are equivalent:*

(i) *The Poincaré conjecture is true;*

(ii) *If* $X \in \mathscr{C}$ *then* X *has a resolution;*

(iii) *If* $X \in \mathscr{C}_0$ *then* X *has a resolution.*

Proof. The implication (i) ⇒ (ii) follows by Theorem 2.4 while (ii) ⇒ (iii) is clear, and to prove (iii) ⇒ (i) consider Example 2.3.

The next result describes the relationship between the softness of a singularity and the existence of a genus zero neighborhood. A subset $Z \subset X$ is Π_1*-negligible* if for each open set $U \subset X$ the inclusion-induced homomorphism $\Pi_1(U - Z) \to \Pi_1(U)$ is one-to-one. A subset $Z \subset X$ is 1-LCC if for every $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the inclusion-induced homomorphism $\Pi_1(V - Z) \to \Pi_1(U - Z)$ is zero.

Theorem 2.6. *Let* X *be a generalized* 3*-manifold with* $S(X) \subset Z$, *where* $Z \subset X$ *is a compact* 0*-dimensional set. Then the following statements are equivalent:*

(i) Z *is* 1-LCC *in* X;

(ii) Z *is* Π_1*-negligible in* X;

(iii) *For every* $x \in X$, $g(X, x) = 0$.

Furthermore, anyone of the statements (i)–(iii) *implies:*

(iv) *All singularities of* X *are soft.*

Proof.

(i) ⇒ (iii). Follows as Theorem 4 of [5].

(iii) ⇒ (ii). Choose an open $U \subset X$ and any loop $J \subset U - Z$. Since $\dim Z = 0$ and since $g(X, z) = 0$ for all $z \in Z$ there is a covering $V_1, \ldots, V_t$ of $Z \cap U$ with pairwise disjoint compact generalized 3-manifolds with boundary $\dot{V}_i = S^2 \subset M(X)$ for all i. Suppose now that J

bounds a (singular) disk in U. With techniques described in detail in [19, Ch. III] we can make this disk locally PL near the surface $S = \bigcup_{i=1}^{t} \dot{V}_i$, put it in general position with respect to S and cut it off at S, thus pushing it into $U - Z$, or just get it off $V_i \cap Z$ for each i.

(ii) ⇒ (i). Let $x \in X$ be any point and choose a neighborhood $U \subset X$ of x. Since X is an ANR it is 1-LC. Thus there is a neighborhood $V \subset U$ of x such that the inclusion-induced homomorphism $\Pi_1(V) \to \Pi_2(U)$ is zero. Since Z is Π_1-negligible, the homomorphisms $\Pi_1(V - Z) \to \Pi_1(V)$ and $\Pi_1(U - Z) \to \Pi_1(U)$ are both one-to-one. Consider the commutative diagram:

$$\begin{array}{ccc} \Pi_1(V - Z) & \xrightarrow{\;i_*\;} & \Pi_1(U - Z) \\ \downarrow & & \downarrow \\ \Pi_1(V) & \xrightarrow{\;0\;} & \Pi_1(U) \end{array}$$

Clearly, $i_* = 0$.

(i) ⇒ (iv). By [5, Theorem 4] no open subset $V \subset X$ has KF unless $V \subset M(X)$. This implies X can only have soft singularities, if any.

Let X be a compact generalized 3-manifold with boundary and suppose that the double DX of X is a 3-manifold. Then X need not be itself a 3-manifold: e.g., R.H. Bing proved that the double of the solid Alexander horned sphere yields S^3 [1]. But we can prove something:

Theorem 2.7. *Let X be a compact generalized 3-manifold with boundary such that DX is a 3-manifold. Then X has no isolated singularities.* (Note that $S(X)$ must lie in $\dot{X}$.)

Proof. Let $p \in \dot{X}$ be a point with a neighborhood $U \subset X$ such that $U \cap S(X) \subset \{p\}$. Consider p as a (potentially wild) point on the surface $\dot{X}$ in the 3-manifold DX. By O.G. Harrold and E.E. Moise [10, Theorem I] the surface $\dot{X}$ can be wild at p from at most one side in DX. But in DX the two sides of $\dot{X}$ are symmetric hence $\{p\} \subset DX$ is 1-LCC. Also, since X is contained in the 3-manifold DX it has KF. By [5, Theorem 4], $p \in M(X)$.

3. EPILOGUE

When one considers a resolution $f: M \to X$ of a generalized 3-manifold with boundary there naturally arises the question whether $Df: DM \to DX$ is also a resolution (of DX), where D means "the double" in the obvious sense. This gives rise to the following question:

Problem 3.1. Let M be a compact 3-manifold with boundary and $f: M \to X$ a proper cell-like map of M onto an ANR X. Let $DX = X \cup X/R$, where R is the equivalence relation on $X \times X$ given by "$x R y$ if and only if $f^{-1}(x) \cap \partial M = f^{-1}(y) \cap \partial M \neq \phi$, $x, y \in X$". Is the associated double $Df: DM \to DX$ also cell-like *if* $\dim X = 3$?

If the answer to 3.1 is affirmative then one can prove an analogue of Brin–McMillan's embedding criterion [4, Theorem 1] which says that for every resolvable compact generalized 3-manifold X with 0-dimensional singular set, the manifold set $M(X)$ embeds in a closed 3-manifold. For, given a resolution $f: M \to X$ of a compact generalized 3-manifold with boundary X such that $\dim S(X) \leqslant 0$, by 3.1, $Df: DM \to DX$ would be a resolution for DX so by [4], $M(DX)$ would embed in some closed 3-manifold, hence so would $M(X) \subset M(DX)$.

Brin–McMillan's result [4] says that the condition "$M(X)$ embeds in a closed 3-manifold" is not only necessary but also sufficient for X to be resolvable (assuming $\dot{X} = \phi$). (Bryant and Lacher proved the sufficiency doesn't depend on the $\dim S(X) \leqslant 0$ condition [5, Theorem 1].) Therefore one may ask the same question if $\dot{X} \neq \phi$:

Problem 3.2. Let X be a compact generalized 3-manifold with boundary and suppose that $\dim S(X) \leqslant 0$. Assume that $M(X)$ embeds in some closed 3-manifold. Does X have a resolution?

Let X be a compact generalized 3-manifold and let Y be X plus a collar C attached to $\dot{X}$. If Y is a 3-manifold with boundary then

(i) $S(X) \subset \dot{X}$; and

(ii) X has a resolution.

(To see that (ii) holds map Y onto X by shrinking out the fibers of the collar C.) The converse need not be true as the next example demonstrates. Take a noncellular arc $A \subset B^3$ with one wild point p^* (e.g., [9, Example 3.1] so that $A \cap \partial B^3 = \{p\}$, where $\partial A = \{p, p^*\}$. Then $X = B^3 / A$ is a compact generalized 3-manifold with boundary and satisfies both properties (i) and (ii). If $Y = X + C$ were a 3-manifold with boundary then A would have to be cellular, a contradiction (see Figure 3).

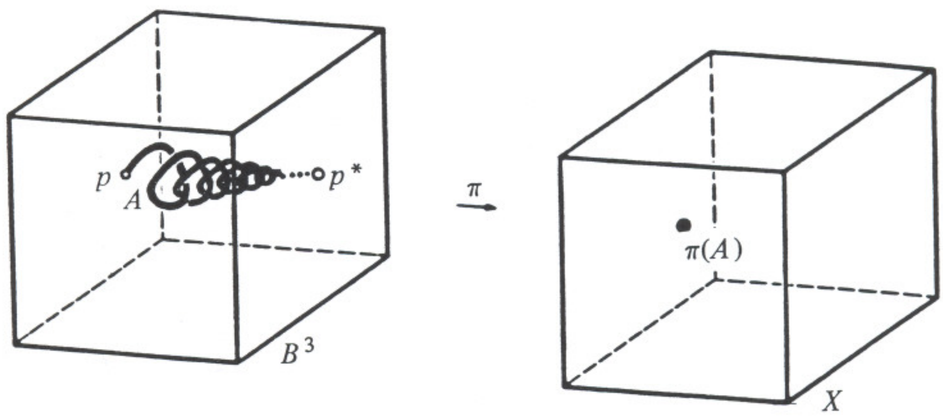

Figure 3

Hence, in general, one can only conclude that $Y = X + C$ is a compact generalized 3-manifold with boundary and that $S(Y) \subset S(X)$, and moreover that if Y has a resolution then X has it, too. Note that an affirmative solution of 3.1 would show that given a resolution $f: M \to X$ one can resolve Y, too. For, one could just add a collar D to M, thus getting a 3-manifold with boundary $N = M + D$ and a map $g: N \to Y$, obtained from f by extending the latter over D fiberwise. By 3.1, g would be cell-like, thus a resolution for Y (see Figure 4).

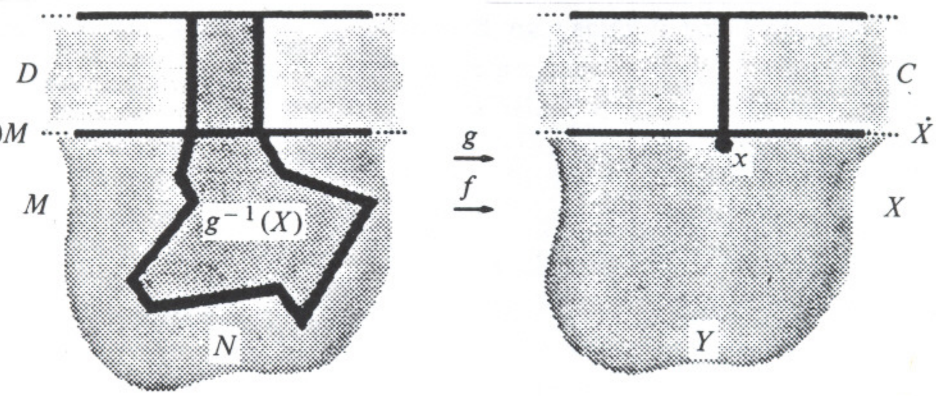

Figure 4

Problem 3.3. Let X be a compact generalized 3-manifold with boundary and suppose that $\dim S(X) \leqslant 0$. If DX is a 3-manifold, is X plus a collar on $\dot{X}$ necessarily a 3-manifold with boundary?

Note that in 3.3 the singular set of X may be nonvoid (recall that solid Alexander horned sphere example from Chapter II) in this case $S(X)$ is a Cantor set. If $S(X) \neq \phi$ then $S(X)$ has at least a Cantor set worth of points, for Theorem 2.7 asserts no genuine singularity can be isolated.

Another interesting question is about the existence of conservative resolutions. It is known that every resolvable generalized n-manifold admits a conservative resolution: for $n \leqslant 2$ this is obvious, $n = 3$ is due to Bryant and Lacher [5], $n = 4$ to Quinn [18], and $n \geqslant 5$ to L.C. Siebenmann [22]. We are asking what happens if $\dot{X} \neq \phi$:

Problem 3.4. Suppose that X is a resolvable generalized 3-manifold with boundary. Does X have a conservative resolution?

Suppose we are given $f\colon M \to X$ as in 3.4 above. If 3.1 had an affirmative answer then $Df\colon DM \to DX$ would be a resolution (as has already been observed earlier in this chapter) so it could be replaced by a conservative one, $g\colon N \to DX$. The problem is how to split N along a surface in N, sufficiently close to the (possibly wild) surface $g^{-1}(\dot{Y})$, in order to get a resolution for X.

Problem 3.5. Suppose that X is a compact generalized 3-manifold with boundary and that there is a proper cell-like onto map $f\colon M \to X$, where M is a compact 3-manifold with boundary. Assume that $f \mid \partial M$ is proper and cell-like, that $f^{-1}(\dot{X}) = \partial M$, and that $S(X) \subset \dot{X}$. Is then X a 3-manifold with boundary?

The remaining problems are related to the search for an appropriate disjoint disks property (DDP) for 3-manifolds with boundary. An acceptable DDP should meet the following criteria:

(i) Every 3-manifold with boundary should have the DDP;

(ii) Given a generalized 3-manifold with boundary, having the DDP, it should be possible to first, resolve it, and second, shrink the associated cell-like decomposition (everything modulo the Poincaré conjecture);

(iii) This DDP should be reasonably easy to check.

(Compare [15].) Edwards and Quinn proved Cannon's DDP [7] works for higher dimensions [8], [17]. In [20] it was shown that the so-called map separation property (MSP), which was introduced by H. W. Lambert and R. B. Sher [16], in the late 1960's, meets the criteria (i)–(iii) if one restricts to the case $\dim S(X) \leqslant 0$ and $\dot{X} = \phi$. (It is interesting to observe that the resolution results of Quinn [17], [18] $(n \geqslant 4)$ and of Thickstun [24] $(n = 3)$ do not require any kind of DDP.)

Recall that a map means only a continuous, hence not necessarily PL, map. A map $f\colon B^2 \to X$ is a *Dehn disk* if $\partial B^2 \cap S_f = \phi$, where $S_f = \overline{\{x \in B^2 \mid f^{-1}f(x) \neq x\}}$ is the *singular set* of f. A space X is said to have the *map separation property* (MSP) if given any collection $f_1, \dots, f_k\colon B^2 \to X$ of Dehn disks such that if $i \neq j$, then $f_i(\partial B^2) \cap f_j(B^2) = \phi$, and given a neighborhood $U \subset X$ of $\bigcup_{i=1}^{k} f_i(B^2)$ there exist maps $F_1, \dots, F_k\colon D \to U$ such that for each i, $F_i \mid \partial B^2 = f_i \mid \partial B^2$ and if $i \neq j$, then $F_i(B^2) \cap F_j(B^2) = \phi$. Using classical results from the 3-manifolds topology, such as [2], [12] and [26], it was shown in [20] that every 3-manifold (possibly with boundary) has the MSP and that, modulo the Poincaré conjecture, generalized 3-manifolds with $\dim S(X) \leqslant 0$ and having the MSP cannot have genuine singularities.

Problem 3.6. Let X be a compact generalized 3-manifold with boundary and assume $\dim S(X) \leqslant 0$. Suppose that X has the MSP. Is X a 3-manifold with boundary (modulo the Poincaré conjecture)?

Note that by Theorem 2.5, X has a resolution, $f\colon M \to X$. The problem is how to apply the MSP to shrink the associated cell-like decomposition $G = \{f^{-1}(x) \mid x \in X\}$. An affirmative answer to 3.6 would produce a positive solution to the next, related question: in the case $\dim S(X) \leqslant 0$.

Problem 3.7. Let X be a generalized 3-manifold with boundary and assume X has the MSP. Does DX have the MSP? Does $X + C$, C a collar on $\dot{X}$, have the MSP?

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