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INFINITELY MANY SYMMETRIC SOLUTIONS FOR ANISOTROPIC PROBLEMS DRIVEN BY NONHOMOGENEOUS OPERATORS

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Dedicated to Professor Vicenţiu Rădulescu with deep esteem and admiration

Abstract. We are concerned with the existence of infinitely many radial symmetric solutions for a nonlinear stationary problem driven by a new class of nonhomogeneous differential operators. The proof relies on the symmetric version of the mountain pass theorem.

1. **Introduction and abstract setting.** Given an even functional on an infinite-dimensional Banach space that fulfills natural assumptions, the symmetric mountain pass lemma of P. Rabinowitz [18] establishes the existence of an unbounded sequence of critical values. This result extends to a symmetric framework the initial version of the mountain pass theorem due to A. Ambrosetti and P. Rabinowitz [1]. At the same time, the symmetric mountain pass theorem can be viewed as an extension of the Ljusternik-Schnirelmann theorem in the framework of unbounded functionals defined on Banach spaces. As pointed out by H. Brezis and F. Browder [6], the mountain pass theorem "extends ideas already present at Poincaré and Birkhoff". We refer to Y. Jabri [12] and P. Pucci and V. Rădulescu [17] for more details on the mountain pass theorem and related applications.

We recall the original statement of the symmetric mountain pass theorem.

Theorem 1.1. *Let X be a real infinite-dimensional Banach space and $\mathcal{J} \in C^1(X,\mathbb{R})$ a functional satisfying the Palais-Smale condition and the following hypotheses:*

(i) $\mathcal{J}(0) = 0$ and there are constants $\rho, \alpha > 0$ such that $\mathcal{J}_{|\partial B_\rho} \geq \alpha$;

(ii) $\mathcal{J}$ is even; and

(iii) for all finite-dimensional subspaces $X_0 \subset X$, there exists $R = R(X_0) > 0$ such that

$$\mathcal{J}(u) \leq 0 \quad \text{for all } u \in X_0 \setminus B_R(X_0).$$

Then $\mathcal{J}$ has an unbounded sequence of critical values.

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This result is an efficient tool for proving multiplicity properties in semilinear or quasilinear elliptic problems with odd nonlinearities and Dirichlet boundary condition. The standard application of Theorem 1.1 concerns the following boundary value problem (see Y. Jabri [12, pp. 122-124])

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary and $f : \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the following properties:
(i) f is odd in u, that is, $f(x,-u) = -f(x,u)$;
(ii) there exists $p \leq 2^* := 2N/(N-2)$ such that f satisfies the growth condition

$$|f(x,u)| \leq C(1 + |u|^{p-1}) \quad \text{a.e. } (x,u) \in \Omega \times \mathbb{R};$$

(iii) there are constants $\mu > 2$ and $r > 0$ such that for almost every $x \in \Omega$ and all $|u| \geq r$

$$0 < \mu F(x,u) \leq u f(x,u), \quad \text{where } F(x,u) := \int_0^u f(x,t)dt.$$

Under these hypotheses, Theorem 1.1 yields the existence of an unbounded sequence of solutions of problem (1).

The present paper was inspired by recent advances in the study of nonlinear stationary problems driven by nonhomogeneous differential operators. Important pioneering contributions to this field are due to T.C. Halsey [11] and V.V. Zhikov [20] who studied the behaviour of non-Newtonian electrorheological fluids and anisotropic materials. These models strongly rely on partial differential equations with variable exponent, which have been intensively studied in the last few decades. We refer to the recent monograph by V. Rădulescu and D. Repovš [19] for a comprehensive qualitative analysis of nonlinear PDEs with variable exponent by means of variational and topological methods. These problems (with possible lack of uniform convexity) are essentially described by the differential operator

$$\Delta_{p(x)} u := \operatorname{div}\,(|\nabla|^{p(x)-2}\nabla u),$$

which changes its growth properties according to the point. More precisely, the variable exponent $p(x)$ describes the geometry of a material that is allowed to change its hardening exponent according to the point. Recently, I.H. Kim and Y.H. Kim [13] introduced a new class of nonhomogeneous differential operators, which extend the standard operators with variable exponent. We refer to S. Baraket, S. Chebbi, N. Chorfi, and V. Rădulescu [4] and N. Chorfi and V. Rădulescu [8] for contributions in this new abstract setting.

In order to introduce the problem studied in this paper and our main result, we need to recall some basic notions and properties. We refer to V. Rădulescu and D. Repovš [19], resp. I.H. Kim and Y.H. Kim [13] for more details.

1.1. **Lebesgue and Sobolev spaces with variable exponent.** Let

$$C_+(\mathbb{R}^N) := \{p : \mathbb{R}^N \to \mathbb{R};\ p \text{ is continuous and } 2 \leq N < \inf_{x\in\mathbb{R}^N} p(x) \leq \sup_{x\in\mathbb{R}^N} p(x) < \infty\}.$$

If $p \in C_+(\mathbb{R}^N)$, we set $p^- := \inf_{x\in\mathbb{R}^N} p(x)$ and $p^+ := \sup_{x\in\mathbb{R}^N} p(x)$.

For all $p \in C_+(\mathbb{R}^N)$, let $L^{p(x)}(\mathbb{R}^N)$ be the Lebesgue space with variable exponent defined by

$$L^{p(x)}(\mathbb{R}^N) := \left\{u : \mathbb{R}^N \to \mathbb{R};\ u \text{ is measurable and } \int_{\mathbb{R}^N} |u(x)|^{p(x)}dx < +\infty\right\}$$

and endowed with the norm

$$|u|_{p(x)} := \inf\left\{\mu > 0;\ \int_{\mathbb{R}^N} \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Let $L^{p'(x)}(\mathbb{R}^N)$ be the dual space of $L^{p(x)}(\mathbb{R}^N)$, where $1/p(x)+1/p'(x)=1$. Then for all $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p'(x)}(\mathbb{R}^N)$ the following Hölder-type inequality holds:

$$\left|\int_{\mathbb{R}^N} uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)}\,. \tag{2}$$

Next, we define the corresponding Sobolev function space with variable exponent by

$$W^{1,p(x)}(\mathbb{R}^N) := \{u \in L^{p(x)}(\mathbb{R}^N);\ |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\}.$$

This space is endowed with the norm

$$\|u\|_{p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

The critical Sobolev exponent of $p \in C_+(\mathbb{R}^N)$ is defined by

$$p^*(x) := \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

The function spaces with variable exponent have some striking properties, namely:
(i) If $p \in C_+(\mathbb{R}^N)$ and $p^+ < \infty$, then the formula

$$\int_{\mathbb{R}^N} |u(x)|^p dx = p\int_0^\infty t^{p-1}\,|\{x \in \mathbb{R}^N;\ |u(x)| > t\}|\,dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do not have the mean continuity property. More precisely, if p is continuous and nonconstant in an open ball B, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are *never* translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f * g|_{p(x)} \le C\,|f|_{p(x)}\,\|g\|_{L^1}$$

holds if and only if p is constant.

We refer to [19] for additional properties.

1.2. **A new nonhomogeneous differential operator.** Assume that $p \in C_+(\mathbb{R}^N)$ and consider the mapping $\phi : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ that satisfies the following condiitons:

(H1) the function $\phi(\cdot,\xi)$ is measurable for all $\xi \ge 0$ and $\phi(x,\cdot)$ is locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$;

(H2) there exist $a \in L^{p'(x)}(\mathbb{R}^N)$ and $b > 0$ such that

$$|\phi(x,|v|)v| \le a(x) + b|v|^{p(x)-1}$$

for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$;

(H3) there exists $c > 0$ such that

$$\phi(x,\xi) \ge c\xi^{p(x)-2}, \quad \phi(x,\xi) + \xi\frac{\partial\phi}{\partial\xi}(x,\xi) \ge c\xi^{p(x)-2}$$

for almost all $x \in \Omega$ and for all $\xi > 0$.

For ϕ with the above properties we set

$$A_0(x,t) := \int_0^t \phi(x,s)sds. \tag{3}$$

Consider the functional $A : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$A(u) := \int_{\mathbb{R}^N} A_0(x, |\nabla u|)dx.$$

Assume that hypotheses (H1) and (H2) hold. Then by [13, Lemma 3.2], the nonlinear operator A is of class C^1 and its Gâteaux derivative is given by

$$\langle A'(u), v\rangle = \int_{\mathbb{R}^N} \phi(x, |\nabla u|)\nabla u \nabla v dx, \quad \text{for all } u, v \in W^{1,p(x)}(\mathbb{R}^N).$$

Let us now assume that hypotheses (H1)–(H3) are fulfilled. By [13, Lemma 3.4], the operator $A : W^{1,p(x)}(\mathbb{R}^N) \to W^{1,p'(x)}(\mathbb{R}^N)$ is strictly monotone and a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle A'(u_n) - A'(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$ as $n \to \infty$.

The nonhomogeneous differential operator $\operatorname{div}(\phi(x, |\nabla u|)\nabla u)$, where ϕ satisfies (H1)–(H3) was introduced in [13]. This operator generalizes the usual operators with variable exponent. For instance, if $\phi(x, \xi) = \xi^{p(x)-2}$ then we obtain the standard $p(x)$-Laplace operator, that is, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. The new abstract setting includes the case $\phi(x, \xi) = (1 + |\xi|^2)^{(p(x)-2)/2}$, which corresponds to the generalized mean curvature operator

$$\operatorname{div}\left[(1 + |\nabla u|^2)^{(p(x)-2)/2}\nabla u\right].$$

The capillarity equation corresponds to

$$\phi(x, \xi) = \left(1 + \frac{\xi^{p(x)}}{\sqrt{1 + \xi^{2p(x)}}}\right)\xi^{p(x)-2}, \quad x \in \Omega,\ \xi > 0,$$

hence the corresponding capillary phenomenon is described by the differential operator

$$\operatorname{div}\left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right].$$

2. **Main result.** Throughout this paper we shall assume that $p \in C_+(\mathbb{R}^N)$ and p is a radial function, that is, $p(x) = p(|x|)$ for all $x \in \mathbb{R}^N$. Let

$$W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N) := \{u \in W^{1,p(x)}(\mathbb{R}^N);\ u \text{ is radial}\}.$$

We are concerned with the study of the following nonlinear problem

$$-\operatorname{div}(\phi(x, |\nabla u|)\nabla u) + \phi(x, |u|)u = V(x)f(u) \quad \text{in } \mathbb{R}^N, \tag{4}$$

where the potential $V : \mathbb{R}^N \to [0, +\infty)$ and the nonlinearity f satisfy the following hypotheses:

$$V \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ is radial and } \exists\, r_0 > 0 \text{ such that } \inf_{|x|\leq r_0} V(x) > 0, \tag{5}$$

$$f \text{ is odd and } \lim_{u\to 0} f(u)/|u|^{p^+-1} = 0, \tag{6}$$

and

$$\text{there exist } \mu > \tfrac{bp^+}{c} \text{ and } M > 0 \text{ such that } 0 < \mu F(u) \leq uf(u) \text{ for all } |u| \geq M, \tag{7}$$

where $F(u) := \int_0^u f(t)dt$.

In this paper, due to the symmetry assumptions imposed to p and V, we are looking for radial solutions of problem (4).

We say that $u \in W^{1,p(x)}_{\rm rad}(\mathbb{R}^N) \setminus \{0\}$ is a solution of problem (4) if

$$\int_{\mathbb{R}^N} [\phi(x,|\nabla u|)\nabla u \cdot \nabla v + \phi(x,|u|)uv]\, dx = \int_{\mathbb{R}^N} V(x)f(u)vdxdx,$$

for all $v \in W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$.

The energy functional associated to problem (4) is $\mathcal{E} : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{E}(u) := \int_{\mathbb{R}^N} [A_0(x,|\nabla u|) + A_0(x,|u|)]\, dx - \int_{\mathbb{R}^N} V(x)F(u)dx.$$

Let $\mathcal{E}_0$ denote $\mathcal{E}$ restricted to the function space $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$. By the isometric Palais principle [16] (see also [14, Theorem 1.50]), any critical point of $\mathcal{E}_0$ is also a critical point of $\mathcal{E}$. This shows that finding radially symmetric solutions of problem (4) reduces to finding the nontrivial critical points of the energy functional $\mathcal{E}_0$.

Theorem 2.1. *Assume that hypotheses (H1)–(H3), (5), (6), and (7) are fulfilled. Then problem (4) has infinitely many solutions.*

As we shall see in the proof of this result, problem (4) still has at least one (radially symmetric) solution, provided that the oddness symmetry hypothesis on f is removed.

Theorem 2.1 extends the pioneering multiplicity result of A. Ambrosetti and P. Rabinowitz [1, Theorem 3.13] in the following directions:

(i) the standard (linear) second order uniformly elliptic operator $\Sigma_{i,j=1}^N (a_{ij}(x) u_{x_i})_{x_j}$ is replaced by the nonhomogeneous differential operator $\mathrm{div}\,(\phi(x,|\nabla u|)\nabla u)$;

(ii) our study is performed in the entire Euclidean space, instead of a bounded domain with smooth boundary. However, in our abstract setting, the lack of compactness of $\mathbb{R}^N$ is compensated by the compactness of the embedding of the space $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$, provided that $N < p^- \le p^+ < +\infty$.

2.1. **Proof of Theorem 2.1.** We first check that $\mathcal{E}_0$ satisfies the geometric hypotheses of the mountain pass theorem.

Step 1. There exist positive constants ρ and α such that $\mathcal{E}_0(u) \ge \alpha$ for all $u \in W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ with $\|u\| = \rho$.

For $\rho \in (0,1)$ (to be prescribed later), we fix $u \in W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ satisfying $\|u\|_{p(x)} = \rho$. By hypothesis (H3), we have

$$\int_{\mathbb{R}^N} [A_0(x,|\nabla u|) + A_0(x,|u|)]\, dx \ge \frac{c}{p^+} \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + |u|^{p(x)})dx.$$

Next, we use relation (1.8) in [19, p. 11]. Thus, since $\|u\|_{p(x)} = \rho < 1$, we have

$$\int_{\mathbb{R}^N} [A_0(x,|\nabla u|) + A_0(x,|u|)]\, dx \ge \frac{c}{p^+} \|u\|^{p^+}_{p(x)}. \tag{8}$$

On the other hand, assumption (6) implies that $F(u)/|u|^{p^+} \to 0$ as $u \to 0$. Fix $\varepsilon > 0$. It follows that if $\rho > 0$ is small enough then

$$\int_{\mathbb{R}^N} V(x)F(u)dx \le \varepsilon \|V\|_{L^1} \|u\|^{p^+}_{L^\infty}.$$

Since $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ is continuously embedded into $L^\infty(\mathbb{R}^N)$, we deduce that there exists $C>0$ such that for all $u\in W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ with $\|u\|_{p(x)}=\rho$ we have

$$\int_{\mathbb{R}^N} V(x)F(u)\le C\varepsilon\,\|u\|^{p^+}_{p(x)}. \tag{9}$$

Combining relations (8) and (9), we obtain

$$\begin{aligned}\mathcal{E}_0(u)\ &\ge \frac{c}{p^+}\,\|u\|^{p^+}_{p(x)}-C\varepsilon\,\|u\|^{p^+}_{p(x)}\\ &=\tfrac{c}{p^+}\,\rho^{p^+}-C\varepsilon\,\rho^{p^+}.\end{aligned}$$

Choosing $\varepsilon=c/(2Cp^+)>0$, we have

$$\mathcal{E}_0(u)\ge\frac{c}{2p^+}\,\rho^{p^+}=:\alpha>0.$$

Step 2. For all finite-dimensional subspaces $X_0\subset W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ there exists $R=R(X_0)>0$ such that

$$\mathcal{E}_0(u)\le 0\quad\text{for all } u\in X_0\setminus B_R(X_0).$$

We first claim that for all $w\in W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ with $\|w\|_{p(x)}=1$ there exists $\lambda(w)>0$ such that

$$\mathcal{E}_0(\lambda w)<0 \text{ for all } \lambda\in\mathbb{R} \text{ with } |\lambda|\ge\lambda(w). \tag{10}$$

We observe that hypothesis (7) implies that there are positive constants C_1 and C_2 such that

$$f(u)\ge C_1u^{\mu-1}-C_2\quad\text{for all } u\ge 0.$$

Therefore

$$F(u)\ge\frac{C_1}{\mu}\,u^\mu-C_2u\quad\text{for all } u\ge 0.$$

It follows that there exists $C_0>0$ such that

$$F(u)\ge C_0u^\mu\quad\text{for all } |u|\ge M. \tag{11}$$

Fix $w\in W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ with $\|w\|_{p(x)}=1$ and $\lambda\in\mathbb{R}$ (with $|\lambda|>1$).

In order to find an upper estimate for $\mathcal{E}_0(\lambda w)$, we first observe that using hypothesis (H2), we have

$$\begin{aligned}&\int_{\mathbb{R}^N}[A_0(x,|\lambda\nabla w)+A_0(x,|\lambda w|)]dx\le\\ &|\lambda|\int_{\mathbb{R}^N}a(x)(|\nabla w|+|w|)dx+b\int_{\mathbb{R}^N}\left(\int_0^{|\lambda\nabla w|}s^{p(x)-1}ds+\int_0^{|\lambda w|}s^{p(x)-1}ds\right)dx\le\\ &|\lambda|\int_{\mathbb{R}^N}a(x)(|\nabla w|+|w|)dx+\frac{b|\lambda|^{p^+}}{p^-}\int_{\mathbb{R}^N}(|\nabla w|^{p(x)}+|w|^{p(x)})dx=\\ &\frac{b|\lambda|^{p^+}}{p^-}+C|\lambda|,\end{aligned} \tag{12}$$

where $C>0$ is a constant depending only on $|a|_{p'(x)}$ and the best constant of the continuous embedding $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)\hookrightarrow L^{p(x)}(\mathbb{R}^N)$.

By (11) we have

$$\int_{|\lambda w|\ge M}V(x)F(\lambda w)dx\ge C_0\int_{|\lambda w|\ge M}|\lambda|^\mu|w|^\mu dx=C_0|\lambda|^\mu\int_{|\lambda w|\ge M}V(x)|w|^\mu dx. \tag{13}$$

Since F is bounded on the interval $[-M, M]$, there exists $C > 0$ such that $F(t) \geq -C$ for all $t \in [-M, M]$. It follows that

$$\int_{|\lambda w|<M} V(x)F(\lambda w)dx \geq -C \int_{|\lambda w|<M} V(x)dx \geq -C \int_{\mathbb{R}^N} V(x)dx = -C\|V\|_{L^1} . \tag{14}$$

Estimates (12), (13), and (14) imply

$$\mathcal{E}_0(\lambda w) \leq \frac{b}{p^-} |\lambda|^{p^+} + C|\lambda| - C_0|\lambda|^\mu \int_{|\lambda w|\geq M} V(x)|w|^\mu dx + C\|V\|_{L^1},$$

hence $\mathcal{E}_0(\lambda w) \to -\infty$ as $|\lambda| \to \infty$. This proves our claim (10).

Next, since the space $W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^\infty(\mathbb{R}^N)$, we deduce that there exists $C > 0$ such that

$$|w(x)| \leq C \text{ for all } w \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N) \text{ with } \|w\|_{p(x)} = 1.$$

This fact implies that our initial claim (10) can be improved as follows: for all $w \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ with $\|w\|_{p(x)} = 1$ there exist $\lambda(w) > 0$ and $\eta(w) > 0$ such that

$$\mathcal{E}_0(\lambda z) < 0 \quad \forall\, |\lambda| \geq \lambda(w), \forall\, z \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N),\ \|z\|_{p(x)} = 1,\ \|z - w\| \leq \eta(w). \tag{15}$$

Returning to Step 2, let $X_0 \subset W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ be a finite-dimensional subspace. Thus the set $X_0 \cap \{w \in W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N);\ \|w\|_{p(x)} = 1\}$ is compact. Next, using (15), we deduce that there exists $\lambda_0 > 0$ depending only on X_0 such that

$$\mathcal{E}_0(\lambda w) \leq 0 \quad \text{for all } |\lambda| \geq \lambda_0 \text{ and for all } w \in X_0,\ \|w\|_{p(x)} = 1.$$

Choosing $R(X_0) = \lambda_0$, we obtain the statement contained in our Step 2.

Step 3. Any Palais-Smale sequence of $\mathcal{E}_0$ is bounded.

We recall that $(u_n) \subset W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ is a Palais-Smale sequence of $\mathcal{E}_0$ if

$$\mathcal{E}_0(u_n) = O(1) \quad \text{and} \quad \|\mathcal{E}_0'(u_n)\| = o(1) \quad \text{as } n \to \infty. \tag{16}$$

We also recall that $\mathcal{E}_0$ satisfies the Palais-Smale condition if any Palais-Smale sequence $(u_n) \subset W^{1,p(x)}_{\text{rad}}(\mathbb{R}^N)$ of $\mathcal{E}_0$ is relatively compact.

Arguing by contradiction and normalizing, we assume that (up to a subsequence) $u_n = \lambda_n v_n$, where $\lambda_n = \|u_n\|_{p(x)} \to \infty$ and $\|v_n\|_{p(x)} = 1$. By (16) we deduce that

$$\begin{aligned} o(\|u_n\|) &= \langle \mathcal{E}_0'(u_n), u_n\rangle = \int_{\mathbb{R}^N} [\phi(x, |\nabla u_n|)|\nabla u_n|^2 + \phi(x, |u_n|)u_n^2]dx \\ &\quad - \int_{[\lambda_n|v_n|\geq M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx - \int_{[\lambda_n|v_n|<M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx. \end{aligned} \tag{17}$$

Using hypothesis (H2), relation (17) yields

$$\begin{aligned} \lambda_n o(1) &\leq \lambda_n \int_{\mathbb{R}^N} a(x)(|\nabla v_n| + |v_n|)dx + b \int_{\mathbb{R}^N} \lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx \\ &\quad - \int_{[\lambda_n|v_n|\geq M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx - \int_{[\lambda_n|v_n|<M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx. \end{aligned}$$

It follows that

$$\int_{[\lambda_n|v_n|\geq M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx \leq \lambda_n \int_{\mathbb{R}^N} a(x)(|\nabla v_n| + |v_n|)dx +$$
$$b\int_{\mathbb{R}^N} \lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx - \int_{[\lambda_n|v_n|<M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx + \lambda_n o(1). \tag{18}$$

By hypothesis (7), we deduce that

$$\int_{[\lambda_n|v_n|\geq M]} V(x)F(\lambda_n v_n)dx \leq \frac{1}{\mu}\int_{[\lambda_n|v_n|\geq M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx. \tag{19}$$

Using hypothesis (H3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} [A_0(x,|\nabla u_n|) + A_0(x,|u_n|)]\,dx = \\
&\int_{\mathbb{R}^N}\left(\int_0^{|\nabla u_n|} s\phi(x,s)ds + \int_0^{|u_n|} s\phi(x,s)ds\right)dx \geq \\
&c\int_{\mathbb{R}^N}\frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx = c\int_{\mathbb{R}^N}\frac{\lambda_n^{p(x)}}{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx \geq \\
&\frac{c}{p^+}\int_{\mathbb{R}^N}\lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx.
\end{aligned} \tag{20}$$

Combining relations (19) and (20) in relationship with (16), we deduce

$$\begin{aligned}
O(1) \quad &= \mathcal{E}_0(u_n) \geq \frac{c}{p^+}\int_{\mathbb{R}^N}\lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx \\
&-\frac{1}{\mu}\int_{[\lambda_n|v_n|\geq M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx - \int_{[\lambda_n|v_n|<M]} V(x)F(\lambda_n v_n)dx.
\end{aligned}$$

Using now the estimate established in (18), we obtain

$$\begin{aligned}
&O(1) \geq \frac{c}{p^+}\int_{\mathbb{R}^N}\lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx - \frac{\lambda_n}{\mu}\int_{\mathbb{R}^N} a(x)(|\nabla v_n| + |v_n|)dx \\
&-\frac{b}{\mu}\int_{\mathbb{R}^N}\lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx \\
&+\frac{1}{\mu}\int_{[\lambda_n|v_n|<M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx - \int_{[\lambda_n|v_n|<M]} V(x)F(\lambda_n v_n)dx + \lambda_n o(1) \\
&= \left(\frac{c}{p^+} - \frac{b}{\mu}\right)\int_{\mathbb{R}^N}\lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx - \frac{\lambda_n}{\mu}\int_{\mathbb{R}^N} a(x)(|\nabla v_n| + |v_n|)dx \\
&+\frac{1}{\mu}\int_{[\lambda_n|v_n|<M]} V(x)f(\lambda_n v_n)\lambda_n v_n dx - \int_{[\lambda_n|v_n|<M]} V(x)F(\lambda_n v_n)dx + \lambda_n o(1).
\end{aligned}$$

Therefore

$$\begin{aligned}
O(1) \quad &= \mathcal{E}_0(u_n) \geq \left(\frac{c}{p^+} - \frac{b}{\mu}\right)\int_{\mathbb{R}^N}\lambda_n^{p(x)}(|\nabla v_n|^{p(x)} + |v_n|^{p(x)})dx \\
&-\frac{\lambda_n}{\mu}\int_{\mathbb{R}^N} a(x)(|\nabla v_n| + |v_n|)dx + \lambda_n o(1).
\end{aligned} \tag{21}$$

Combining hypothesis (7) with relation (21) and the fact that $\lim_{n\to\infty}\lambda_n = +\infty$, we deduce that $\mathcal{E}_0(u_n) \to +\infty$ as $n \to \infty$, which contradicts (16). We conclude that the Palais-Smale sequence (u_n) is bounded in $W^{1,p(x)}_{\mathrm{rad}}(\mathbb{R}^N)$. We shall prove that in fact, the sequence converges strongly (up to a subsequence) in $W^{1,p(x)}_{\mathrm{rad}}(\mathbb{R}^N)$.

For this purpose, we apply some ideas developed by R. Filippucci, P. Pucci and V. Rădulescu [10, pp. 712-713].

Step 4. Any Palais-Smale sequence of $\mathcal{E}_0$ is relatively compact in $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$.

Let $(u_n) \subset W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$ be an arbitrary Palais-Smale sequence of $\mathcal{E}_0$. By Step 3, (u_n) is bounded. Using the compact embedding $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we can assume that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } W^{1,p(x)}_{\rm rad}(\mathbb{R}^N) \tag{22}$$

$$u_n \to u \quad \text{in } L^\infty(\mathbb{R}^N). \tag{23}$$

By hypotheses (H2) and (H3) in conjunction with relations (22) and (23), we deduce that

$$\int_{\mathbb{R}^N} \phi(x,|u_n|)u_n(u_n-u)dx - \int_{\mathbb{R}^N} \phi(x,|u|)u(u_n-u)dx \to 0 \quad \text{as } n\to\infty \tag{24}$$

and

$$\int_{\mathbb{R}^N} \phi(x,|\nabla u_n|)\nabla u_n\cdot\nabla(u_n-u)dx - \int_{\mathbb{R}^N} \phi(x,|\nabla u|)\nabla u\cdot\nabla(u_n-u)dx \to 0 \quad \text{as } n\to\infty. \tag{25}$$

Recall that Proposition 3.3 in [13] established that under hypotheses (H1) and (H3) the following Simon-type inequality holds:

$$\phi(x,|u|)u(u-v) - \phi(x,|v|)v(u-v) \geq 4^{1-p^+}c\,|u-v|^{p(x)} \quad \text{for all } x\in\Omega.$$

Using this inequality, relations (24) and (25) imply that

$$u_n \to u \quad \text{in } W^{1,p(x)}_{\rm rad}(\mathbb{R}^N).$$

We conclude that any Palais-Smale sequence of $\mathcal{E}_0$ is relatively compact in $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$, hence $\mathcal{E}_0$ satisfies the Palais-Smale condition.

Next, we observe that $\mathcal{E}_0$ is even and $\mathcal{E}_0(0)=0$. The above steps show that $\mathcal{E}_0$ fulfills the hypotheses of Theorem 1.1. We deduce that problem (4) has infinitely many solutions in $W^{1,p(x)}_{\rm rad}(\mathbb{R}^N)$. □

Remarks and perspectives. 1. The methods developed in this paper can be applied to other recent classes of nonhomogeneous differential operators. For instance, the operator $\operatorname{div}(\phi(x,|\nabla u|)\nabla u)$ that describes problem (4) can be replaced with $\operatorname{div}[\phi'(|\nabla u|^2)\nabla u]$, where for some $1<p<q<N$, the function $\phi\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ satisfies the following conditions:

(ϕ_1) $\phi(0)=0$;

(ϕ_2) there exists $c_1>0$ such that $\phi(t)\geq c_1 t^{p/2}$ if $t\geq 1$ and $\phi(t)\geq c_1 t^{q/2}$ if $0\leq t\leq 1$;

(ϕ_3) there exists $c_2>0$ such that $\phi(t)\leq c_2 t^{p/2}$ if $t\geq 1$ and $\phi(t)\leq c_2 t^{q/2}$ if $0\leq t\leq 1$;

(ϕ_4) there exists $0<\mu<1/s$ such that $2t\phi'(t)\leq s\mu\phi(t)$ for all $t\geq 0$;

(ϕ_5) the mapping $t\mapsto\phi(t^2)$ is strictly convex.

This operator was introduced by A. Azzollini, P. d'Avenia, and A. Pomponio [3] and it is described by a potential with different growth near zero and at infinity (the double-power growth hypotheses). We refer to A. Azzollini [2] and N. Chorfi and V. Rădulescu [7] for recent contributions in connection with the abstract setting generated by this operator.

2. In a general framework, the presence of two variable exponents $p_1(x)$ and $p_2(x)$ dictates the geometry of a composite that changes its hardening exponent according to the point. Problems with nonstandard growth conditions of (p,q)-type have been initially studied by P. Marcellini [15] who was interested in the properties of the integral energy functional $\int_{\mathbb{R}^N} F(x,\nabla u)dx$, where $F:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ satisfies unbalanced polynomial growth conditions, namely

$$|\xi|^p \lesssim F(x,\xi) \lesssim |\xi|^q \quad \text{with } 1<p<q.$$

We believe that the main result of this paper can be extended to "unbalanced" anisotropic differential operators of the type

$$-\mathrm{div}\,(\phi(x,|\nabla u|)\nabla u) - \mathrm{div}\,(a(x)\psi(x,|\nabla u|)\nabla u)$$

and

$$-\mathrm{div}\,(\phi(x,|\nabla u|)\nabla u) - \mathrm{div}\,(a(x)\psi(x,|\nabla u|)\log(e+|x|)\nabla u).$$

This abstract setting is in close relationship with the recent contributions of G. Mingione *et al.* [5, 9], who studied non-autonomous problems with associated energies of the type

$$\int_{\mathbb{R}^N} [|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)}]dx$$

and

$$\int_{\mathbb{R}^N} [|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)}\log(e+|x|)]dx,$$

where $p_1(x)\le p_2(x)$, $p_1\ne p_2$, and $a(x)\ge 0$.

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