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SELECTIONS OF MAPS WITH NONCLOSED VALUES AND TOPOLOGICALLY REGULAR MAPS

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ABSTRACT. We introduce the concept of a topologically regular map as a map with homeomorphic fibers, whose multivalued inverse map is continuous with respect to the Fréchet metric. Using E. Michael's selection theorem, we prove that every topologically regular map with the fibers homeomorphic to $[0,1]$, of a locally σ-compact metric space onto a perfectly normal space, is a locally trivial bundle.

Let $p : E \to B$ be a continuous map. We are interested in finding conditions which guarantee that p is a locally trivial fibration. Clearly, a necessary condition is that the fibers $p^{-1}(b)$, $b \in B$, are homeomorphic.

Denote by $\exp^M(E)$ the class of all closed subsets of the topological space E which are homeomorphic to a fixed topological space M. Usually the set $\exp^M(E)$ is equipped with the Vietoris topology, which in the case when E is metrizable and M is compact coincides with the topology, induced by the Hausdorff metric. However, such a topology doesn't take into account the uniqueness (up to homeomorphism) of the elements of $\exp^M(E)$. We shall introduce a metric in $\exp^M(E)$ which will not have this defect:

Definition 1. The *Fréchet distance* between two homeomorphic closed subsets A and B of the metric space E is the infimum of all $\varepsilon > 0$ for which there exists a homeomorphism $h : A \to B$ (or $h : B \to A$) which doesn't move points for more than ε, i.e., h is an ε-move.

In [**3**] this distance was called the *homeomorphic distance*. In the present paper we chose the new term having in mind the analogy

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with the Fréchet distance in the space $\mathcal{C}(I,X)$ of the continuous maps of the unit interval $I = [0,1]$ into a metric space X (cf. [**6, 9, 10**]). Note that the Fréchet distance in the sense of [**6, 9, 10**] is only a pseudometric on the space $\mathcal{C}(I,X)$. It becomes a metric only after a factorization with respect to the following equivalence relation: $f_0 \sim f_1 \Leftrightarrow f_0 = fh_0,\ f_1 = fh_1$, for some $f \in \mathcal{C}(I,X)$, $h_i \in \{h \in \mathcal{C}(I,I) \mid h(0) = 0, h(1) = 1, t_1 < t_2 \Rightarrow h(t_1) \le h(t_2)\}$. The proof of this fact uses the so-called Whitney-Morse *μ-parametrization* of continuous curves (cf. [**6, 10**]).

Definition 2. A map $p : E \to B$ is said to be *topologically regular* if its multivalued inverse $p^{-1} : B \to \exp^M(E)$ maps B into $\exp^M(E)$ for some M, and p^{-1} is continuous with respect to the Fréchet metric in $\exp^M(E)$.

The purpose of the present paper is to prove the following result:

Theorem 1. *Let $p : E \to B$ be a topologically regular map between compact metric spaces E and B such that for every $b \in B$, $p^{-1}(b)$ is homeomorphic to the unit interval $I = [0,1]$. Then p is a locally trivial fibration.*

We believe that not just Theorem 1 but also the methods of its proof are of some interest because we use a selection theorem of E.A. Michael [**7**, Theorem (3.1$'''$)] for convex-valued but nonclosed-valued maps:

Theorem 2 (E.A. Michael [**7**]). *For a Hausdorff space X, the following conditions are equivalent*:

(1) *X is perfectly normal*; *and*

(2) *Every lower semicontinuous map of X into convex $\mathcal{D}$-type subsets of a separable Banach space, has a continuous univalued selection.*

Recall that a convex subset of a Banach space is said to be *convex D-type* if it contains all interior (in the convex sense) points of its closure. (A point of a closed convex subset of a Banach space is said to be *interior* (*in the convex sense*) if it isn't contained in any supporting hyperplane.) Standard examples of convex $\mathcal{D}$-type sets are: (1) closed convex sets; (2) convex subsets of Banach spaces which contain at least one interior (in the usual metric sense) point; and (3) finite-dimensional convex sets.

We shall need the following example of a convex $\mathcal{D}$-type set in the Banach space $\mathcal{C}(X)$ of all bounded continuous functions on a completely regular space X. $\mathrm{Auth}_0(I)$ denotes the collection of all homeomorphisms of the unit interval $I = [0,1]$ onto itself which are identity on the boundary ∂I.

Lemma 1. *Let X be a completely regular space, $h : I \to X$ an embedding, and let*

$$\mathcal{C}_h(X) = \{f \in \mathcal{C}(X) \mid fh \in \mathrm{Auth}_0(I)\}.$$

Then $\mathcal{C}_h(X)$ is a convex $\mathcal{D}$-type subset of the space $\mathcal{C}(X)$.

Proof. The convexity of the set $\mathcal{C}_h(X)$ follows immediately from the convexity of the set $\mathrm{Auth}_0(I)$:

$$((1-\lambda)f + \lambda g) \circ h = (1-\lambda)(f \circ h) + \lambda(g \circ h),$$
$$0 \le \lambda \le 1, \;\; f, g \in \mathcal{C}_h(X).$$

The inequality $||f_0 \circ h - f_n \circ h||_{\mathcal{C}(I)} \le ||f_0 - f_n||$ implies that the closure of $\mathcal{C}_h(X)$ lies in the set

$$\{f \in \mathcal{C}(X) \mid f \circ h \in \mathrm{Cl}\,(\mathrm{Auth}_0(I))\}.$$

Consider an arbitrary element $f \in \mathrm{Cl}\,(\mathcal{C}_h(X)) \backslash \mathcal{C}_h(X)$. Then there exist numbers $0 \le a < b \le 1$ such that $f(h(a)) = f(h(b)) = f \circ h|_{[a,b]}$. In the Banach space $\mathcal{C}(X)$, the set $\Pi = \{g \in \mathcal{C}(X) \mid g(h(a)) = g(h(b))\}$ can easily be shown to be a codimension 1 hyperplane. This hypersurface Π will be supporting the closed convex set $\mathrm{Cl}\,(\mathcal{C}_h(X))$ since: (i) it passes through the point $f \in \mathrm{Cl}\,(\mathcal{C}_h(X))$; and (ii) the whole set $\mathrm{Cl}\,(\mathcal{C}_h(X))$ lies in the closed halfspace $\{g \in \mathcal{C}(X) \mid g(h(a)) \le g(h(b))\}$. Therefore, f isn't interior (in the convex sense) point of $\mathrm{Cl}\,(\mathcal{C}_h(X))$. Consequently, the convex set $\mathcal{C}_h(X)$ contains all interior (in the convex sense) points of its closure, i.e., $\mathcal{C}_h(X)$ is a convex $\mathcal{D}$-type set. $\square$

Proof of Theorem 1. Let $p : E \to B$ be a topologically regular map between compacta with fibers all homeomorphic to the unit interval $I = [0, 1]$. Let $b_0 \in B$ be any point, and let $\partial(p^{-1}(b_0)) = \{c_0, d_0\}$ be the endpoints of the arc $p^{-1}(b_0) \approx I$. Choose $\varepsilon_0 > 0$ such that $0 < 2\varepsilon_0 < \operatorname{dist}_E(c_0, d_0)$. Since, by hypothesis, the map p is topologically regular, there exists a neighborhood $U(b_0) \subset B$ of b_0 such that, for every point $b \in U(b_0)$, one of the endpoints of the arc $p^{-1}(b)$ lies in the ε_0-neighborhood of the point c_0 (denote this endpoint by $c(b)$) and the other endpoint (denoted by $d(b)$) lies in the ε_0-neighborhood of the point d_0. For every $b \in U(b_0)$, define the set

$$F(b) = \{f \in \mathcal{C}(E) \mid f(c(b)) = 0,\ f(d(b)) = 1,\ f|_{p^{-1}(b)} \text{ is one-to-one}\}.$$

Then it follows by Lemma 1 that $F(b)$ is a convex $\mathcal{D}$-type subset of the separable Banach space $\mathcal{C}(E)$. Let us verify the lower semicontinuity of the multivalued map $F : U(b_0) \to C(E)$, given by $b \mapsto F(b)$. Fix the point $b \in U(b_0)$, a map $f \in F(b)$, and an $\varepsilon > 0$. We must find a neighborhood $U(b)$ of b such that for all $z \in U(b)$ there exists $g \in F(z)$ such that $||f - g|| < \varepsilon$.

Since f is uniformly continuous, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x, y \in E$ such that $\rho(x, y) < \delta$ it follows that $|f(x) - f(y)| < \varepsilon$. Since $p : E \to B$ is topologically regular, there exists a neighborhood $U(b) \subset U(b_0)$ of the point b such that the Fréchet distance between the fibers $p^{-1}(b)$ and $p^{-1}(z)$ is less than δ for all $z \in U(b)$. In other words, for any $z \in U(b)$ there exists a homeomorphism $h_z : p^{-1}(b) \to p^{-1}(z)$ which is a δ-move such that $h_z(c(b)) = c(z)$ and $h_z(d(b)) = d(z)$. In these circumstances, the inequality $|f(x) - f(h_z^{-1}(x))| < \varepsilon$ holds for all $x \in p^{-1}(z)$. By the Tietze extension theorem, there exists an extension $\varphi \in \mathcal{C}(E)$ of the function $f - f \circ h_z^{-1}$ from the fiber $p^{-1}(z)$ over the whole compactum E, and we may assume that $||\varphi|| < \varepsilon$.

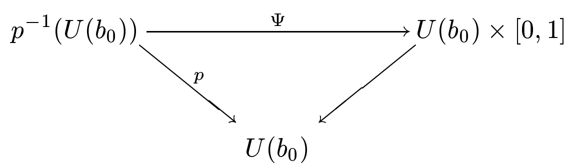
Consider the function $g = f - \varphi$. By construction, it is ε-close to f and its restriction onto the fiber $p^{-1}(z)$ agrees with $f \circ h_z^{-1}$ and therefore it is injective on this fiber. Consequently, $g \in F(z)$, which demonstrates the lower semicontinuity of the multivalued map $b \mapsto F(b)$, defined on the neighborhood $U(b_0)$ of $b_0 \in B$.

We can now invoke Theorem 2 to obtain a continuous univalued map $\varphi_0 : U(b_0) \to \mathcal{C}(E)$ such that $\varphi_0(b) \in F(b)$ for all $b \in U(b_0)$, i.e., $[\varphi_0(b)](c(b)) = 0$, $[\varphi_0(b)](d(b)) = 1$, and $\varphi_0(b)$ is injective on the fiber $p^{-1}(b)$.

The trivialization of the map $p : E \to B$ over the neighborhood $U(b_0)$ of the point $b_0 \in B$ is then given in the standard way. For $x \in p^{-1}(U(b_0))$ we set

$$\Psi(x) = (p(x), [\varphi_0(p(x))](x)) \in U(b_0) \times [0,1].$$

Then Ψ maps the preimage $p^{-1}(U(b_0))$ of the neighborhood $U(b_0)$ homeomorphically onto the Cartesian product $U(b_0) \times [0,1]$, and the diagram



is commutative. □

Remarks. 1) Theorem 1 remains true for locally compact spaces E. In this case our proof must be modified only in the following point: in the definition of the sets $F(b)$ one must not consider the whole space $\mathcal{C}(E)$ of continuous functions on E, but rather its separable subspace, consisting of those functions, whose supports are contained in the fixed neighborhood of the fiber $p^{-1}(b_0)$, the closure of which is compact.

2) Theorem 1 is valid also for locally σ-compact spaces E. In that case one must consider the neighborhood U of the fiber $p^{-1}(b_0)$ which is the union $\cup_{n\geq 1} K_n$ of a sequence of compacta $\{K_n\}_{n\geq 1}$ and then (in the standard way) introduce the Fréchet space of continuous functions, with supports, contained in U. Then the sets $F(b)$ must be constructed as subsets of this separable Fréchet space.

3) We never used the compactness of B in our proof; Theorem 1 is valid for all spaces B for which one can apply Theorem 2, i.e., for all perfectly normal spaces B.

4) There is an alternative proof of Theorem 1 based on the concept of the map, universal for all maps with a fixed (up to homeomorphism) preimage. More generally, for a compact manifold M consider the space $\exp^M(Q)$ with the Fréchet metric, where Q is the Hilbert cube. In the Cartesian product $Q \times \exp^M(Q)$ consider the subset $U^M(Q) = \{(x, F) \in Q \times \exp^M(Q) \mid x \in F\}$. The canonical projection u_M :

$U^M(Q) \to \exp^M(Q)$ is a generalization of the Grassman fibration. Thus, u_M is a topologically regular map (all of whose preimages are homeomorphic to M) in which one can embed, in the canonical way, every topologically regular map $p : E \to B$ whose fibers are homeomorphic to M. Indeed, the embedding $i : E \subset Q$ induces an embedding $j : B \to \exp^M(Q)$, given by $j(b) = i(p^{-1}(b))$, for every $b \in B$. Then the map

$$e \mapsto (i(e), j(p(e))), \qquad e \in E$$

defines an embedding of the space E into $U^M(E)$:

$$\begin{array}{ccc} E & \xrightarrow{(i,j)} & U^M(Q) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle u_M} \\ B & \xrightarrow[j]{} & \exp^M(Q) \end{array}$$

In this way, the map p maps homeomorphically onto the restriction of the map u_M onto the preimage of some subset ($\operatorname{Im} j$ in our case) of the space $\exp^M(Q)$. Consequently, the local triviality of the map p can be deduced from the local triviality of the universal map u_M.

5) In the case where $M = I = [0,1]$, the local triviality of the universal map can be deduced by means of the Morse-Whitney μ-parametrization of continuous curves [**9, 10**], invoking an argument analogous to the one in [**6**, Section 1]. Although Geoghegan's proofs [**6**] formally result in the construction of selections of certain multivalued maps, he could not get them (as he points out in [**6**]) by an application of Theorem 2. The problems arose in connection with finding a "convex structure" in the space $\mathcal{C}(I, X)$. We believe that our method of proof of Theorem 1 presents an interesting application of Michael's selection theorem for convex-valued but nonclosed-valued maps (Theorem 2).

6) We should point out that for finite-dimensional spaces B, results analogous to Theorem 1 are well known: the first one belongs to E. Dyer and M.E. Hamstrom [**4**] (see also [**1, 2, 5**]). We hope that our techniques will be useful also in other situations with the infinite-dimensional basis B.

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REFERENCES

1. T.A. Chapman and S.C. Ferry, *Hurewicz fiber maps with ANR fibers*, Topology **16** (1977), 131–143.

2. ———, *Approximating homotopy equivalences by homeomorphisms*, Amer. J. Math. **101** (1979), 583–607.

3. A.N. Dranišnikov and E.V. Ščepin, *Cell-like mappings. The dimension raising problem*, Uspehi Mat. Nauk **41** (6) (1986), 49–90; English translation in Russian Math. Surveys.

4. E. Dyer and M.E. Hamstrom, *Completely regular mappings*, Fund. Math. **45** (1957), 103–118.

5. S. Ferry, *Strongly regular mappings with compact ANR fibers are Hurewicz fibrations*, Pacific J. Math. **75** (1978), 373–381.

6. R. Geoghegan, *On spaces of homeomorphisms, embeddings and functions*, I, Topology **11** (1972), 159–177.

7. E.A. Michael, *Continuous selections*, I, Annals Math. **63** (1956), 361–382.

8. ———, *Convex structures and continuous selections*, Canad. J. Math. **11** (1959), 556–575.

9. M. Morse, *A special parametrization of curves*, Bull. Amer. Math. Soc. **42** (1936), 915–922.

10. H. Whitney, *Regular families of curves*, Annal Math. **34** (1933), 244–270.

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