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ON CALCULATION OF THE WITTEN INVARIANTS OF 3-MANIFOLDS

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Abstract

In this paper we present a short definition of the Witten invariants of 3-manifolds. We also give simple proofs of invariance of those obtained for $r = 3$ and $r = 4$. Our definition is extracted from the 1993 paper of Lickorish and the Prasolov-Sossinsky book, where it is dispersed over 20 pages. We show by several examples that it is indeed convenient for calculations.

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1. Definition of the Witten invariant

The construction of Witten invariants of 3-manifolds and the proof of their invariance use deep ideas from the quantum field theory and the theory of Temperley-Lieb algebras and are not short. But a mathematician might want to calculate and apply these invariants without necessarily understanding their origin. The definition of the Witten invariants in [6, page 660] is direct and short, but is not so convenient for calculations. In this paper we present a short definition of the Witten invariants (Theorem 1.3) which is extracted from [8] (where it is dispersed over 20 pages, mixed with the proof of invariance) and we show by several examples that it is indeed more convenient for calculations. In Section 2 we give a new simple proof of invariance for $r = 4$.

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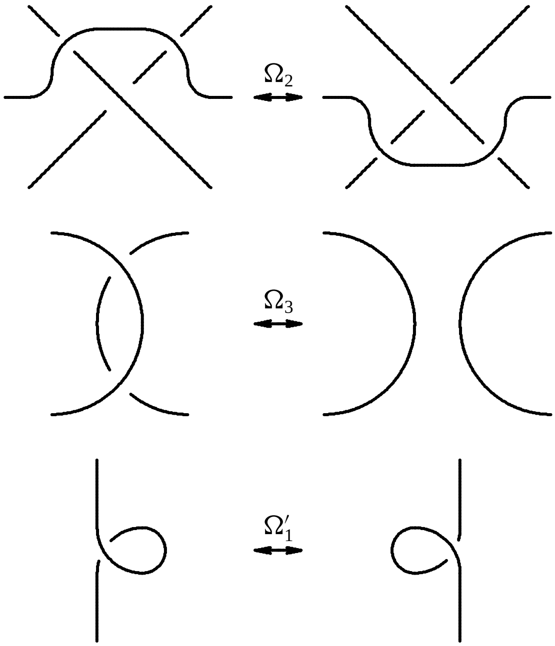

FIGURE 1.

The definition of the Witten invariant is based on the representation of 3-manifolds by (unoriented) plane diagrams. By a *plane diagram* we understand a set of circles in $\mathbb{R}^2$ in general position, with chosen undercrossing and overcrossing at each intersection point. For every single component D_k of the plane diagram D we can determine its integer framing as follows. Choose any orientation of D_k. Define the framing as the sum of the signs (± 1) of all of its crossings. Note that this number is independent of the choice of orientation on D_k.

Suppose that L is an unoriented link in S^3 and that an integer $g(k)$ is assigned to each component L_k of L. Then the pair (L, g) is called a *framed link*. We say that a framed link (L, g) is represented by a plane diagram D, if D is a diagram for L in the usual sense and $g(k)$ equals the framing of D_k, for every single component D_k of D.

It is well known that every closed oriented 3-manifold can be obtained from the 3-sphere S^3 by the Dehn surgery on some framed link (L, g). Denote by χ_D the 3-manifold obtained by the Dehn surgery along the framed unoriented link, corresponding to D.

PROPOSITION 1.1 ([3, 1]). *Suppose that D and D' are plane diagrams. Then $\chi_D \cong \chi_{D'}$ if and only if D' can be obtained from D by a sequence of the Reidemeister moves Ω_1', Ω_2, and Ω_3 shown in Figure 1 and the Fenn-Rourke moves shown in Figures 4* (a)–(b).

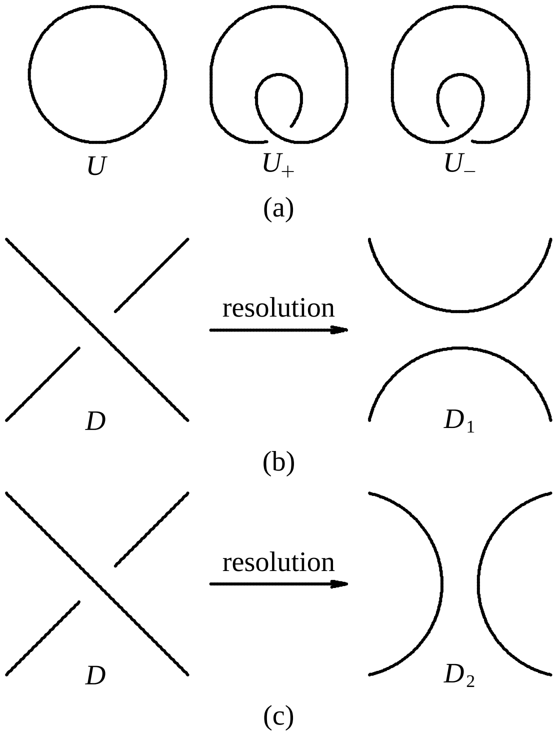

FIGURE 2.

For a plane diagram $D=(D_1,\ldots,D_n)$, consider any oriented link $L=(L_1,\ldots,L_n)$ in S^3, whose plane projection coincides with D. Let $b_{pq} = \mathrm{lk}(L_p, L_q)$ for $p \neq q$ and let b_{kk} equal the framing of D_k. Denote by $b_+(D)$ and $b_-(D)$ the numbers of positive and negative eigenvalues of the linking coefficients matrix (b_{pq}) of L. Let $\sigma(D) = b_+(D) - b_-(D)$ be the signature of (b_{pq}) and $D \cdot D = \sum_{p,q} b_{pq} \pmod 4$. Clearly, the above numbers depend only on D, not on L and its orientation. We set $\sigma = \sigma(D)$ and $b_\pm = b_\pm(D)$ when D is fixed and no confusion can arise. Let $|D|$ be the number of components in D. Then $\mathrm{rk}\, H_1(\chi_D, \mathbb{Z}) = |D| - b_+(D) - b_-(D)$. Denote by $\#D$ the number of crossings in D. Let $|D|_+$ and $|D|_-$ be the numbers of the connected components after resolution of all the crossings as shown in Figures 2 (b) and (c), respectively.

In what follows capital Latin letters denote (unoriented) plane diagrams (in [8] they are sometimes called *framed diagrams*). Let U_+, U and U_- be the diagrams representing the unknot with framings $+1$,0 and -1, respectively (see Figure 2 (a)).

Everywhere below we suppose that diagrams in the equalities coincide except where shown in corresponding figures.

The *Kauffman bracket* is a function $\langle\cdot\rangle$: {plane diagrams} $\to \mathbb{Z}[a^{\pm 1}]$, defined by the following three properties (see for example [8, Section 26, (1)–(3)]):

(a) $\langle D\rangle = a\langle D_1\rangle + a^{-1}\langle D_2\rangle$, where the diagrams D, D_1 and D_2 are shown in Figures 2 (b)–(c);
(b) $\langle D \sqcup U\rangle = (-a^2 - a^{-2})\langle D\rangle$; and
(c) $\langle\emptyset\rangle = 1$.

The normalization of (c) is not entirely standard, but in this paper it is more convenient to use $\langle D\rangle$ instead of the *original Kauffman bracket* $\langle D\rangle/(-a^2 - a^{-2})$.

PROPOSITION 1.2 ([2, 5, 8, Section 26]). *The Kauffman bracket is unchanged by the Reidemeister moves* Ω_1', Ω_2, *and* Ω_3.

THEOREM 1.3 ([8, 7, 27.3, 28.2] cf. [6]). *Fix integers* $r \geq 3$ *and* $k = 1, \ldots, 4r-1$ *relatively prime to* $2r$. *Define the polynomial*

$$\omega(\alpha) = \prod_{\substack{l=1 \\ k\pm l\neq r,3r}}^{r-1} \left(\alpha - 2\cos\frac{\pi l}{r}\right).$$

For a plane diagram D *with* $n = |D|$ *components, let* $D^{(k_1,\ldots,k_n)}$ *be the diagram obtained from* D *by taking* k_i *curves, close and parallel to the* i*-th component. Define a polylinear map* $f_D : (\mathbb{C}[\alpha])^n \to \mathbb{C}$ *on the basic elements by setting* $f_D(\alpha^{k_1}, \ldots, \alpha^{k_n}) = \langle D^{(k_1,\ldots,k_n)}\rangle$ *at* $a = \exp(\pi i k/2r)$. *Then the following number* (*the Witten invariant for* r *at* a) *depends only on the oriented* χ_D:

$$W(D) = f_{U_+}^{-b_+(D)}(\omega) \cdot f_{U_-}^{-b_-(D)}(\omega) \cdot f_D(\omega, \ldots, \omega).$$

REMARK 1.4. It follows from [Lic93, Lemma 4] or [PrSo97, Proposition 29.4] that $f_{U_\pm}(\omega) \neq 0$. For $r = 3$ and $r = 4$, we easily verify it below.

REMARK 1.5. It is easier to calculate the polynomial ω not by the explicit formula of Theorem 1.3 but by the following algorithm. Define the (renormalized Chebyshev) polynomials $S_n(\alpha)$ by the recurrence formula $S_0 = 1$, $S_1 = \alpha$ and $S_{n+1} = \alpha S_n - S_{n-1}$. Then

$$\omega = (-1)^{r-k+1} \sum_{n=0}^{r-2} (-1)^n \frac{\sin(\pi k(n+1)/r)}{\sin(\pi k/r)} S_n.$$

Indeed, it suffices to show that the above sum has exactly $r-2$ roots $2\cos(\pi l/r)$, where $1 \leq l \leq r-1$ and $k \pm l \neq r, 3r$ (there are exactly $r-2$ numbers l with these properties). Note that $\sin x \cdot S_n(2\cos x) = \sin(n+1)x$. Then

$$\begin{aligned}
&2\sin(\pi k/r)\sin(\pi l/r)\omega(2\cos(\pi l/r))\\
&\quad = 2\sum_{n=0}^{r-2}(-1)^n \sin(\pi k(n+1)/r)\sin(\pi l(n+1)/r)\\
&\quad = \sum_{n=1}^{r-1}(-1)^{n+1}\cos(\pi(k+l)n/r) - \sum_{n=1}^{r-1}(-1)^{n+1}\cos(\pi(k-l)n/r)\\
&\quad = (-1)^{r+k+l} - (-1)^{r+k-l} = 0.
\end{aligned}$$

REMARK 1.6. For odd r in Theorem 1.3, one can also take $k = 1, \ldots, 2r-1$ relatively prime to $2r$, $a = e^{\pi i k/r}$ and

$$\omega(\alpha) = \prod_{\substack{l=1\\ 2k\pm l\neq r,3r}}^{r-1}\left(\alpha - 2\cos\frac{\pi l}{r}\right).$$

EXAMPLE 1.7. $W(S^3) = W(U_\pm) = 1$.

EXAMPLE 1.8. It follows from [9, 3.4] that a changing of the orientation of 3-manifold has the effect of complex conjugation on the Witten invariants.

EXAMPLE 1.9. For $a = e^{\pi i/3}$, we have $\langle D\rangle = 1$. This can be verified by induction on the number of crossings in D using the definition of the Kauffman bracket.

EXAMPLE 1.10. Suppose $r = 3$, $k = 1$ and $a = e^{\pi i/3}$. Then $\omega = 1+\alpha$ (see Remark 1.5) and by Example 1.9, $\langle D\rangle = 1$. Hence

$$W(D) = 2^{-b_+}\cdot 2^{-b_-}\sum_{P\subset D} 1 = 2^{|D|-b_+-b_-} = 2^{\operatorname{rk} H_1(\chi_D)}.$$

REMARK 1.11. Observe that if ω is replaced in Theorem 1.3 throughout by $\mu\omega$, where μ is a constant complex number, then another invariant is obtained. The new invariant is the old one multiplied by $\mu^{\operatorname{rk} H_1(\chi_D,\mathbb{Z})}$. Choose $\mu \in \mathbb{C}$ so that $\mu^{-2} = f_{U_+}(\omega)\cdot f_{U_-}(\omega)$. This means that $f_{U_+}(\mu\omega)^{-1} = f_{U_-}(\mu\omega)$. So we obtain the Witten invariant $R(D) = f_D(\mu\omega, \mu\omega, \ldots, \mu\omega) f_{U_-}(\mu\omega)^\sigma$.

LEMMA 1.12. *For the Kauffman bracket at $a = e^{\pi i/6}$, we have*

$$\langle D\rangle = (-1)^{|D|_+}\cdot i^{\#D} = (-1)^{|D|_-}\cdot i^{-\#D} = i^{2|D|-D\cdot D}.$$

PROOF. First we prove that

$$(*)\qquad \langle D\rangle = i\langle D_1\rangle = -i\langle D_2\rangle,$$

where the diagrams D, D_1 and D_2 differ as shown in Figures 2 (b)–(c). This can be verified by induction on $\#D$. It follows from (a) that we must only prove that $\langle D_1\rangle = -\langle D_2\rangle$. The base $\#D = 0, 1$ is easy. If $\#D \geq 2$, then D_1 and D_2 have a crossing point. The induction hypothesis then gives $\langle D_1\rangle = i\langle D_{11}\rangle = -i\langle D_{21}\rangle = -\langle D_2\rangle$, where the diagrams D_i and D_{i1} are identical except where shown in Figure 2 (b) ($i = 1, 2$) and ($*$) is proved. From this at once we obtain the first two equalities of Lemma 1.12.

Now we prove that $\langle D\rangle = i^{2|D|-D\cdot D}$. The equality is evident for trivial diagrams D (that is, for diagrams without any crossings). By Proposition 1.2 it also holds for diagrams of the unoriented trivial link. There exists an orientation $\overline{D} = (\overline{D}_1, \ldots, \overline{D}_k)$ of D such that b_{pq} equals the sum of the signs ± 1 of all the crossings where $\overline{D}_p$ overcrosses $\overline{D}_q$. It is well known that D can be obtained from the diagram of the trivial link by changing some overcrossings by undercrossings and reverse operations. Clearly, $i^{-D\cdot D}$ is multiplied by -1 under such operation. It follows from ($*$) that $\langle D\rangle$ is also multiplied by -1 and we are done. □

EXAMPLE 1.13. Suppose $r = 3$, $k = 1$ and $a = e^{\pi i/6}$. Then $\omega = 1 - \alpha$, $f_{U_+}(\omega) = 1 - i$, $f_{U_-}(\omega) = 1 + i$, $\mu = 1/\sqrt{2}$ and $f_{U_-}(\mu\omega) = e^{\pi i/4}$. Hence, by Lemma 1.12 the Witten invariant of Remark 1.11 equals

$$R(D) = 2^{-|D|/2} e^{\pi i\sigma/4} \sum_{P\subset D} (-1)^{|P|} \langle P\rangle = 2^{-|D|/2} e^{\pi i\sigma/4} \sum_{P\subset D} i^{-P\cdot P}.$$

Note that $R(D)$ is obtained from $\tau_3(D)$ of [4, page 521] by complex conjugation.

EXAMPLE 1.14. Let $r = 4$, $k = 1$ and $a = e^{\pi i/8}$. We have $\omega = \alpha^2 - \sqrt{2}\alpha$, $\langle U_+^2\rangle = \langle U_-^2\rangle = 0$, $f_{U_+}(\omega) = -2e^{3\pi i/8}$, $f_{U_-}(\omega) = 2e^{5\pi i/8}$ and $\mu = 1/2$. Therefore, the Witten invariant from Remark 1.11 equals

$$R(D) = (-1)^{|D|} 2^{-|D|/2} e^{5\pi i\sigma/8} \sum_{P\subset D} \left(-\sqrt{2}\right)^{-|P|} \langle D\circ P\rangle,$$

where $D \circ P$ is the diagram obtained from D by drawing circles, parallel and close to the components of P, see for example [4, Section 6].

2. Simple proofs of Theorem 1.3 for $r = 3$ and $r = 4$

We only consider the case $k = 1$. The case of arbitrary k (for given r) is proved analogously.

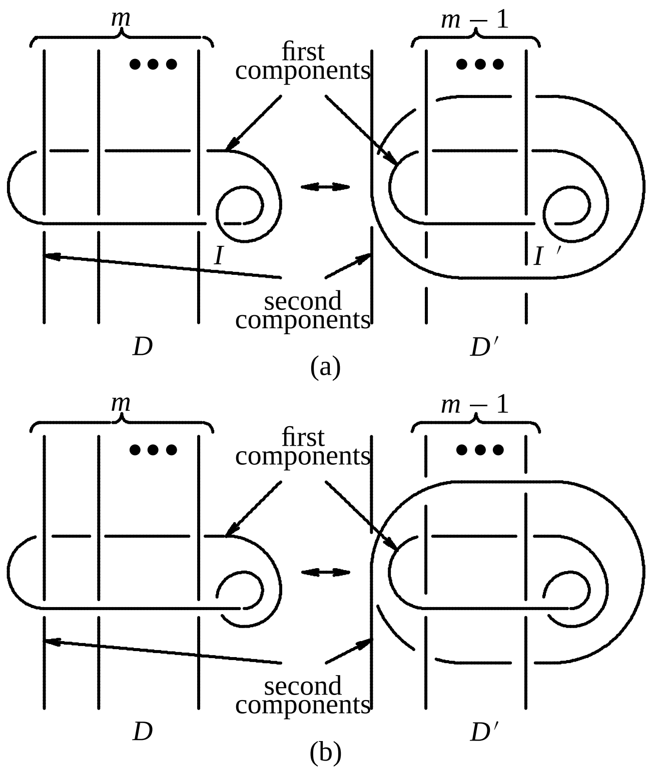

FIGURE 3.

LEMMA 2.1. *The numbers $b_+(D)$ and $b_-(D)$ remain unchanged under the moves in Figures* 3 (a)–(b).

PROOF. Let D and D' be the diagrams shown in Figures 3 (a)–(b). It is easy to see that $(b_{pq}) = (x_{pq})^t(b'_{pq})(x_{pq})$ for $x_{pp} = 1$, $x_{12} = \pm 1$ and $x_{pq} = 0$ otherwise, where the first two components of D and D' are specified. Hence the lemma follows. □

It follows from Proposition 1.1 and Proposition 1.2 that for proving the invariance of $W(D)$ one need only verify the invariance under the Fenn-Rourke moves in Figure 4 (a)–(b).

PROOF OF THEOREM 1.3 FOR $r = 3$ AND $k = 1$. Let $a = e^{\pi i/6}$. The proof is essentially the same as in [4, page 521], where invariance under the Kirby transformations was verified. It follows from Lemma 1.12 and Example 1.13 that

$$R(D) = 2^{-|D|/2} e^{\pi i\sigma/4} \sum_{P\subset D} (-1)^{|P|+|P|_+} i^{\#D} = 2^{-|D|/2} e^{\pi i\sigma/4} \sum_{P\subset D} (-1)^{|P|+|P|_-} i^{-\#D}.$$

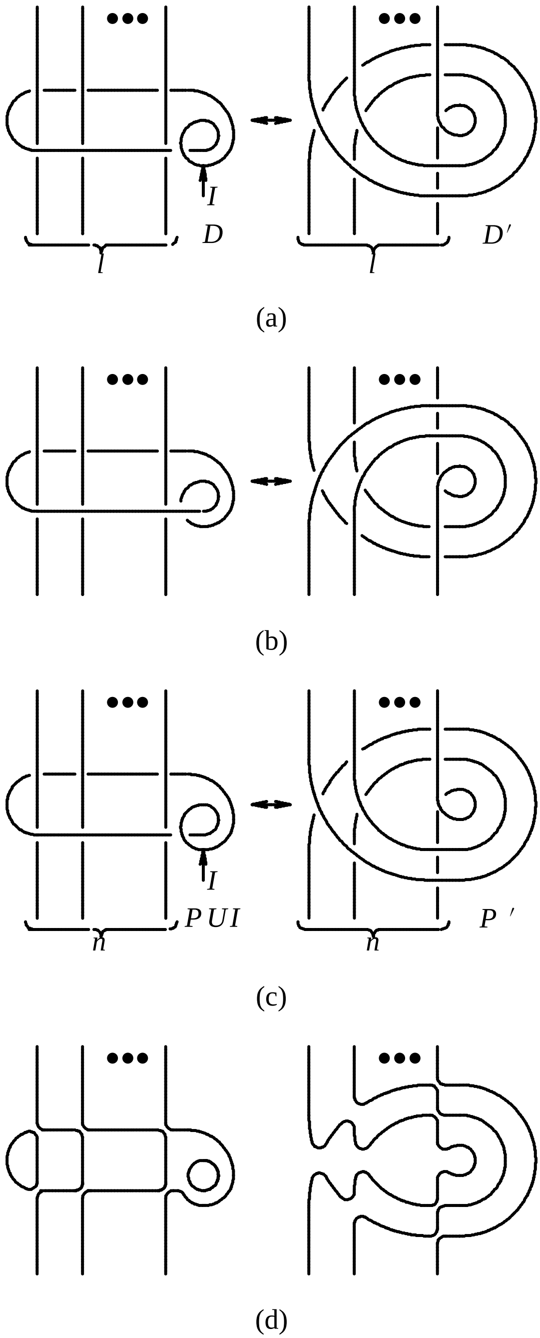

FIGURE 4.

We prove the invariance under the move in Figure 4 (a) using the formula for $R(D)$ involving $|\cdot|_+$. The invariance under the move in Figure 4 (b) is verified analogously using the formula for $R(D)$ involving $|\cdot|_-$. Denote by D, D', and I the diagrams shown in Figure 4 (a). Clearly, the Fenn-Rourke move in Figure 4 (a) is decomposed into l second Kirby moves in Figure 3 (a) (for $m = l, \dots, 1$) and one first Kirby move in Figure 5. Since $\sigma(D' \cup U_+) = \sigma(D') + 1$, it follows from Lemma 2.1 that $\sigma(D) = \sigma(D') + 1$. Let P denote an arbitrary subdiagram of $D \setminus I$. Clearly, $|P \cup I| = |P| + 1$ and $|P \cup I|_+ = |P| + 2$. Hence, we have

$$R(D) = \frac{2^{-|D'|/2} e^{\pi i \sigma(D')/4}}{1-i} \sum_{P \subset D \setminus I} \left((-1)^{|P|+|P|_+} i^{\#P} + (-1)^{|P\cup I|+|P\cup I|_+} i^{\#(P\cup I)}\right)$$

$$= 2^{-|D'|/2} e^{\pi i \sigma(D')/4} \sum_{P \subset D \setminus I} (-1)^{|P|+|P|_+} \frac{i^{\#P} + i^{\#(P\cup I)}}{1-i}.$$

There exists a natural correspondence between the subdiagrams of D' and $D \setminus I$. If P' and P are the corresponding subdiagrams, then (by Figures 4 (c)–(d)), $|P| = |P'|$, $|P|_+ = |P'|_+$, $\#P = \#P' - n^2$, $\#(P \cup I) = \#(P') - n^2 + 2n + 1$, where $n \geq 0$ is the number of components in the part of P corresponding to the part of D shown in Figure 4 (a). Since $i^{-n^2} - i^{-n^2+2n+1} = 1 - i$, it follows that $R(D') = R(D)$. □

LEMMA 2.2. *$W(D)$ remains unchanged under the first Kirby move in Figure 5.*

PROOF. Clearly, $b_\pm(D \cup U_\pm) = b_\pm(D) + 1$, $b_\pm(D \cup U_\mp) = b_\pm(D)$, and

$$f_{D\cup U_\pm}(\omega, \dots, \omega) = f_D(\omega, \dots, \omega) \cdot f_{U_\pm}(\omega).$$

Hence $W(D \cup U_-) = W(D) = W(D \cup U_+)$. □

LEMMA 2.3. *$W(E)$ remains unchanged under the Fenn-Rourke moves of the diagram E in Figures 4 (a)–(b) if for arbitrary diagrams D and D' that differ as in Figures 3 (a)–(b) the following equality holds*

$$f_D(\omega, \alpha, \alpha, \dots, \alpha) = f_{D'}(\omega, \alpha, \alpha, \dots, \alpha).$$

PROOF. Clearly, the Fenn-Rourke moves in Figures 4 (a)–(b) are decomposed into l second Kirby moves in Figures 3 (a)–(b) (for $m = l, \dots, 1$) respectively, and one first Kirby move in Figure 5. Thus it follows from Lemma 2.1 and Lemma 2.2 that we only need to check the equality $f_D(\omega, \dots, \omega) = f_{D'}(\omega, \dots, \omega)$, where D and D' are shown in Figure 3 (a) or 3 (b) and their first two components are specified. Let $n = |D| = |D'|$ and $k_2, k_3, \dots, k_n \geq 0$ be arbitrary integers. It suffices to verify that

$$f_D(\omega, \alpha^{k_2}, \alpha^{k_3}, \dots, \alpha^{k_n}) = f_{D'}(\omega, \alpha^{k_2}, \alpha^{k_3}, \dots, \alpha^{k_n}).$$

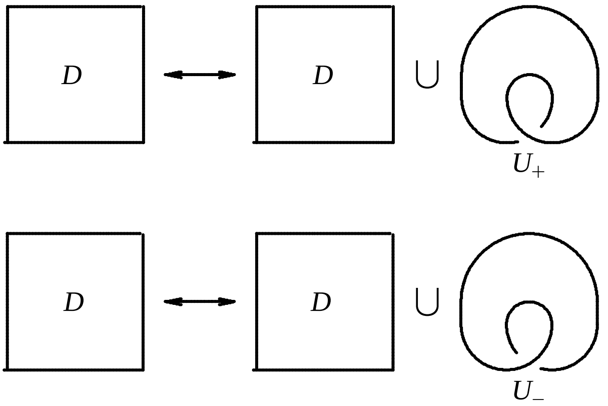

FIGURE 5.

This equality is clear for $k_2 = 0$. If $k_i = 0$, for some $i \geq 3$, we may consider $D \setminus D_i$ and $D' \setminus D'_i$ instead of D and D'. Therefore we may assume that $k_3, \dots, k_n \neq 0$. Let C and C' be the diagrams obtained from D and D' by taking k_i curves, for each $i \geq 3$, close and parallel to the i-th component. Considering C and C' instead of D and D' we may assume that $k_3, \dots, k_n = 1$. By induction on k_2 it follows that the above equality for $k_2 = 1$ implies the analogous equation for arbitrary k_2. Indeed, suppose that $k_2 \geq 2$. Let $K = D'^{(1,2,1,\dots,1)}$ with $|K| = n + 1$ and J' be the second component of D'. Obviously, we have $D'^{(k_1,k_2,1\dots,1)} = K^{(k_1,k_2-1,1,1\dots,1)}$. The induction hypothesis for diagrams K and $D \cup J'$ then gives that

$$\begin{aligned} f_{D'}(\omega, \alpha^{k_2}, \alpha, \dots, \alpha) &= f_K(\omega, \alpha^{k_2-1}, \alpha, \alpha, \dots, \alpha) \\ &= f_{D \cup J'}(\omega, \alpha^{k_2-1}, \alpha, \alpha, \dots, \alpha) \\ &= f_D(\omega, \alpha^{k_2}, \alpha, \dots, \alpha). \end{aligned}$$ □

PROOF OF THEOREM 1.3 FOR $r = 4$ AND $k = 1$. From now on assume that the Kauffman bracket is calculated at $a = e^{\pi i/8}$. We prove the invariance of $W(D)$ under the move in Figure 4 (a). The invariance under the move in Figure 4 (b) is verified analogously. Let D and D' be the diagrams shown in Figure 3 (a). By I and I' we denote their first components.

Since $\omega = \alpha^2 - \sqrt{2}\alpha$ it follows by Lemma 2.3 that we must only show that

$$(**) \qquad \langle D \circ I \rangle - \sqrt{2}\langle D \rangle = \langle D' \circ I' \rangle - \sqrt{2}\langle D' \rangle.$$

Applying (a) to the crossings marked in Figure 6 (a), we obtain $-\sqrt{2}\langle D \rangle = \sqrt{2}a^3\langle S \rangle$, $\langle D \circ I \rangle = 2\langle Q \rangle - \langle T \rangle$, $-\sqrt{2}\langle D' \rangle = -\sqrt{2}\langle S' \rangle$ and $\langle D' \circ I' \rangle = -2a^{-3}\langle Q' \rangle + a^{-3}\langle T' \rangle$.

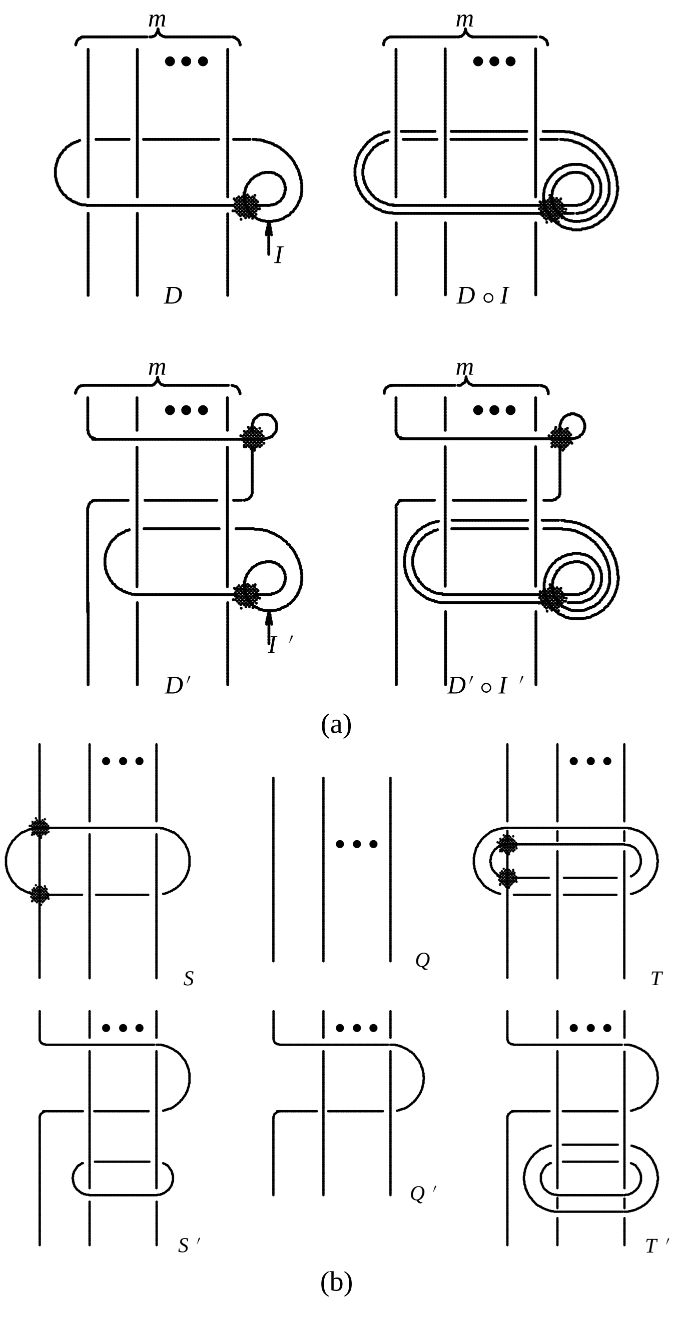

FIGURE 6.

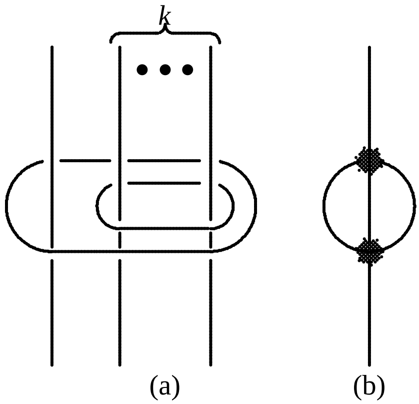

FIGURE 7.

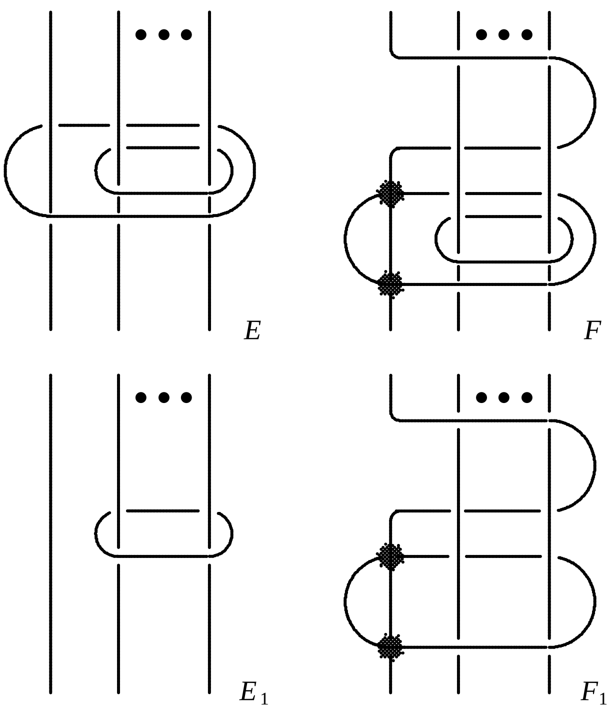

FIGURE 8.

To complete the proof of Theorem 1.3 for $r = 4$ and $k = 1$ we need the following simple lemma.

LEMMA 2.4. *Suppose that the diagram A contains the part shown in Figure 7 (a), where $k \geq 0$. Then $\langle A \rangle = 0$.*

PROOF. By property (a) of the Kauffman bracket, we may assume that A has no crossings outside the part shown. It is easy to see that A contains the part shown in Figure 7 (b). Applying (a) to the two marked crossings in Figure 7 (b) and using (b) one can easily obtain that $\langle A \rangle = 0$. □

Applying (a) to the crossings of T and F_1 marked in Figure 6 (b) and Figure 8, using Proposition 1.2 (for the first and the last equalities) and Lemma 2.4 (for the second equality) we get that

$$\langle T \rangle = (1+i)\langle F_1 \rangle + \frac{1+i}{\sqrt{2}}\langle E \rangle = (1+i)\langle F_1 \rangle = 2\langle Q \rangle + \sqrt{2}\langle S' \rangle.$$

Hence, $\langle D \circ I \rangle - \sqrt{2}\langle D \rangle = \sqrt{2}a^3\langle S \rangle - \sqrt{2}\langle S' \rangle$. Clearly, (∗∗) is equivalent to the equality $\sqrt{2}a^3\langle S \rangle = -2a^{-3}\langle Q' \rangle + a^{-3}\langle T' \rangle$. Using Lemma 2.4 (for first equality), applying (a) to the crossings of F and S marked in Figure 6 (b) and Figure 8 and using Proposition 1.2 (for the last two equalities) we obtain that

$$\begin{aligned} \sqrt{2}a^3\langle S \rangle &= \sqrt{2}a^3\langle S \rangle + a^{-1}\langle F \rangle \\ &= \sqrt{2}a^3\langle S \rangle + a^{-3}\langle T' \rangle + \sqrt{2}a^{-3}\langle E_1 \rangle \\ &= -2a^{-3}\langle Q' \rangle + a^{-3}\langle T' \rangle \end{aligned}$$

and we are done. □

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