UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 544 — № 576 (1976), 93—94.

560. DECOMPOSITION THEOREM FOR NATURAL NUMBERS*

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1. The Decomposition Theorem. Let's take an arbitrary natural number $N\,(N \neq 0)$ and write it as the sum of k integers $N = a_1 + a_2 + \cdots + a_K$, where $\alpha \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_k$ and $\alpha \in \mathbf{N}$ such that $1 \leqq \alpha \leqq [N/k]$ (hereafter $[x]$ will denote the greatest integer not exceeding x). We shall denote the number of all such decompositions for given N, k and α by $S\,(N, k, \alpha)$.

Theorem 1.

$$S\,(N, k, \alpha) = \sum_{N_{k-1}=[((k-1)N)/k]}^{N-\alpha} \; \sum_{N_{k-2}=[((k-2)N_{k-1})/(k-1)]}^{2N_{k-1}-N} \cdots$$

$$\cdots \sum_{N_3=[(3N_4)/4]}^{2N_4-N_5} \; \sum_{N_2=[(2N_3)/3]}^{2N_3-N_4} ([N_2/2]-(N_3-N_2-1)).$$

Proof. The proof goes by induction on k. Obviously $S\,(N, 1, \alpha) = 1$. We shall not immediately proceed by $k=n$ for we shall rather work also on the proof for $k=2$ and $k=3$ to get the general idea of how the formula was evaluated.

1° Let $k=2$. Since $a_1 \leq a_2$ N can be decomposed in the following ways only: $N = 1 + (N-1) = 2 + (N-2) = \cdots = (\alpha-1) + (N-(\alpha-1)) = \alpha + (N-\alpha) = \cdots = m + (N-m)$ where $m = \max\{n \in \mathbf{N} \setminus \{0\} \mid n \leqq N-n\}$. Clearly $m = [N/2]$. Since in the first $\alpha - 1$ decompositions $a_1 \leqq \alpha$ they cannot count. Therefore $S(N, 2, \alpha) = [N/2] - (\alpha - 1)$.

2° Let $k=3$. Observe that $1 \leqq \alpha \leqq [N/3]$. Take a_1 from the set $\{\alpha, \alpha+1, \ldots, [N/3]\}$. Then N can be decomposed in the following way $N = a_1 + (N - a_1)$. What we are now looking for is-how many pairs (a_2, a_3) satisfying the condition $a_1 \leqq a_2 \leqq a_3$ and the equation $a_2 + a_3 = N - a_1$ there exist? Denote $N_2 = N - a_1$ and observe that $[(2N)/3] \leqq N_2 \leqq N - \alpha$. From 1° we know that $S(N_2, 2, a_1) = [N_2/2] - (N - N_2 - 1)$. Therefore

$$S(N, 3, \alpha) = \sum_{N_2=[(2N)/3]}^{N-\alpha} ([N_2/2] - (N - N_2 - 1))$$

since N_2 can take an arbitrary value from the set $\{[(2N)/3], \ldots, N-\alpha\}$.

3° Suppose theorem holds for $k=n$. Let $k=n+1$. Observe $1 \leqq \alpha \leqq [N/(n+1)]$. Take a_1 from the set $\{\alpha, \alpha+1, \ldots, [N/(n+1)]\}$ and rewrite N as $N = a_1 + (N - a_1)$.

* Presented June 1, 1976 by M. S. POPADIĆ.

Denote by N_s the sum $N_s = a_{k-s+1} + a_{k-s+2} + \cdots + a_k$, so $N = a_1 + N_n$ and by our supposition of induction

$$S(N_n, n, a_1) = \sum_{N_{n-1}=[((n-1)N_n)/n]}^{N_n-(N-N_n)} \sum_{N_{n-2}=[((n-2)N_{n-1})/(n-1)]}^{2N_{n-1}-N} \cdots$$

$$\cdots \sum_{N_3=[(3N_4)/4]}^{2N_4-N_5} \sum_{N_2=[(2N_3)/3]}^{2N_3-N_4} ([N_2/2] - (N_3 - N_2 - 1)).$$

Therefore

$$S(N, n+1, \alpha) = \sum_{N_n=[(nN)/(n+1)]}^{N-\alpha} S(N_n, n, a_1).$$

This also proves the Theorem 1.

2. The Product Theorem. Observe all the decompositions of N discussed in Part 1. for $\alpha = 1$. When is the product of all sumands $a_1 a_2 \cdots a_k$ maximal and what is its value for an arbitrary N? The answer will be given by the following theorem. Denote the maximal product by $P_{\max}(N)$.

Theorem 2. *Let* $N \in \mathbf{N}$, *then*

$$P_{\max}(N) = \begin{cases} 3^b & \text{if } N = 3b \\ 2.3^b & \text{if } N = 3b+2 \\ 4.3^b & \text{if } N = 3b+4 \end{cases}$$

Proof. Let $N = a_1 + \cdots + a_p$. When can the corresponding product $a_1 a_2 \cdots a_p$ be increased? Obviously this can be done if for some a_j $(1 \leqq j \leqq p)$ there exist such $a_{i,j}$ $(i = 1, \ldots, m_j)$ that $a_j = a_{1,j} + a_{2,j} + \cdots + a_{m_j,j}$ and the inequality $a_{1,j} a_{2,j} \cdots a_{m_j,j} > a_j$ hold.

This way we get that $(a_{1,1} \cdots a_{m_1,1}) \cdots (a_{1,p} \cdots a_{m_p,p}) > a_1 a_2 \cdots a_p$.

Clearly $a_{i,j} \geqq 2$, therefore $P_{\max}(N) = 2^a 3^b$. Since $3^2 > 2^3$ and $3+3 = 2+2+2$ a can take only the following values: 0, 1 or 2. Taking into the consideration the fact that $2a + 3b = N$ we get the formula for $P_{\max}(N)$.

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