THE HISTORY OF THE RECOGNITION PROBLEM FOR MANIFOLDS

Dušan Repovš

Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P. O. Box 2964, Jadranska 19, 1001 Ljubljana, Slovenia

e-mail: dusan.repovs@uni-lj.si

The definition of a *topological n-manifold* M ($n \in \mathbb{N}$) requires, besides separability and metrizability, that every point $x \in M$ must possess a neighborhood $U \subset M$ which is *homeomorphic* to $\mathbb{R}^n$. (We shall only consider *closed* topological manifolds M, i.e. M is connected, compact and $\partial M = \emptyset$.) However, in practice, the verification of the existence (or nonexistence) of such homeomorphism $h : U \to \mathbb{R}^n$ is a problem. So is it possible to find a characterization of topological manifolds which does not mention homeomorphisms, but is reasonably simple to state and not too difficult to verify? This is the so-called *Recognition problem for topological manifolds.* In this paper we survey the history of this problem. For previous surveys see [13],[26] and[34]-[36].

Topological manifolds of dimensions 1 and 2 have very simple characterizations: S^1 is the only compact, connected metric space containing at least 2 points, which is separated by every pair of its points [28], and S^2 is the only nondegenerate locally connected, connected, compact metric space which is separated by no pair of its points but is separated by each of its simple closed curves [3]. In which *class of topological spaces* do we want to detect higher-dimensional topological manifolds? The most appropriate seems to be the class of so-called *generalized manifolds.* The main *difference* between topological n-manifolds and generalized n-manifolds is that the latter may fail to possesses sufficient *general position properties.*

Generalized manifolds were first introduced into topology in the 1930's – one of the major motivations was the discovery that they were the proper framework to generalize classical theorems of the *Jordan-Schoenflies* type from dimension 2 to higher dimensions (since the examples like the *Alexander horned sphere* [1] makes a direct generalization impossible). Since then they have played an important role in various parts of topology, e.g. theory of transformation groups [4], theory of cell-like decompositions of manifolds

[17], taming theory [12], suspensions of homology spheres [14], compactifications of open topological manifolds [8], manifold factors [15], etc.

Definition 1. *A locally compact Hausdorff space X is said to be a* generalized n-manifold *($n \in \mathbb{N}$) if X satisfies the following properties: (i) X is an* Euclidean neighborhood retract (ENR), i.e. *for some integer m, X embeds in $\mathbb{R}^m$ as a retract of an open subset of $\mathbb{R}^m$ (equivalently, X is a locally compact, finite-dimensional separable, metrizable ANR); and (ii) X is a* $\mathbb{Z}$-homology n-manifold, i.e. *for every point $x \in X$, $H_*(X, X \setminus \{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$.*

Let X be a generalized n-manifold. If $n \leq 2$ then it follows by classical results that X is a topological n-manifold. On the other hand, if $n \geq 3$ then X need not be a genuine n-manifold anymore, in fact, it may fail to possess Euclidean n-dimensional neighborhoods at all points $x \in X$. Such points are called *singularities* of X and they form the *singular set* $S(X)$ of X, i.e. $S(X) = \{x \in X \mid x$ does not have any neighborhood in X homeomorphic to $\mathbb{R}^n\}$. Its complement, $M(X) = X \setminus S(X)$, is called the *manifold set* of X and, if $S(X) \neq X$, it is clearly an open n-manifold. (For many *totally singular* generalized manifolds X, i.e. $S(X) = X$, the singularities completely vanish upon multiplication of X by the real line, i.e. $X \times \mathbb{R}$ is a genuine manifold – see [15].)

A *resolution* of an n-dimensional ANR X is a proper, cell-like map $f : M \to X$ from a topological n-manifold M onto X. It follows by classical results that if X admits a resolution, X *must* be a generalized n-manifold [24]. *Cell-like* maps were introduced in [24] and are connected with *cellularity* [9]. They are defined as those maps $f : M \to X$ whose point-preimages $f^{-1}(x)$ are *cell-like sets*, i.e. continua with the (Borsuk) shape of a point [5]: $\mathrm{Sh}(f^{-1}(x)) = \mathrm{Sh}(pt)$. Cell-like maps play an important role in topology and they have been significantly applied in solutions of several very difficult problems, e.g. the 4-dimensional Poincaré Conjecture. They also play a key role in the Recognition problem [27] and [34]-[36].

The following is the first of the two key problems – the *Resolution problem: Does every generalized manifold have a resolution?* Given the resolution $f : M \to X$, one considers the associated cell-like, upper semicontinuous decomposition $G_f = \{f^{-1}(x) | x \in X\}$ of M, consisting of the preimages of the map f, and tries to establish some *general position properties* of X which would allow the controlled, simultaneous shrinking of the elements of the decomposition G_f to arbitrary small sizes. If such a manipulation can be carried out then the classical Bing Shrinking theorem [25] tells us that f is a *near-homeomorphism*, i.e. f can be approximated arbitrarily closely by homeomorphisms $h : M \to X$. The best result so far in dimensions ≥ 5 is the following Resolution theorem (see the survey [35]):

Theorem 2. (F. S. Quinn [31]–[33]) *Let X be a connected generalized n-manifold, $n \geq 5$. Then there is an integral invariant $I(X) \in H_0(X;\mathbb{Z})$ of X such that: (i) $I(X) \equiv 1 \pmod 8$; (ii) For every open subset $U \subset X$, $I(X) = I(U)$; (iii) For every generalized m-manifold Y, $m \geq 5$, $I(X \times Y) = I(X) \times I(Y)$; and (iv) $I(X) = 1$ if and only if X admits a resolution.*

Quinn's local surgery obstruction $I(X)$ can be *nontrivial* (see [22] and [29] where the part missing from [10] is provided):

Theorem 3. (J. L. Bryant, S. C. Ferry, W. Mio and S. Weinberger [10]) *For every integers $n \geq 6$ and $m \geq 1$, and for every simply connected, closed n-manifold M, there exists a generalized n-manifold X such that: (i) $I(X) = m$ (hence X does not admit a resolution and is totally singular); and (ii) X is homotopy equivalent to M.*

Theorem 4. (J. L. Bryant, S. C. Ferry, W. Mio and S. Weinberger [10]) *For every integer $n \geq 6$, there exists a generalized n-manifold X such that: (i) X does not admit a resolution; and (ii) X is not homotopy equivalent to any topological manifold.*

Essentially nothing is known in dimension 4, except for the fact that a generalized 4-manifold X has a resolution if and only if $X \times \mathbb{R}$ has one. This follows from the following result:

Theorem 5. (F. S. Quinn [30]) *Let X be a generalized n-manifold $(n \geq 4)$. Then the following statements are equivalent: (i) X has a resolution; (ii) For some $k \in \mathbb{N}$, $X \times \mathbb{R}^k$ has a resolution; and (iii) $X \times \mathbb{R}^2$ is a topological $(n+2)$-manifold.*

In dimension 3, the Resolution problem is entangled with the Poincaré conjecture, e.g. if there exist *fake* 3-cells it's easy to construct a *nonresolvable* generalized 3-manifold X, homotopy equivalent to S^3, with just one singularity (see [11] and [34]). The following is the current status (although there has been some progress announced – see [20] and [40]):

Theorem 6. (T. L. Thickstun [39]) *If the Poincaré conjecture is true then every generalized 3-manifold X with $\dim S(X) = 0$ admits a (conservative) resolution.*

The Resolution problem remains open (modulo the Poincaré conjecture) for generalized 3-manifolds X with $\dim S(X) \geq 1$: *Suppose that there exist no fake cubes. Does there exist a nonresolvable generalized 3-manifold X?* Note that in order to resolve a generalized 3-manifold X it suffices to find an almost $\mathbb{Z}_2$-acyclic resolution of X:

Theorem 7. (D. Repovš and R. C. Lacher [38]) *Let $f : M \to X$ be a closed, monotone map from a 3-manifold M onto a locally simply connected*

$\mathbb{Z}_2$-homology 3-manifold X. Suppose that there is a 0-dimensional (possibly dense) set $Z \subset X$ such that for every point $x \in X \setminus Z$, $\check{H}^1(f^{-1}(x);\mathbb{Z}_2) = 0$. Then the set $C = \{x \in X | f^{-1}(x)$ is not cell-like$\}$ is locally finite in X. Moreover, X is a resolvable generalized 3-manifold.

The second key problem is the *General position problem*: *Which general position property for a finite-dimensional ANR X, where X is the image of a cell-like map $f : M \to X$ on an n-manifold M, implies that f is a near-homeomorphism?* Higher dimensional (≥ 5) topological manifolds possess the following general position property:

Definition 8. *A metric space X is said to have the* disjoint disks property (DDP) *if for every pair of maps $f, g : B^2 \to X$ of the closed 2-cell B^2 into X and every $\varepsilon > 0$ there exist maps $f', g' : B^2 \to X$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$ and $f'(B^2) \cap g'(B^2) = \emptyset$.*

This property is also *characteristic* for manifolds in this dimension range:

Theorem 9. (R. D. Edwards [21]) *Let M be a topological n-manifold, $n \geq 5$, and let $f : M \to X$ be a surjective cell-like map of M onto a finite-dimensional ANR X. Then X is a topological n-manifold if and only if X has the DDP.*

As a corollary we immediately get the solution of the Recognition problem for dimensions ≥ 5: *For every $n \geq 5$, the class of topological n-manifolds $\mathcal{M}_n$ is equal to the class of generalized n-manifolds $\mathcal{G}_n$ with the DDP and vanishing Quinn's local surgery obstruction $I(X)$.*

Let X be any generalized n-manifold, $n \geq 6$, which doesn't admit a resolution. Then by Quinn's Theorem, the product $X \times T^2$ is also a generalized $(n+2)$-manifold without a resolution. However, by [16], $X \times T^2$ has the DDP, so [10] implies that there exist generalized m-manifolds, $m \geq 8$, which are *not* topological m-manifolds *although* they do possess the DDP.

In dimension 3 the appropriate versions of DDP for 3-manifolds was introduced in [18] and [19]: Recall that a subset $Z \subset X$ of space X is *locally simply co-connected* (1- LCC) if every $x \in X$ and every neighborhood $U \subset X$ of x, there is a neighborhood $V \subset U$ of x such that the inclusion-induced homomorphism $\Pi_1(V \setminus Z) \to \Pi_1(U \setminus Z)$ is trivial.

Definition 10. *A metric space X is said to have the* Spherical simplicial approximation property *(SSAP) if for every map $f : S^2 \to X$ and every $\varepsilon > 0$, there exist a map $f' : S^2 \to X$ and a finite topological 2-complex $K_{f'} \subset X$ such that (i) For every $t \in S^2$, $d(f(t), f'(t)) < \varepsilon$; (ii) $f'(S^2) \subset K_{f'}$; and (iii) $X \setminus K_{f'}$ is 1-FLG (free local fundamental group) in X.*

We define that $X \setminus K_{f'}$ is 1-FLG in X if for every $y \in K_{f'}$ and for every sufficiently small neighborhood $U \subset X$ of y, there exists another neighborhood $V \subset U$ of y, such that for every connected open neighborhood $W \subset V$ of y, for each nonempty component $W' \subset W$ of $W \setminus K_{f'}$, the inclusion-induced image of $\Pi_1(W') \to \Pi_1(U')$ is a free group on $m-1$ generators, where $U' \subset U$ is the component of $U \setminus K_{f'}$ containing W' and m is the number of components of $\mathrm{st}(y) \setminus y$ that meet $\mathrm{Cl}(W')$. Note that for any finite, connected 2-complex K, having no local separating points and lying in a generalized 3-manifold X, the following are equivalent: (i) $X \setminus K$ is 1-FLG in X; (ii) K is 1-LCC in X; and (iii) Each 2-simplex of K is 1-LCC in X (cf. [19]). It is easy to see that every topological (≥ 3)-manifold has SSAP. The main result from [19] which solves the General position problem for 3-manifolds is:

Theorem 11. (R. J. Daverman and D. Repovš [19]) *A resolvable generalized 3-manifold is a topological 3-manifold if and only if it possesses the SSAP.*

The *Dehn's lemma property* and the *Map separation property* are another kind of general position properties of 3-manifolds which were used earlier to shrink certain cell-like decompositions [37]:

Theorem 12. (W. Jakobsche and D. Repovš [23]) *Suppose that there exist fake cubes. Then there exists a compact homogeneous ANR X with the following properties: (i) X is a generalized 3-manifold and $S(X) = X$; (ii) X does not admit a resolution; (iii)X has the Dehn's lemma property; (iv) X has the Map separation property; (v)$X \times S^1$ is homeomorphic to $S^3 \times S^1$.*

The following interesting question arises (see also [6] and [7]): *Does the example from [23] also possess any of the following position properties: (i) LMSP(*); or (ii) (W)SAP; or (iii) SSAP?* In dimension 4 very little is known (see [2] and [19] for partial results) both Resolution problem as well as General position problem are still open, while in dimensions ≥ 5 there also remain some questions, e.g. *Does there exist a nonresolvable generalized 5-manifold?*

Finally, the following is a related, very difficult problem from *cohomological dimension theory*, equivalent to the celebrated Cell-like mapping problem in dimension 4 (for more see the survey [26]): *Suppose that $f : M \to X$ is a cell-like map of a topological 4-manifold M onto a space X. Is* $\dim X < \infty$ *(equivalently,* $\dim X{=}4$*)?* Note that by theorem of W. J. R. Mitchell, D. Repovš and E. V. Ščepin [27], $\dim X < \infty$ if and only if X has a certain kind of general position property, called the *disjoint Pontryagin triples property.*

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