On regular neighborhoods of homotopic embeddings of polyhedra in manifolds

Eugene Rafikov[1] and Dušan Repovš[2]

This paper is devoted to the following problem, raised independently in [8], [20] and also by Štan'ko (unpublished): Find conditions for a compact k-polyhedron K and a PL m-manifold M under which the following property holds:

(*) $R_M(f(K)) \cong R_M(g(K))$, for every two homotopic PL embeddings $f, g : K \to M$.

For a polyhedron $K \subset M$ the notation $R_M(K)$ denotes a regular neighborhood of K in M. A version of this problem, when the homeomorphism between the regular neighborhoods is required to be an extension of $f \circ g^{-1} : g(K) \to f(K)$ over M, is better known and easier [13].

The special cases when $M = \mathbf{R}^m$ (then f and g are always homotopic) or when K is a PL manifold are the most interesting. For $M = \mathbf{R}^m$ and $m \geq 2k+1$ (in particular, for $k = 2$, $M = \mathbf{R}^m$ and $m \geq 5$), the property (*) holds due to [13] (since m-thickenings are classified by their tangent bundles). For $k = 1$ the property (*) also holds. The only case not covered by the above remark is $m = 2$ and we prove it below. It is clear that homotopic embeddings f and g are not necessarily isotopic, i.e. do not always have the same rotation systems (which would trivially imply that the regular neighborhoods are homeomorphic).

Lemma 1. *If M is a 2-surface (not necessarily closed), K is a graph and $f, g : K \to M$ are homotopic PL embeddings, then the regular neighborhoods of $f(K)$ and $g(K)$ in M are homeomorphic.*

Proof. In this proof we omit $\mathbf{Z}_2$ coefficients from the notation of homology groups. Let $N = R_M(f(K))$. We identify $H_1(K)$ and $H_1(N)$ by the inclusion-induced isomorphism. Suppose first that M is orientable. Apply the Euler formula to the graph $f(K)$, embedded into the surface obtained from N by patching all holes by

[1]Department of Differential Geometry, Faculty of Mechanics and Mathematics, Moscow State University, 119899, Moscow, Russia, e-mail: rafikov@mccme.ru

[2]The second author was supported in part by the Ministry for Science and Technology of the Republic of Slovenia research grant No. J1-0885-0101-98, Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P. O. Box 2964, 1001 Ljubljana, Slovenia, e-mail: dusan.repovs@fmf.uni-lj.si

2-disks. We obtain that the number of holes in N is uniquely determined by the genus of N, i.e., by $\operatorname{rk} H_1(N)$. Therefore the topological type of N is uniquely determined by $\operatorname{rk} H_1(N)$. But $\operatorname{rk} H_1(N)$ equals to the rank of the intersection form on $H_1(N)$. Let u, v be two embedded circles in $f(K)$, representing homology classes $u, v \in H_1(N)$. Clearly, the map $f_* : H_1(K) \to H_1(M)$ is uniquely determined by the homotopy class of f. Then $u \cap v = u|_{H_1(M)} \cap v|_{H_1(M)}$ is uniquely determined by the homotopy class of f. Therefore the intersection form on $H_1(N) \cong H_1(K)$ is uniquely determined by the homotopy class of f, and we are done.

Suppose now that M is nonorientable. Clearly, orientability of N is determined by the homotopy class of f. By 'genus' of N we mean the nonorientable genus (i.e. $\operatorname{rk} H_1(N)$). By the same argument we are done. □

If $m = k + 1 \geq 3$ and K is a special polyhedron, then the property (*) holds by [2], [3], [4], [17], [20]. The example $M = \mathbf{R}^m$, $m \geq 3$ and $K = S^{m-1} \vee S^1 \vee S^1$ shows that the property (*) fails for $k = m - 1$ [20]. The Dunce Hat example shows that the property (*) is false for $M = \mathbf{R}^4$ and a certain 2-polyhedron K [26], [9].

The regular neighborhood $R_{\mathbf{R}^m}(K)$ of a smooth k-submanifold $K \subset \mathbf{R}^m$ is the space of a normal vector bundle. Therefore $R_{\mathbf{R}^m}(K)$ does not depend on the embedding $K \subset \mathbf{R}^m$ if either $k \geq m - 1$ or ($k = m - 2$ and K is orientable [12], [22]) or ($K = S^k$ and $m \geq \frac{3k}{2} + 1$ [11], [6], [15]). On the other hand, there are smooth embeddings $K \to \mathbf{R}^4$ of a closed nonorientable 2-manifold K with different normal bundles: if K is a connected sum of l copies of $\mathbf{R}P^2$, then $\bar{e}(K \to \mathbf{R}^4)$ can assume every value from $\{-2l, -2l + 4, \dots, 2l\}$, and only these values [16], [21], see also [1], [18]. Haefliger also constructed a smooth embedding $S^{11} \to \mathbf{R}^{17}$ with a nontrivial normal bundle [7]. It would be interesting to know whether the regular neighborhoods, i.e. the *spaces* of normal bundles, are the same for such embeddings.

Analogously, the regular neighborhood $R_{\mathbf{R}^m}(K)$ of a PL locally flat k-submanifold $K \subset \mathbf{R}^m$ is the space of a normal block bundle [23]. Therefore $R_{\mathbf{R}^m}(K)$ does not depend on the embedding $K \subset \mathbf{R}^m$ if either $k \geq m - 1$ or ($k = m - 2$ and K is orientable [23] Corollary 6.8, [24] Theorem 2) or $K = S^k$. In particular, $R_{\mathbf{R}^4}(S^2) \cong S^2 \times D^2$ for any *smooth* or *PL locally flat* embedding $S^2 \subset \mathbf{R}^4$.

Conjecture 2.

a) Let f, g be any two smooth embeddings of the Klein bottle K into $\mathbf{R}^4$ with the normal Euler classes 0 and 4, respectively. Then the regular neighborhoods of the images of f and g are not homeomorphic.

b) [cf. [10]] *The regular neighborhood of $f(\mathbf{R}P^2)$ does not depend on a smooth embedding $f : \mathbf{R}P^2 \to \mathbf{R}^4$.*

Conjecture 3. *There exists a (nonlocally flat) PL embedding $S^2 \subset \mathbf{R}^4$ such that $R_{\mathbf{R}^4}(S^2) \not\cong S^2 \times D^2$.*

Construction of a candidate for Conjecture 3. Let B^4 be the standard 4-ball in $\mathbf{R}^4$ and $S^3 = \partial B^4$. Take a (sliced) knot $S^1 \subset S^3$ concordant to the trivial knot, i.e. bounding a proper PL locally flat disk $D^2 \subset \mathbf{R}^4 \setminus \overset{\circ}{B}{}^4$. Take an embedded cone on $S^1 \subset S^3 = \partial B^4$ inside B^4. The union of this cone and the disk $D^2 \subset \mathbf{R}^4 \setminus \overset{\circ}{B}{}^4$ is a candidate for the required 2-sphere $S^2 \subset \mathbf{R}^4$. By Lemma 4, $N = R_{\mathbf{R}^4}(S^2)$ is obtained from D^4 by Dehn surgery along the knot $S^1 \subset S^3 = \partial D^4$ with framing 0. So it remains to prove Conjecture 5 below.

Lemma 4. *Let $B^4 \subset \mathbf{R}^4$ be the standard 4-ball and $D^2 \subset \mathbf{R}^4 \setminus \overset{\circ}{B}{}^4$ a proper locally flat 2-disk. Let $S^2 \subset \mathbf{R}^4$ be the 2-sphere formed by the union of D^2 and the cone over ∂D^2 with a vertex in $\overset{\circ}{B}{}^4$. Let $N = R_{\mathbf{R}^4}(S^2)$. Then ∂N is obtained from S^3 by Dehn surgery along ∂D^2 with framing 0.*

Proof. In this proof we omit $\mathbf{Z}$ coefficients from the notation of homology groups. Let $D^4 = R_{\mathbf{R}^4 \setminus \overset{\circ}{B}{}^4}(D^2)$. Then $B^4 \cup D^4 = R_{\mathbf{R}^4}(S^2)$. Next, $B^4 \cap D^4 = \partial B^4 \cap \partial D^4 = R_{\partial B^4}(\partial D^2)$ is a solid torus. Since the embedding $D^2 \subset \mathbf{R}^4$ is locally flat, it follows that the pair (D^4, D^2) is standard [25]. Clearly, $\partial D^4 \setminus (B^4 \cap \partial D^4)$ is also a solid torus. Since the solid tori $\partial B^4 \cap \partial D^4$ and $\partial D^4 \setminus (B^4 \cap \partial D^4)$ have common boundary, it follows that ∂N is obtained by Dehn surgery with some integer frame k (this is clear by the definition from [14], which is equivalent to that from [19]). To prove that $k = 0$ we assume that $k \neq 0$ and obtain that $H_1(\partial N) = \mathbf{Z}_{|k|}$. From this we deduce that ∂N cannot be embedded in $\mathbf{R}^4$, which is a contradiction. See the details below.

Recall that $\partial D^2 \subset S^3$ is a knot. Let $A = S^3 \setminus (B^4 \cap D^4) \simeq S^3 \setminus \partial D^2$, $B = \partial D^4 \setminus (B^4 \cap \partial D^4) \cong D^2 \times S^1$. Then $A \cup B = \partial N$ and $A \cap B \cong S^1 \times S^1$. We have the Mayer-Vietoris sequence:

$$\begin{array}{ccccccccc} H_2(A) \oplus H_2(B) & \to & H_2(A \cap B) & \to & H_1(A \cap B) & \xrightarrow[\alpha]{} & H_1(A) \oplus H_1(B) & \to & H_1(A \cap B) \to 0 \\ \| & & \| & & \| & & \| & & \| \\ 0 & \to & H_2(\partial N) & \to & \mathbf{Z} \oplus \mathbf{Z} & \xrightarrow[\alpha]{} & \mathbf{Z} \oplus \mathbf{Z} & \to & H_1(\partial N) \to 0 \end{array}$$

The homomorphism α is given by $\alpha(a, b) = (a + kb, -a)$, where the basis in $H_1(A \cap B)$ is formed by the meridian of $A \cap B$ and the boundary of the meridian disk of B. So $H_1(\partial N) = \mathbf{Z}_{|k|}$, for every $k \neq 0$.

Now we prove that a 3-manifold M with odd torsion does not embed in $\mathbf{R}^4$ [5]. Suppose to the contrary, that $M \subset \mathbf{R}^4$ were such an embedding. Let W_1 and

W_2 be the closures of the connected components of $\mathbf{R}^4 \setminus M$. By Alexander and Poincaré duality theorems, we would have

$$\operatorname{Tors} H_1(W_1) \oplus \operatorname{Tors} H_1(W_2) \cong \operatorname{Tors} H_1(\mathbf{R}^4 \setminus M) \cong \operatorname{Tors} H_1(M).$$

By Alexander duality, $\operatorname{Tors} H_1(W_1) \cong \operatorname{Tors} H_1(W_2)$. Therefore all torsion coefficients of M must be even. Contradiction. $\square$

Conjecture 5. *There exists a knot L, concordant to the trivial knot, but such that L with the zero framing is not Kirby-equivalent to the trivial knot with the zero framing.*

Remark 6. We use notation and conventions from Lemma 4 and its proof. Let us prove that $H_1(\partial N) = \mathbf{Z}$, $H_2(\partial N) = \mathbf{Z}$ and the intersection product $H_2(\partial N) \times H_2(\partial N) \to H_1(\partial N)$ is trivial (by the same argument higher Massey products on $H_1(\partial N)$ are also trivial). Particularly, they do not depend on the embedding $S^2 \subset \mathbf{R}^4$ for the class of embeddings from Lemma 4. In fact, the homology groups can be calculated applying the Mayer-Vietoris sequence. To prove that the intersection product on ∂N is trivial, observe that the generator of $H_2(\partial N)$ is represented by an orientable 2-surface, that is the union (along the common boundary) of the meridian disk of B and Seifert surface in A. But the self-intersection of an orientable 2-surface in a 3-manifold is trivial.

Conjecture 7.

a) For every $k \geq 3$ there exists a k-polyhedron K and PL embeddings $f, g : K \to \mathbf{R}^{2k}$ such that the regular neighborhoods of $f(K)$ and $g(K)$ in $\mathbf{R}^{2k}$ are not PL homeomorphic.

b) For every $k \geq 3$ there exists a PL embedding $f : S^n \to \mathbf{R}^{n+2}$ such that the regular neighborhood of $f(S^n)$ in $\mathbf{R}^{n+2}$ is not PL homeomorphic to $S^n \times D^2$.

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