On nerves of fine coverings

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Abstract. A basis $\mathcal{U}$ of a topological space is said to be *fine* if the closures of every two of its disjoint elements are disjoint. The following are our main results: (1) Every metric space has a fine open basis; (2) Let X and Y be compact Hausdorff spaces with fine bases $\mathcal{U}$ and $\mathcal{V}$, respectively. Suppose that the nerves $\mathcal{N}(\mathcal{U})$ and $\mathcal{N}(\mathcal{V})$ are simplicialy isomorphic. Then X and Y are homeomorphic; and (3) Let (X_1, τ_1) and (X_2, τ_2) be separable metric spaces without isolated points. Then $\mathcal{N}(\tau_1)$ is simplicially isomorphic to $\mathcal{N}(\tau_2)$.

1. Introduction

Definition 1.1. A covering (not necessarily open) of a topological space is said to be *fine* if the closures of every two of its disjoint elements are disjoint. An open basis $\mathcal{U}$ is said to be *fine* if it is a fine covering.

BANDT [2] introduced the concept of a Δ-*basis* and proved that every compact Hausdorff space with weight $\omega(X) \leq \aleph_1$ has a Δ-basis. If $\mathcal{F} = \{F\}$ is a Δ-basis then $\mathcal{U} = \{X \setminus F\}$ is a fine basis. It is easy to see that any product of compact Hausdorff spaces with fine bases has a fine basis. However, as it was proved by L. SHAPIRO (cf. [12]), I^τ does not have any Δ-basis, if $\tau \geq \aleph_2$. There exist Hausdorff spaces without fine bases (see e.g. [1] and [11]; Double origin topology, pp. 92-93]). The following interesting question remains open:

Question 1.2. *Does there exist a normal space without any fine basis?*

Here is our first main result:

Theorem 1.3. *Every metric space has a fine open basis.*

Furthermore, it is well-known that two compact Hausdorff spaces are homeomorphic if the corresponding rings of continuous functions are isomorphic as algebraic objects [6], [7]. Similar to this is our second main result:

Theorem 1.4. *Let X and Y be compact Hausdorff spaces with fine bases $\mathcal{U}$ and $\mathcal{V}$, respectively, and suppose that the nerves $\mathcal{N}(\mathcal{U})$ and $\mathcal{N}(\mathcal{V})$ are simplicialy isomorphic. Then the spaces X and Y are homeomorphic.*

Recall that the concept of a nerve of an infinite open covering was introduced by DOWKER [3] (see also [4], [8]), in connection with the definition of Čech homology groups for general spaces.

The requirement in Theorem 1.4 that the bases of spaces X and Y be fine, is essential, as our third main result shows:

Theorem 1.5. *Let (X_1, τ_1) and (X_2, τ_2) be separable metric spaces without isolated points. Then the nerves $\mathcal{N}(\tau_1)$ and $\mathcal{N}(\tau_2)$ are simplicialy isomorphic.*

2. Preliminaries

For every covering $\mathcal{W}$ we shall denote by $\overline{\mathcal{W}}$ the covering $\{\overline{W} \mid W \in \mathcal{W}\}$.

Lemma 2.1. *Let $\mathcal{U} = \{U_s\}_{s \in S}$ be an open locally finite covering and let $\mathcal{F}$ be a locally finite family of closed subsets of a paracompact space X. Then there exists an open covering $\mathcal{W}_S$ such that $\overline{\mathcal{W}}_S$ is a locally finite refinement of $\mathcal{U}$ and $\mathcal{F} \cup \mathcal{W}_S$ is a fine covering of X.*

PROOF. Since $\mathcal{U}$ is a locally finite open covering of X, there exists a closed covering $\mathcal{G} = \{G_s\}_{s \in S}$ such that $G_s \subset U_s$, for every $s \in S$ (see, e.g. [5; p. 301, Remark 5.1.7]). Let A be a subset of the index set S and let $\mathcal{W}_A = \{W_s\}_{s \in A}$ be a family of open subsets $W_s \subset X$ such that $G_s \subset W_s$ and $\overline{W_s} \subset U_s$, for every $s \in A$. The covering

$$\mathcal{O}_A = \mathcal{W}_A \cup \{G_s \mid s \in S \setminus A\} \cup \mathcal{F}$$

is called the *modification* of $\mathcal{G}$ generated by $\mathcal{W}_A$, if it is fine. The set of all modifications is nonempty because every closed covering and, in particular $\{G_s \mid s \in S\} \cup \mathcal{F}$, is fine.

Consider the following partial ordering on the set of all modifications: we say that $\mathcal{O}_A \leq \mathcal{O}'_{A'}$ if $A \subset A'$ and for every $s \in A$, $W_s = W'_s$. Obviously, any partially ordered set of modifications, generated by the family $\{\mathcal{W}_A\}_{A\in\mathcal{A}}$, where $\mathcal{A} \subset 2^S$ is a subset of 2^S, has the upper bound

$$\bigcup_{A \in \mathcal{A}} \mathcal{W}_A \cup \left\{ G_s \mid s \in S \quad \text{and} \quad s \notin \bigcup_{A \in \mathcal{A}} A \right\} \cup \mathcal{F}.$$

By the Kuratowski–Zorn lemma, there is a maximal modification $\mathcal{O}_{A_0}$ of $\mathcal{G}$, generated by some $\mathcal{W}_{A_0}$. We shall prove that $A_0 = S$. Suppose to the contrary, that there exists an element $s_0 \notin A_0$. Let the closed set H_{s_0} be the union of all elements of the locally finite covering $\overline{\mathcal{O}_{A_0}} = \overline{\mathcal{W}_{A_0}} \cup \{G_s \mid s \notin A_0\} \cup \mathcal{F}$ which do not intersect G_{s_0}. Since every paracompact space is normal, there exists an open set W_{s_0} such that $G_{s_0} \subset W_{s_0}$, $\overline{W_{s_0}} \subset U_{s_0}$, and $\overline{W_{s_0}} \cap H_{s_0} = \emptyset$. Consider the system $\mathcal{O}_{A_0 \cup \{s_0\}}$ generated by $\mathcal{W}_{A_0} \cup \{W_{s_0}\}$. Let O be any element of $\mathcal{O}_{A_0}$. Suppose that $\overline{W_{s_0}} \cap \overline{O} \neq \emptyset$. Since $\overline{W_{s_0}} \cap H_{s_0} = \emptyset$, it follows by definition of H_{s_0} that $G_{s_0} \cap \overline{O} \neq \emptyset$. Therefore $W_{s_0} \cap \overline{O} \neq \emptyset$ and $W_{s_0} \cap O \neq \emptyset$.

So if we replace in the covering $\mathcal{O}_{A_0}$ the element G_{s_0} by W_{s_0} we get the modification of $\mathcal{G}$ which is larger than $\mathcal{O}_{A_0}$. Contradiction. That is why $A_0 = S$ and $\mathcal{F} \cup \mathcal{W}_S$ is a fine covering. $\square$

Lemma 2.2. *Suppose that in a topological space Y there exist two finite systems of open sets $V_0, V_1, \ldots, V_n$ and $W_0, W_1, \ldots, W_n$ such that $\overline{V_i} = \overline{W_i}$, for every $i \in \{0, 1 \ldots, n\}$, and the intersection $\bigcap_{i=0}^n V_i$ is nonempty. Then the intersection $\bigcap_{i=0}^n W_i$ is also nonempty.*

PROOF. Since $\bigcap_{i=0}^n V_i \neq \emptyset$ and $\bigcap_{i=0}^n V_i \subset \bigcap_{i=0}^n \overline{V_i} = \bigcap_{i=0}^n \overline{W_i}$ it follows that $\bigcap_{i=0}^n \overline{W_i} \neq \emptyset$. Let y be any element of $\bigcap_{i=0}^n V_i$. Then $y \in \overline{W_0}$ and there exists an open nonempty subset O_0 in $(\bigcap_{i=0}^n V_i) \cap W_0$.

Suppose that for some number $k < n$, there exists a nonempty open subset O_k in $(\bigcap_{i=0}^n V_i) \cap (\bigcap_{i=0}^k W_i)$. Choose the open set O_{k+1} in the following manner: Let $y_k \in O_k$. Then $y_k \in \overline{W_{k+1}}$ and $O_k \cap W_{k+1} \neq \emptyset$. Therefore, there exists a nonempty open set O_{k+1} in $O_k \cap W_{k+1}$. By induction, we get a nonempty subset O_n which lies in $\bigcap_{i=0}^n W_i$, so $\bigcap_{i=0}^n W_i \neq \emptyset$. $\square$

3. Proof of Theorems 1.3–1.5

PROOF of Theorem 1.3. It suffices to prove that there exists a countable family of open coverings $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \ldots$, such that for every index n, $\bigcup_{i=1}^{n} \mathcal{W}_i$ is a fine covering and the mesh of $\mathcal{W}_n$ is less than $\frac{1}{n}$.

We shall prove the existence of such a family by induction. In addition, all constructed coverings $\mathcal{W}_n$ will be locally finite. Let $n = 1$. Since X is a metric space, there exists a locally finite open covering $\mathcal{U}_1$ with the mesh less than 1, by the Stone theorem [5]. By Lemma 2.1, there exists a fine open covering $\mathcal{W}_1$ such that $\overline{\mathcal{W}_1}$ is a locally finite refinement of $\mathcal{U}_1$.

Suppose inductively, that locally finite coverings $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n$, such that the family $\{\bigcup_{i=1}^{n} \mathcal{W}_i\}$ is fine and mesh $\mathcal{W}_i < \frac{1}{i}$, $1 \le i \le n$, have already been constructed. Then we construct the covering $\mathcal{W}_{n+1}$ in the following way. Consider a locally finite open covering $\mathcal{U}_{n+1}$ of X of mesh less than $\frac{1}{n+1}$ and the locally finite covering $\mathcal{F}_{n+1}$ of X, consisting of the closures of the elements of the covering $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_n$. By Lemma 2.1, there exists a covering $\mathcal{W}_{n+1}$ such that $\overline{\mathcal{W}_{n+1}}$ is a locally finite refinement of $\mathcal{U}_{n+1}$ and the covering $\mathcal{F}_{n+1} \cup \mathcal{W}_{n+1}$ is fine. Then the covering $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_{n+1}$ is also fine. By induction we can conclude that the coverings $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of the space X have been constructed and that $\mathcal{W} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$ is a fine open basis of X. $\square$

Recall that simplicial complex P is said to be *full* if for every finite set of vertices of the complex P there exists a simplex of P which is spanned by these vertices (see e.g. [9; p. 101]).

PROOF of Theorem 1.4. Consider any point $x \in X$. Let $\mathcal{U}_x$ be the family of all elements of the basis $\mathcal{U}$ which contains the point x (hence $\bigcap \mathcal{U}_x = \{x\}$). To $\mathcal{U}_x$ there corresponds a full subcomplex N_x of the nerve $\mathcal{N}(\mathcal{U})$. By hypothesis of the theorem, there exists a simplicial isomorphism $\varphi : \mathcal{N}(\mathcal{U}) \to \mathcal{N}(\mathcal{V})$. The image $\varphi(N_x)$ is a full subcomplex of $\mathcal{N}(\mathcal{V})$ and to it there corresponds a system $\mathcal{W}$ of elements of the basis $\mathcal{V}$, having the finite intersection property.

Since the space Y is compact and Hausdorff and the basis $\mathcal{V}$ is fine, the intersection of the closures of the elements of the system $\mathcal{W}$ consists of just one point. Indeed, by compactness, this intersection is nonempty; if it contained more than one point then there would be elements V_1 and V_2 of the basis $\mathcal{V}$ which would not intersect each other but would intersect with all elements of the system $\mathcal{W}$. Then the elements U_1 and U_2 of the covering

$\mathcal{U}$ which correspond to V_1 and V_2, respectively, would not intersect each other. However, $x \in \overline{U_1} \cap \overline{U_2}$, so $\overline{U_1} \cap \overline{U_2} \neq \emptyset$. Contradiction.

Since x is an arbitrary point of X, we obtain a mapping $f : X \to Y$. It follows immediately from the construction of f that this mapping is injective. Let us show that it is also surjective. Indeed, let $y \in Y$ be an arbitrary point and consider the set $\mathcal{V}_y = \{V \mid V \in \mathcal{V} \text{ and } y \in V\}$. Take any $x \in \bigcap\{\overline{\varphi^{-1}(V)} \mid V \in \mathcal{V}_y\}$ and pick an arbitrary $U \in \mathcal{U}$ such that $x \in U$. Then $\varphi(U) \cap V \neq \emptyset$, for every $V \in \mathcal{V}_y$, where by $\varphi(U)$ we mean the element of the basis $\mathcal{V}$ which by φ corresponds to U. It follows that $y \in \overline{\varphi(U)}$ and $f(x) = y$, as asserted.

By definition, $f(x) = \{\bigcap \overline{\varphi(U)} \mid U \in \mathcal{U}_x\}$. Consider the system

$$\{Y \setminus V\} \cup \{\overline{\varphi(U)} \mid U \in \mathcal{U}_x\}.$$

Since Y is compact and $(Y \setminus V) \cap \{\bigcap \overline{\varphi(U)} \mid U \in \mathcal{U}_x\} = \emptyset$, there exists a finite system of elements $U_1, U_2, \ldots, U_n$ of the basis $\mathcal{U}_x$ such that

$$(Y \setminus V) \cap \left\{ \bigcap_{i=1}^{n} \overline{\varphi(U_i)} \,\Big|\, U_i \in \mathcal{U}_x \right\} = \emptyset$$

(the system does not have the finite intersection property). Therefore $\bigcap_{i=1}^{n} \overline{\varphi(U_i)} \subset V$.

Consider now any point $x' \in \bigcap_{i=1}^{n} U_i$. By definition, $f(x') = \{\bigcap \overline{\varphi(U)} \mid U \in \mathcal{U}_{x'}\}$. Hence $f(x') \in \bigcap_{i=1}^{n} \overline{\varphi(U_i)}$ and $f(\bigcap_{i=1}^{n} U_i) \subset \bigcap_{i=1}^{n} \overline{\varphi(U_i)} \subset V$, i.e. f is continuous. Since X is compact and Y is Hausdorff it follows that f is a homeomorphism. $\square$

Remark. We observe that Theorem 1.4 is not valid if the space Y fails to be compact, as the following simple example shows: Let X be the unit segment $[0, 1]$ and let $\mathcal{U}$ be any fine basis of connected sets on X. Let $Y = X \setminus \{x\}$, where $x \in (0, 1)$ is any point, and define $\mathcal{V} = \{U \cap (X \setminus \{x\}) \mid U \in \mathcal{U}\}$. Obviously, $\mathcal{N}(\mathcal{U}) = \mathcal{N}(\mathcal{V})$ and $\mathcal{V}$ is a fine basis. However, X is not homeomorphic to Y.

PROOF of Theorem 1.5. Let (X, τ) be a separable metric space and (M, τ_μ) any countable dense subspace without isolated points. Consider the following equivalence relations on τ and τ_μ. Two open sets in a space will be called *equivalent* if their closures are equal. Since (X, τ) and (M, τ_μ)

do not contain isolated points, any equivalence class has the power of continuum. Indeed, in every space with countable basis, the cardinality of the set of all open subsets is at most the continuum. Since our spaces have no isolated points, there exists in every open set a convergent sequence of distinct points. The set of all subsequences of this sequence has the cardinality the continuum. The complements of these subsequences, together with the limit point, are equivalent, and have the cardinality at least the continuum.

Consider any such equivalence class $[U]$ in τ. Associate to it the class $[U \cap M]$ in τ_μ. Obviously, this correspondence is well-defined and it is a bijective mapping of all equivalence classes of τ to all equivalence classes of τ_μ. Since all equivalence classes have cardinality the continuum, we can fix, for every class $[U]$, a bijection from $[U]$ to $[U \cap M]$. In this way we get a bijective mapping from the set of all vertices of $N(\tau)$ to the set of all vertices of $N(\tau_\mu)$.

Let us now prove that this bijective mapping can be extended to a simplicial isomorphism from $\mathcal{N}(\tau)$ to $\mathcal{N}(\tau_\mu)$. It suffices to prove that, if to the elements $U_0, U_1, \ldots, U_n$ of τ there correspond $V_0, V_1, \ldots, V_n$ of τ_μ, then $\bigcap_{i=0}^n U_i \neq \emptyset$ if and only if $\bigcap_{i=0}^n V_i \neq \emptyset$.

Suppose first that $\bigcap_{i=0}^n U_i \neq \emptyset$. Since M is dense in X it follows that $\bigcap_{i=0}^n (U_i \cap M) \neq \emptyset$. Since $\overline{U_i \cap M} = \overline{V_i}$ in (M, τ_μ) it follows by Lemma 2.2 that $\bigcap_{i=0}^n V_i \neq \emptyset$.

Suppose now that $\bigcap_{i=0}^n V_i \neq \emptyset$. Since $\overline{U_i \cap M} = \overline{V_i}$ in (M, τ_μ), it follows by Lemma 2.2 that $\bigcap_{i=0}^n (U_i \cap M) \neq \emptyset$ and hence also $\bigcap_{i=0}^n U_i \neq \emptyset$.

So we can conclude that $\mathcal{N}(\tau) = \mathcal{N}(\tau_\mu)$. Now, let $(\mathcal{M}_1, \tau_{\mu_1})$ and $(\mathcal{M}_2, \tau_{\mu_2})$ be countable dense subspaces of (X_1, τ_1) and (X_2, τ_2), respectively. By the Sierpinski theorem [10], all countable metrizable spaces without isolated points are homeomorphic. Therefore we have the following isomorphisms $\mathcal{N}(\tau_1) \approx \mathcal{N}(\tau_{\mu_1}) \approx \mathcal{N}(\tau_{\mu_2}) \approx \mathcal{N}(\tau_2)$. $\square$

4. Epilogue

It follows from Theorem 1.3 that the nerve of a fine basis of a compact Hausdorff space contains all information about its topology.

Question 4.1. *Which topological invariants (e.g. dimension, metriz-ability, Čech cohomology group, ...) have nice descriptions in terms of nerves of fine coverings?*

As it was observed in the introduction, every compact Hausdorff space X of weight $\omega(X) \leq \aleph_1$ has a fine basis. We say that a basis $\mathcal{U}$ is an f-basis if the natural mapping $\mathcal{N}(\mathcal{U}) \to \mathcal{N}(\overline{\mathcal{U}})$ is a simplicial isomorphism.

Question 4.2. *Does every normal space of weight $\omega \leq \aleph_1$ have an f-basis?*

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