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PROCEEDINGS OF THE GRADUATE WORKSHOP IN MATHEMATICS AND ITS APPLICATIONS IN SOCIAL SCIENCES

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PREFACE

The cooperation between Slovene and Soviet mathematicians already has a long history. The first contacts go back to 1960's when France Križanič visited the Moscow State University. In the 1970's, first Žiga Knap, then Dušan Pagon and lastly, Dušan Repovš followed the suit.

The first research institution to start joint mathematical research with the Soviets was our Institute for Sociology — the team lead by Stane Saksida and Žiga Knap has been cooperating with the Moscow State University for over 15 years by now.

The Institute of Mathematics, Physics and Mechanics of the University of Ljubljana at present has two long term programs of cooperation with the Steklov mathematical Institute of the Soviet Academy of Sciences — through the kind of cooperation the Slovene Academy of Arts and Sciences. The first one is the Complex Analysis and it is directed by Josip Globevnik whereas the second one is the Geometric Topology and is directed by Dušan Repovš.

Towards the end of the year 1990 we started our initiative with the Slovene Ministry of Science and Technology. A delegation consisting of Valerij B. Kudryavcev, Žiga Knap, Stane Saksida, and Dušan Repovš held a meeting with Ciril Baškovič, Councelor to the Minister, in which the idea of holding a graduate workshop in mathematics and its applications in social sciences was first presented.

The idea of organizing such a workshop was based on our common goal to promote, broaden and intensify the cooperation particulary between Ljubljana and Maribor on our side and Moscow and Tbilisi on the other side.

The democratic changes which have taken part in both countries recently have made such plans far more realistic than they would have been some years ago. Slovenia, on its way to independence and international recognition, was seeking new avenues of quality international cooperation. No longer shall we have to send all our project proposals to the Belgrade and wait (sometimes for years) to have the results of the Belgrade-Moscow negotiations, which for most of the time resulted in minimal concessions for Slovenian partners.

On the other hand, Soviet Union has been increasingly opening up to the world. For the Soviet science this is a great chance but also a great hazard. On one hand the Soviet scientists can now freely travel abroad. However, too many have already failed to return home, having extended their visit abroad indefinitely. Now is therefore the time when a meaningful bilateral cooperation between the two countries can at last be organized in such a way that both parties see some benefit in it.

It was this main idea, which led us to set up the framework for such a workshop. It's main goal was to present to our graduate students of mathematics and (to a slightly lesser degree) to our research community, the topics in theoretical mathematics as well as in its applications in which a quality joint research is already being done in the network Ljubljana-Maribor-Moscow-Tbilisi. It is our hope that the future cooperation would first expand along those lines and then include all subjects as the number of participating scientists would grow.

Another important idea was intensively discussed with the Ministry — the idea of opening one or two positions at the Universities of Ljubljana and Maribor for Visiting Professors from the Soviet Union. Such a visitor would not only give a yearlong graduate course for our students of mathematics on some vital subject of modern mathematics but also actively join in the research of our Institute for Mathematics, Physics an Mechanics and the Institute for Social Sciences (formely Institute for Sociology). We are very pleased that also this idea was met with an overall approval of the Ministry.

The Graduate Workshop in Mathematics and Its applications in Social Sciences was held in Ljubljana from 23 to 27 September 1991. We have invited some leading Soviet mathematicians to present one-hour lectures on a very broad variety of subjects. Joining in from the Slovenian side, several of our mathematicians were also invited to present general surveys on the subject of their speciality.

The organisation of the workshop became an almost impossible task after the sudden June agression of the Federal army on the independent Republic of Slovenia, followed by the breakout of the Serbo-Croatian war. As a result, the lines of communication between Slovenia and Soviet Union were almost completely cut. The closure of airports in Slovenia and Croatia as well as the blockade of the international trains made it virtually impossible for our invited speakers to come to Ljubljana. Nevertheless, they did and the workshop could be run as scheduled. One should note, that this was the first international scientific meeting in Slovenia after the Declaration of Independence — all others, scheduled for this period were either canceled or postponed or moved to a location outside Slovenia.

All invited speakers prepared their contributions for these Proceedings. The Editor wishes to acknowledge their kind cooperation as well as the help of the anonymous referees which critically examined all the manuscripts and in some instances helped to improve the exposition. We also wish to express our acknowledgements to the Slovene Minister for Science and Technology Peter Tancig for his enthusiastic support of the workshop and his active participation at the opening ceremony. We have also received the media attention we usually do not expect for mathematics and it is our hope that this will benefit the promotion of Slovene mathematics. Our thanks also go to Minister's councelor Ciril Baškovič who was instrumental in the outlining the framework of the workshop. Finally, we acknowledge the kind cooperation of Ministry's staff in the technical realization of the workshop, in particular the help of Sonja Štamcar. The minute details of preparing the Proceedings were supervised by Ciril Velkovrh whose help was, as always, instrumental for the successful completion of the job.

Ljubljana, 31 October 1991 *The Editor*

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CONTENTS

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RECENT DEVELOPMENTS IN THE THEORY OF VARIETIES OF LIE ALGEBRAS AND LIE SUBALGEBRAS

YU. BAHTURIN

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Key words: Lie algebra, colour Lie superalgebra, the variety of Lie algebras or superalgebras, identical relation, the basis of the variety, the growth of the variety, finiteness conditions.

Abstract. In the introduction we give the basical definitions of colour Lie superalgebra and the variety of Lie algebras or superalgebras. Then the next problems connected with the varieties of Lie superalgebras induced by identity relations are discussed: finiteness of the variety bases, the growth of a variety, finiteness conditions on Lie algebras and special Lie algebras.

The theory of varieties of algebraic systems dates back to the thirties although a number of researches into this area had been made earlier without mentioning the word *variety*. This can be illustrated, of course, by W. Burnside's work on groups with the law $x^n = 1$ as well as by B. Nielsen and O. Schreier work on free groups. The same can be said about the research into varieties of Lie algebras where the important results of E. Witt, W. Magnus, M. Hall, A. I. Shirshov and A. I. Kostrikin now fitting into the framework of the theory had been obtained some decades before it became a theory in the proper sense.

The first books where the theory of varieties of Lie algebras was first dealt to a certain extent were R. Amayo and I. Stewart *Infinite-dimensional Lie algebras* (1974) and Y. Bahturin *Lectures on Lie algebras* (1978). In 1985 we published and in 1987 translated into English a monography entirely devoted to varieties of Lie algebras called *Identical Relations in Lie Algebras* (Nauka, Moscow 1985 ; VNU Science Press, Utrecht 1987). In 1989 Yu. P. Razmyslov published *Identities of Algebras and Their Representations* where he exposes his approach to the solution to a number of difficult problems on varieties of Lie and associative algebras.

By a Lie algebra we will mean a vector space over a field F endowed with a bilinear operation $(x, y) \mapsto [x, y]$ satisfying the identities

$[x, x] = 0$ implying the anticommutativity $[x, y] = -[y, x]$ (1)

and the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 . \tag{2}$$

A regular way for obtaining Lie algebras is to consider the bracket operation (called the commutator)

$$[a,b] = ab - ba \tag{3}$$

in an associative algebra A over F. A generalization of the class of Lie algebras can be obtained if we consider F-algebras graded by a commutative group G:

$$L = \bigoplus_{g \in G} L_g \tag{4}$$

The gradeness means that $[L_g, L_h] \subset L_{g+h}$. The generalization is achieved if we consider a bilinear alternating form

$$\varepsilon : G \times G \to F$$

and replace (1) and (2) by a set of identities for graded elements

$$[x,y] = -\varepsilon(g,h)[y,x] \tag{5}$$

$$[[x,y],z] = [x,[y,z]] - \varepsilon(g,h)[y,[x,z]], \quad x \in L_g,\ y \in L_h \tag{6}$$

If $G = \{0\}$ then $\varepsilon(0,0) = 1$ and (5), (6) amount to (1) and (2). An algebra (4) satisfying (5) and (6) is called a colour Lie superalgebra. As above, a regular way for obtaining colour Lie superalgebras is to considera G-graded associative algebra

$$A = \bigoplus_{g \in G} A_g$$

and to introduce the colour comutator by setting

$$[x,y] = xy - \varepsilon(g,h)yx \tag{7}$$

It has been shown by M. Scheunert that if we replace the operation of L in (4) by setting

$$[x,y]_\delta = \delta(g,h)[x,y], \quad x \in L_g,\ y \in L_h$$

then, for a suitable choice of two-cocycle $\delta : G \times G \to F$ it is possible to reduce to ordinary Lie superalgebras, i.e. with the grading group $\mathbf{Z}_2 = G/G_+$, $G_+ = \{g \mid \varepsilon(g,g) = 1\}$ and with the bilinear function $\bar{\varepsilon} : \mathbf{Z}_2 \times \mathbf{Z}_2 \to F$ given by $\varepsilon(1,1) = -1$. Ordinary Lie superalgebras were the class first considered in a detail beyond the class of usual Lie algebras (see [Berezin 83], [Kac 77], [Schneuert 73]).

Already long time ago N. Jacobson has noticed that some theorems about sets of operators closed under ordinary commutator $ab - ba$ (e.g. Engel's Theorem) remain valid if the ordinary commutator is replaced by a more general bracket. Recent developments show that this observation is a very general pattern although the exact transfer of theorems from Lie algebras to Lie superalgebras is by no means automatic and a number of theorems do not extend to the more general situation. A remarkable feature is that Lie superalgebras over fields of zero characteristic behave like modular Lie algebras, i.e. over fields with positive characteristic $p > 0$. Latest progress in this latter theory makes it possible to develop the theory of the new level.

We are not going to dwell here upon all questions of the theory of varieties of Lie algebras and superalgebras but rather survey some developments which recently have led to some interesting results.

To begin with we define the variety of Lie algebras over a field F as a class of all Lie algebras over F satisfying a fixed system of identical relations. For example, the class $\mathcal{A}$ of all abelian Lie algebras, i.e. with trivial bracket is the variety defined by the identical relation $[x, y] = 0$. Nilpotent Lie algebras are those defined by the law $[x_1, x_2, ..., x_{c+1}] = 0$. The class of such algebras is denoted by $\mathcal{A}_c$. Soluble Lie algebras satisfy on of the laws $\delta_n(x_1, ..., x_{2^n}) = = 0$ where $\delta_1(x_1, x_2) = [x_1, x_2]$ and

$$\delta_{n+1}(x_1, ..., x_{2^{n+1}}) = [\delta_n(x_1, ..., x_{2^n}), \quad \delta_n(x_{2^n+1}, ..., x_{2^{n+1}})] .$$

The notation for this class is $\mathcal{S}_n$. To add to the list of identical relations we mention two types of relations: the Negel identity $(\operatorname{ad} x)^n = 0$ where $\operatorname{ad} x$ is the operator sending each y into $[x, y]$, and the standard identity

$$\sum_{\sigma \in \mathcal{S}_n} (\operatorname{sgn} \sigma) \operatorname{ad} x_{\sigma(1)} \operatorname{ad} x_{\sigma(2)} \cdots \operatorname{ad} x_{\sigma(n)} = 0 \tag{8}$$

where $\mathcal{S}_n$ is the symmetric group of n symbols. The Engel identity with $n = p - 1$, p a prime, holds in Lie algebras of groups with identity $x^p = 1$. The standard identity (8) holds in $(n-1)$-dimensional Lie algebras.

An important class of Lie algebras with identities can be obtained if we consider special Lie algebras, i.e. Lie subalgebras under the bracket operation in associative polynomial identity (PI) Lie algebras. This class is often denoted by SPI.

In the case of graded algebras the identities are graded, that is, they have the form

$$f(x_1, x_2, ..., x_n) = 0$$

where $x_1 \in X_{g_1}$, $x_2 \in X_{g_2}, ..., x_n \in X_{g_n}$, $g_1, g_2, ..., g_n \in G$ (we have

$$X = \bigcup_{g \in G} X_g$$

in thecase of graded algebras). This relation holds in a Lie superalgebra (4) if for any $a_1 \in L_{g_1}$, $a_2 \in L_{g_2}, ..., a_n \in L_{g_n}$ we have

$$f(a_1, a_2, ..., a_n) = 0 .$$

According to G. Birkhoff's Theorem a non-empty class of algebras is a variety if, and only if, it is closed under taking subalgebras, quotient-algebras (factor-algebras) and Cartesian products of their members. One of the methods of defining varieties is to consider varieties generated by one or more Lie algebras, i.e. consisting of all algebras satisfying all the identities holding in this or those algebras. A variety generated by a Lie algebra L is denoted by var L.

An important concept is that of the free algebra of a variety $\mathcal{V}$ with free generating set X. This algebra $L = L(X, \mathcal{V})$ is the unique (up to isomorphism) Lie algebra such that any mapping $\phi : X \to M \in \mathcal{V}$ extends uniquely to a homomorphism $\bar{\phi} : L \to M$. A free algebra of the variety $\mathcal{O}$ of all Lie algebras (over F) is called a free Lie algebra (with free generating set X) denoted by $L(X)$.

There are a number of questions concerning varieties which are under consideration now.

1. Finite Basis Problem. It is not a completely solved problem whether any set of identical relations of a Lie algebra over a field F is equivalent to its finite subset? The first succes was achieved in 1970 when M. R. Vaughan-Lee and later V. S. Drensky showed that over prime characteristic fields there exist varieties (and even finite-dimensional algebras, provided that F is not finite) which do not admit finite bases for their laws. In the decades to follow various authors have been proving some theorems in the positive that have culminated in a remarkable result of A. V. Il'tyakov (see [Il'tyakov 91a,b]) who has proved that any finitely generated special Lie algebra over any zero characteristic field has finite basis for its laws. In particular, any finite-dimensional Lie algebra over a field of characteristic zero is finitely based. (The case of finite-dimensional soluble Lie algebras over zero characteristic fields was first settled by A. N. Krasil'nikov.) A. V. Il'takov's result is based on the techniques developed by A. R. Kemer in his proof of the Specht property for arbitrary varieties of associative algebras over zero characteristic fields. Thus, the most important problem in this area now is to settle the finite basis problem for algebras which do not admit finite generation or which are not special. It is not obvious that the answer is in the positive. It is important to remark that in many a situation the associative and Lie algebras behave quite differently. This is the case, for example, in the considerations concerning the growth of varieties and their free algebras.

2. Growth of varieties. Given a set X we consider the monoid $W(X)$ of all words in the alphabet X under juxtaposition of words and linear space $A(X)$ with basis $W(X)$ and the same operation expanded by distributivity. Clearly, $A(X)$ is an associatove algebra and it is free associative in the sense that every mapping of X into an associative algebra uniquely extends to a hoomorphism of associative algebras. The elements of $A(X)$ are called (noncommutative) polynomials. A polynomial which is a linear combination of the words of the same degree (=length) n is called homogenuous of degree

n (including zeropolynomial) is denoted by $A_n(X)$. Thus we have

$$A(X) = \sum_{n=1}^{\infty} A_n(X)$$

If we fix $x_1, ..., x_n \in X$ then an element $a \in A_n(X)$ is called multilinear if it depends on every variable $x_1, ..., x_n$. The subspace of all such polynomials is denoted by $P(x_1, ..., x_n)$.

A remarkable fact is that $A(X)$ is the universal enveloping algebra for the free Lie algebra $L(X)$. It is equivalent to say that under the commutator operation (3) the subset X of $A(X)$ generates a Lie algebra isomorphic to $L(X)$. There is a colour (and super) analog of this result.

Thus $L_n(X) = L(X) \cap A_n(X)$ is the set of all homogenuous Lie polynomials of degree n and

$$PL(x_1, ..., x_n) = P(x_1, ..., x_n) \cap L(x_1, , ..., x_n)$$

is the set of all multilinear polynomials in the variables $x_1, ..., x_n$.

It is a simple result that any identical relation $v = 0$ has a consequence $w = 0$ where w is multilinear of the same degree as v. If F is of zero characteristic then the converse is also true i.e. every system of identities is equivalent to a multilinear system, that is, with all member multilinear. Thus the system of multilinear identities holding in a variety $\mathcal{V}$ is its important characteristic. In fact, it turned out that it is more imoportant to consider the sequence $\{c_n(\mathcal{V})\}$ with

$$c_n(\mathcal{N}) = \dim PL(x_1, ..., x_n)/V(x_1, ..., x_n)$$

where $V(x_1, ..., x_n)$ is the set of multilinear identities in the variables $x_1, ..., x_n$ holding in $\mathcal{V}$.

The sequence $c_n(\mathcal{V})$ belongs to one of three classes as follows:

(1) at most polynomial, i.e. with $c_n(\mathcal{V}) \leq f(n)$ where $f(t)$ is polynomial, for all n;

(2) at most exponential, i.e. $c_n(\mathcal{V}) \leq Cd^n$ for some d;

(3) superexponential, i.e. not fitting into any of the cases (1) or (2).

S. P. Mischenko has shown that there exist no varieties of intermediate growth.

An important theorem of A. Regev says that for a proper variety of associative algebras its growth is at most exponential (a variety is called proper if it is different from the variety of all associative algebras). This important property does not hold for Lie algebras. For example, take $\mathcal{V}$ given by

$$[x_1, x_2, x_3], [x_4, x_5, x_6]] = 0 \tag{9}$$

(Here we use so called right-normed notation of commutators:

$$[x_1, x_2, ..., x_n] = [x_1, [x_2, ..., x_n]]\,.)$$

There are a number of interesting results concerning the growth of varieties. For instance, varieties with polynomial growth over fields of characteristics zero have finite basis property and consist of algebras with nilpotent commutator subalgebra (the linear span of all commutators). Every non-polynomial itself contains a just-non-polynomial variety, i.e. one which is not polynomial itself but with every proper subvariety polynomial. Thus the knowledge of just-non-polynomial varieties enables to determine whether a variety under consideration is polynomial or not. No complete list of such varieties in the general case is known. But in the case of soluble Lie algebra there is a very nice theorem of S. P. Mischenko (see his survey of 91). In the nonsoluble case there is only one just-non-polynomial variety known, viz., var gl_2, generated by 2 by 2 trace zero matrices which we denote by $\mathcal{V}_0$. To introduce the soluble ones we introduce three algebras in the form of matrices. Thus we set

$$A=\begin{pmatrix}\mathcal{I} & \mathcal{I}\\ 0 & 0\end{pmatrix}$$

where $\mathcal{I}$ is the infinite-dimensional Grassmann algebra. Also we set

$$B=\left\{\begin{pmatrix}g & f\\ 0 & 0\end{pmatrix}\,\middle|\, g\in P,\ f\in F[t]\right\},$$
$$C=\left\{\begin{pmatrix}g & f\\ 0 & 0\end{pmatrix}\,\middle|\, g\in Q,\ f\in F[t]\right\}$$

where P is the two-dimensional non-abelian Lie algebra, Q three-dimensional nilpotent non-abelian Lie algebra and $F[t]$ is a natural P- and Q-module, respectively. The operation is given by

$$\left[\begin{pmatrix}g & f\\ 0 & 0\end{pmatrix},\begin{pmatrix}g_1 & f_1\\ 0 & 0\end{pmatrix}\right]=\begin{pmatrix}[g,g_1] & g\circ f_1-g_1\circ f\\ 0 & 0\end{pmatrix}.$$

Now we set $\mathcal{V}_1=A$, $\mathcal{V}_2=B$, $\mathcal{V}_3=C$. We also define $\mathcal{V}_4$ as the variety given by the law

$$[[x_1,x_2],[x_3,x_4],[x_5,x_6]]=0$$

i.e. consisting of algebras with nilpotent commutator subalgebra of nilpotent class two.

Theorem. *Varieties $\mathcal{V}_i$, $i=1,2,3,4$, form the list of all just-non-polynomial soluble varieties over any field of characteristic zero.*

The consequences of the result include the description of soluble varieties whose lattice of subvarieties is distributive, that is, satisfies the identity

$$\mathcal{U}\cap(\mathcal{V}\cup\mathcal{M})=(\mathcal{U}\cap\mathcal{M})\cup(\mathcal{U}\cap\mathcal{V}).$$

Here we come to another piece of techniques of importance for varieties of algebras, not necessarily Lie algebras. Namely, the set $PL(x_1,\ldots,x_n)$ of multilinear Lie polynomials admits the natural action of the symmetric group

S_n (permutation of indices). The lattice of S_n-submodules of $PL(x_1, ..., x_n)$ is anti-isomorphic to the lattice of varieties given by homogenuous identities of degree n. The structure of this module is given in a theorem of A. A. Klyachko (see [Bahturin 85, Chapter 3]). Thus, in the study of varieties of Lie algebras an important role is played by the theory of representations of the symmetric group, e.g. Young diagram, etc. This is true in the case of zero characteristic of the base field and one of the problems in this area (there are many of them) is to apply representation theory in the prime characteristic case as well. Returning to the distributivity we denote by $PV(x_1, ..., x_n)$ the quotient space $PL(x_1, ..., x_n)/V(x_1, ..., x_n)$consisting of polynomials vanishing on V. Then the lattice of subvarieties of $\mathcal{V}$ is distributive if, and only if each $PV(x_1, ..., x_n)$ has no multiple irreducible S_n-submodules of multiplicity greater than 1. An open problem is to describe all varieties with distributive lattice of subvarieties. A corollary of the above theorem says that either $\mathcal{V}$ with such a lattice is polynomial or it is equal to one of the varieties $\mathcal{V}_1$, $\tilde{\mathcal{V}}_1$ where $\tilde{\mathcal{V}}_1$ is an explicitly defined extension of $\mathcal{V}_1$.

Turning out attention to varieties of Lie superalgebras we have to mention that we have no analog of A. V. Il'tyakov's result in the case of Lie superalgebras. However, many partial results on the posiitive solution of the finite basis problem can be extended to this case. To formulate a result of this kind we define the concept of the product of varieties if we define $\mathcal{UV}$ as the class of all Lie (super) algebras L with an ideal $\mathcal{I}$ in $\mathcal{U}$ and the quotient-algebra $L/\mathcal{I}$ in $\mathcal{V}$. We also say that a variety $\mathcal{V}$ has finite axiomatic rank if it can be given by a set of laws depending on a fixed finite set of variables. A result we want to formulate is due to V. V. Stovba and it says that over a field of sufficiently many elements any subvariety of finite axiomatic rank in a variety $\mathcal{M}_c\mathcal{M}_d$ is finitely based.

A complete picture of subvarieties does exist in the case of two-step soluble (so called metabelian) varieties over fields of characteristic zero. Here we can consider colour Lie superalgebras with arbitrary finite grading group G and bilinear form $\varepsilon : G \times G \to F$. This information is contained in the joint paper of V. S. Drensky and the author (see the monograph on infinite dimensional Lie superalgebras in the list of references). It might be of interest to give here one formula related to so called Hilbert series of free algebras of varieties. We know that given a variety $\mathcal{V}$ the free algebra $L = L(X, \mathcal{V})$ decomposes into the direct sum of homogenuous components $L = \bigoplus_{n=1}^{\infty} L_n$. If $d_n = \dim L_n$ then

$$H(L,t) = \sum_{n=1}^{\infty} d_n t^n$$

is called the Hilbert series of L. Now let

$$G_+ = \{g \mid \varepsilon(g,g) = 1\},\ G_- = \{g \mid \varepsilon(g,g) = -1\},$$
$$X_+ = \{x_g \mid g \in G_+\},\ X_- = \{x_h \mid h \in G_-\},\ \ X = X_+ \cup X_-$$

If $\mathcal{A}^2$ is the variety of all metabelian Lie algebras $|X_+| = m$, $|X_-| = n$, $L = L(X, \mathcal{A}^2)$ then we have

$$H(L,t) = 1 + (m+n)t + ((m+n)t - 1)\frac{(1-t)^n}{(1-t)^m}.$$

We also proved that the variety under consideration is Specht, that is, each of its subvarieties is finitely based.

It is of interest to try to solve the rationality problem, i.e. to determine the varieties for which the Hilbert series of free algebras of finite rank are rational, that is, can be presented as the ratio of two polynomials. V. S. Drensky has proved some results in the case of varieties with polinomial growth.

3. Finiteness conditions nad varieties of Lie algebras and Lie superalgebras. A condition on an algebra is called a finiteness condition provided it holds for all finite-dimensional algebras and there exists an infinite-dimensional algebra with this condition. The most important finiteness conditions are probably the following.

(i) Local finiteness. A Lie Algebra L over a field F is called locally finite if every finite subset of L lies in a finite-dimensional subalgebra.

(ii) Residual finiteness. A Lie algebra L over a field F is called residualy finite if for every $x \in L$, $x \neq 0$, there exists a homomorphism $\phi : L \to M$ where M is finite-dimensional and $\phi(x) \neq 0$ in M. A finitely generated residual finite Lie algebra has algorithmically solvable word problem.

(iii) Representability. A Lie algebra L over a field F is called representable if there exists a field extension $K \supset F$ such that the extended algebra $K \bigotimes_F L$ can be embedded in a finite dimensional algebra M over K. In other words L can be represented by matrices of finite order over K.

(iv) Noetherian property. A Lie algebra L over a field F is called Noetherian if any ascending chain of subalgebras in L has finite length. We say that L is weakly Noetherian provided that any ascending chain of ideals of L is of finite length.

(v) Artinian property. The same as just above (iv) but with „ascending" replaced by „descending".

One of the most popular questions is to find out whether a given identical relation ensures the local finiteness of respective algebras or not (in this case we simply speak about the local finiteness of the variety). The most important achievement in this area has become the result of E. I. Zel'manov according to which any Lie algebra with the Engel condition $(\operatorname{ad} x)^n = 0$ is locally finite (and, according to Engel's Theorem, locally nilpotent) solving a problem which remained open for some decades. (An important monograph in this area is A. I. Kostrikin's *Around Burnside* of 1986.)

The most extensive study of locally finite varieties exists in the case of finite fields of coefficients and was initiated in the joint paper of A. Yu. Ol'shansky and the author where we proved that finite Lie algebras have finite bases for their laws. One of the most impressive recent achievements in this area is a theorem of A. A. Premet and K. N. Semyonov in which the authors describe residually finite varieties of Lie algebra. Using the techniques developped in this paper and those arisen in the study of Engel Lie algebras K. N. Semyonov later was able to explicitly write up a basis for the laws of $\mathrm{sl}_2(F)$ with F arbitrary finite of characteristic at least 5.

Theorem. *Let F be a finite field of characteristic $p > 3$. A variety $\mathcal{V}$ over F is residually finite if and only if it is generated by a single finite-dimensional algebra with all nilpotent subalgebras abelian.*

The cases of char $F = 2, 3$ remain so far open.

An easy observation shows that all algebras of a variety over an infinite field are residually finite if, and only if, the variety is abelian. Thus, in the case of infinite field, it is more appropriate to study locally residually finite varieties, i.e. with finitely generated subalgebras residually finite. The first non-trivial variety with this property was found by the author in 1972 when it was shown that the variety $\mathcal{A}^2$ of all metabelian Lie algebras over an arbitrary field is localy residually finite. We have also produced an example showing that the variety of cetre-by- metabelian Lie algebras, i.e. with the quotient-algebra over the centre metabelian, does not enjoy the property under consideration. I. B. Volichenko has shown (in all cases except char $F = 2$) that a variety does not contain the variety of all centre-by-metabelian Lie algebras if, and ony if, it satisfies a non-trivial identity of the form

$$(\mathrm{ad}\, x)(\mathrm{ad}\, y)^n + \sum_{k=1}^{n} \alpha_k (\mathrm{ad}\, y)^k (\mathrm{ad}\, x)(\mathrm{ad}\, y)^{n-k} = 0$$

In 1988 and later M. V. Zaĭcev showed the above identity is crucial in determining whether a variety has one of the finiteness conditions listed some paragraphs earlier for finitely generated algebras. Thus he has proved the following.

Theorem. *Let F be a field of characteristic zero. Then the following conditions are equivalent:*

(i) $\mathcal{V}$ is a locally residually finite variety.
(ii) $\mathcal{V}$ is a locally representable variety.
(iii) $\mathcal{V}$ satisfies some identity as above and every finitely generated algebra in $\mathcal{V}$ has nilpotent commutator subalgebra.

In the case of finite fields of positive characteristic the identity under consideration is equivalent to (i), (ii) and to some other finiteness conditions even without the nilpotency of the commutator subalgebra.

In the joint paper of M. V. Zaĭcev and the author the above result was generalized to the case of Lie superalgebras, sometimes even colour superalgebras. Precise formulations and the proofs can be found in [Bahturin et al. 91].

A particularly interesting question is to find matrix representations for free algebras of certain varieties of Lie algebras and Lie superalgebras. Some examples have been dealt with in earlier papers of D. I. Eidel'kind.

4. Special Lie algebras. Special Lie algebras form a class which generalizes finite-dimensional algebras (as previously, this is a finiteness condition, due to Ado-Iwasawa's Theorem). It has been considered in a detail in the author's monograph of 1985. A question which remains unsettled in this book was V. N. Latyshev's problem on the homomorphic image of a special Lie algebra: is it always special itself? Due to a result of S. A. Pikhtil'kov this question turned out to be equivalent (if char $F = 0$) to a question of the author: is it true that a central extension of an SPI-algebra is SPI itself? Thus, in the course of a seminar for foreign graduates of Moscow University, we produced an example of an SPI algebra whose central extension, presumably, was not special. It turned out to be isomorphic to the commutator subalgebra of the well-known Kac-Moody affine algebra $A_1^{(1)}$. But only in 1988 using our theorem of 1985 on the structure of PI-envelopes of semisimple Lie algebras Yu. V. Billig found an elegant proof of non-speciality of the algebra in question. In fact, he showed that all affine Lie algebras over fields of characteristic zero are not special. However Yu. V. Billig has proved that affine algebras over prime characteristic fields are special. This became an impetus to his describing in 1990 the important class of affine modular Lie algebras. An important question, therefore, is about the homomorphic image of special Lie algebras over prime characteristic fields. Another question of interest is to describe special varieties, that is, generated by special Lie algebras. It is known that customary operations over special varieties seldom lead to special varieties. Also, Yu. V. Billig's results have proved the existence of special varieties with non-special members. But it was shown that all algebras in $\mathcal{V}_0 = \operatorname{var} \mathrm{sl}_2$ have associative envelopes with identities of the matrix algebra of order 2. It would be of interest to prove that all algebras in $\mathcal{V}_1$ are special.

The bibliography that follows is by no means complete. Moreover, in many places in the text of the talk we omitted references. Normally, these can be recovered in one of the monographs [Bahturin 85] or [Bahturin et al. 91].

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Yu. Bahturin
Moscow State University,
Moscow, Russia

Graduate Workshop in Mathematics
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О τ-ПОЛНОТЕ В КЛАССЕ ДЕТЕРМИНОВАННЫХ ФУНКЦИЙ

В. А. БУЕВИЧ

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Abstract. In the article the author describes a solution to the completeness problem for the class of maps of determined functions of words from finite alphabets. The solution consists in the effective description of all τ-precomplete classes.

В заметке решается задача о полноте в классе отображений, осыществляемых детерминоваными функциями на словах длины τ, составленных из букв произвольного конечного алфавита. Это достигается путем эффективного описания множества всех τ-предполных классов. Все неопределяемые в дальнейшем понятия можно найти в [1,2,3].

1. Пусть $k \geq 2$, $E_k = \{0, ..., k-1\}$. Пусть $t \geq 1$, и E_k^t — множество слов длины τ, составленных из элементов E_k. Каждое такое слово a будем представлять в виде $a = (a(1)...(a(t))$. Пусть $E_k^* = \bigcup\limits_{t=1}^{\infty} E_k^t$. Пусть P_g^k — множество всех детерминированных функций, которые зависят от переменных, принимающих значения из E_k^*. На P_g^k обычным образом можно ввести операции суперпозиции, а также понятие замыкания подмножеств P_g^k отностельно этих операций [1,2]. Пусть $\tau \geq 1$. Пусть $\mathcal{M} \subseteq P_g^k$. Множество $\mathcal{M}$ называется *τ-полным*, если для всякой функции $f(x_1, ..., x_n)$ из P_g^k в замыкании $\mathcal{M}$ содержится функция $g(x_1, ..., x_n)$ такая, что для любого набора $(a_1, ..., a_n)$ элементов из $E_k^\tau f(a_1, .., a_n)$ совпадает с $g(a_1, ..., a_n)$. Множество $\mathcal{N} \subset P_g^k$ называется *τ-предполным классом*, если $\mathcal{N}$ не является τ-полным, но для любой функции $f \in P_g^k / \mathcal{N}$ множество $\mathcal{N} \cup \{f\}$ τ-полно. Используя [1,2,3] легко показать, что множество всех τ-препдолных классов образует *минимальную критериальную систему* для распознавания τ-полноты произвольных множеств детерминированых функций. В частитости, при любых $k \geq 2$ минимальная 1-критериальная система изоморфна множеству предполных классов в k-значных логиках [1,4,5].

Пусть $p \geq 1$. Пусть $T = (t_1, ..., t_p)$ — произвольный набор целых неотрицательных чисел. Пусть $E_k^T = E_k^{t_1} \times \cdots \times E_k^{t_p}$. Любое непустое подмножество $R \subseteq E_k^T$ называется отношением, заданным на E_k^T, а число p — арностью этого отношения. Функция $f(x_1, ..., x_n)$ из P_g^k сохраняет отношение R, если для любой совокупности

$$\{(a_1^1, ..., a_p^1), ..., (a_1^n, ..., a_p^n)\}$$

наборов из R набор $(f(a_1^1, ..., a_1^n), ..., f(a_p^1, ..., a_p^n)\}$ также принадлежит R. Множество всех функций, сохраняющих отношение R, обозначим $\mho(R)$. В терминах сохранения отношений дадим описание множества всех τ-предполных классов.

Пусть $k \geq 2$. Определим функцию π, отображеающую $E_k^* \times E_k^*$ на $\{0, 1, ...\}$. Пусть $a_1 \in E_k^{t_1}$, $a_2 \in E_k^{t_2}$. Пусть $t = \min\{t_1, t_2\}$. Тогда $\pi(a_1, a_2) = 0$, если $a_1(1) = a_2(1), ..., a_1(t) = a_2(t)$; $\pi(a_1, a_2) = i$ $(1 \leq i \leq$ $\leq t - 1)$, если $a_1(1) = a_2(1), ..., a_1(t - i) = a_2(t - i)$, но $a_1(t - i + 1) \neq$ $= a_2(t - i + 1)$; $\pi(a_1, a_2) = t$, если $a_1(1) \neq a_2(1)$. На множествое E_k^T определим отношение предпорядка »$\preceq$« [6]. Пусть $A = (a_1, ..., a_p)$, $A' = (a_1', ..., a_p')$ — элементы из E_k^T; $A \preceq A'$, если для любых i, j из $\{1, ..., p\}$ $\pi(a_i', a_j' \leq \pi(a_i, a_j)$.Пусть $\bar{t} = \max\{t_1, ..., t_p\}$, $p \leq K^{\bar{t}}$. Пусть $A = (a_1, ..., a_p)$ — произвольный элемент из E_k^T такой, что для любых i, j из $\{1, ..., p\}$, $i \neq j$, $\pi(a_i, a_j) \neq 0$. Множество всех A' из E_k^T таких, что $A' \preceq A$ назовем *π-множеством, задаваемым элементом A*. Для обозначения π-множеств будем использовать символ Δ. Любое π-множество Δ разбивается на два подмножества: $\Delta^{(M)}$ — множество всех максимальных по порядку »$\preceq$« элементов из Δ, и $\Delta^{(m)}$ — множество всех оставшихся элементов. Если $\Delta \subseteq E_k^T$, то $T = (t_1, ..., t_p)$, $\{t_1, ..., t_p\}$, максимум чисел из $\{t_1, ..., t_p\}$ обозначим соответствено $T(\Delta)$, $\{T(\Delta)\}$, $\max\{T(\Delta)\}$. Число p назовем арностью π-множества Δ и будем обозначить $p(\Delta)$. Нетрудно видеть, что для любых i, j из $\{1, ..., p(\Delta)\}$ значение $\pi(a_i, a_j)$ не зависит от выбора элемента $(a_1, ..., a_{p(\Delta)})$ из $\Delta^{(M)}$. Поэтому число $\pi(a_i, a_j)$ обозначим как $\pi_\Delta(i, j)$. На множестве $\{1, ..., p(\Delta)\}$ определим отношение *Δ-эквивалентности*: числа i, j Δ-эквивалентны, елси $t_i = t_j$, $\pi_\Delta(i, j) \leq$ ≤ 1. Подстановку γ чисел $1, ..., p(\Delta)$ назовем Δ-постановкой, если для любого $(a_1, ..., a_{p(\Delta)})$ из Δ $a_{\gamma(1)}, ..., a_{\gamma(p)})$ также принадлежит Δ.

2. Опишем восемь специальных семейтств отношений. Каждое отношение R из этих семейств является собственным подмножеством некоторого π-множества Δ_R, причем $\max\{T(\Delta_R)\} \leq \tau$. В дальнейшем индекс R в обозначении Δ_R будем опискать и всегда считать, что $T(\Delta) = (t_1, ..., t_{p(\Delta)})$.

Пусть $R \subseteq \Delta$. Отношение R назовем *Δ-рефлексивным*, если $\Delta^{(m)} \subseteq R$. Отношение R назовем Δ-симметричным, если для любих $(a_1, ..., a_{p(\Delta)})$ из R и Δ-подстанкови γ $(a_{\gamma(1)}, ..., a_{\gamma(p(\Delta))})$ также принадлежит R. Для всякого $t \in \{T(\Delta)\}$ определим множество $E_k^t(R) \subseteq E_k^t$: $a \in E_k^t(R)$, если для любого $\alpha \in E_k$ в Δ/R существует элемент $(a_1, ..., a_{p(\Delta)})$ такой, что для некоторого $i \in \{1, ..., p(\Delta)\}$ $t_i = t$, $\pi(a_i, a) \leq 1$, $a_i(t_i) = \alpha$.

Семейство $T_k(\tau)$. ($T_k(\tau) \neq \emptyset$ при любых $k \geq 2, \tau \geq 1$.) Отношение $R \in T_k(\tau)$ тогда и только тогда, когда R Δ-рефлексивно, Δ-симметрично и для любого $t \in \{T(\Delta)\}$ $E_k^t(R) \neq \emptyset$.

Заметим, что каждое $R \in T_k(1)$ суть отношение из семейства »центральных« [1,4,5].

Пусть $R \subset \Delta$. Для всякого $A = (a_1, ..., a_{p(\Delta)})$ из Δ/R определим систему $\{\mathcal{E}_1(A), ..., \mathcal{E}_{p(\Delta)}(A)\}$ подмножеств E_k : $\alpha \in \mathcal{E}_i(A) (1 \leq i \leq p(\Delta))$, если в Δ/R существует элемент $a'_1, ..., a'_p(\Delta))$ такой, что для всякого $j \neq i$, $a'_j = a_j$, $\pi(a_i, a'_i) \leq 1$, $a'(t_i) = \alpha$. Пусть $R \notin T_k(\tau)$. Тогда для некоторого $t \in \{T(\Delta)\}$ $E_k^t(R) \neq \emptyset$. На множестве Δ/R определим отношение $\Sigma(R)$-*эквивалентности*: $A = (a_1^1, ..., a_{p(\Delta)}^1)$ и $A' = (a'_1, ..., a'_{p(\Delta)})$ $\Sigma(R)$-эквивалентны, если существует набор$A_1 = (a_1^1, ..., a_{p(\Delta)}^1), ..., A_n = (a_1^n, ..., a_{p(\Delta)}^n)$ элементов из Δ/R, числа $i_1, j_2, ..., i_{n-1}, j_n$ из $\{1, ..., p(\Delta)\}$ такие, что $A_1 = A$, $A_n = A'$ и для всякого $l \in \{1, ..., n\}$ $a_{i_l}^l \in E_k^{t_{i_l}}(R)$, $t_{j_{l+1}} = t_{i_l}$, $\pi(a_{i_l}^l, a_{j_{l+1}}^{l+1}) \leq 1$.

Семейство $G_k(\tau)$. ($G_k(\tau) \neq \emptyset$ при $\tau \geq 1$, если $k \geq 3$, и при $\tau \geq 2$,если $k = 2$.) Отношение $R \in G_k(\tau)$ тогда и только тогда, когда R Δ-рефлексивно, Δ-симетрично и выполнены следующие условя:

Пусть $A - (a_1, ..., a_{p(\Delta)})$ — элемент из Δ/R. Тогда для любой пары l и j Δ-эквивалентных чисел $\mathcal{E}_l(A) \cap \mathcal{E}_j(A) = \emptyset$. Елси $a_i \in E_k^{t_i}(R)$, то существует класс Δ-эквивалентности $\{i_1, ..., i_s\}$ такой, что $s \geq 2$, $i \in \{i_1, ..., i_s\}$, $\mathcal{E}_{i1}(A) \cup \dots \cup \mathcal{E}_{is}(A) = E_k$. Пусть$A' = (a'_1, ..., a'_{p(\Delta)})$ также принадлежитΔ/R. Тогда, если $a_i \in E_k^{t_i}(R)$, то $a'_i \in E_k^{t_i}(R)$. Пусть A и A' $\Sigma(R)$-эквивалентны. Тогда существует Δ-подстановка γ такая, что для любых i, j из $\{1, ..., p(\Delta)\}$, $\pi(a_i, a_{\gamma(j)}) \geq \pi_\Delta(i, j)$ и при $\pi(a_i, a_{\gamma(i)}) = 0$ $\mathcal{E}_i(A) = \mathcal{E}_{\gamma(i)}(A')$. Если $p(\Delta) = 2$, то $\Delta^{(M)} \cap R \neq 0$.

Заметим, что каждое $R \in G_k(1)$ суть отношение из семейства отношений, гомоморфных »элементарным« [1,4,5]. При этом $h \leq k$, $m = 1$.

Пусть $k \geq 3$, $\tau \geq 1$. Пусть $m \geq 2$, $h \geq 2$. через $Q(m, h)$обозначим множество всех систем$\{R_1, ..., R_m\}$ отношений из $G_k(\tau)$, удовлетворяющих свойству Q.

Существует π-множество Δ такое, что $\bigcup_{l=1}^{m} R_l \subseteq \Delta$, и числа $1, ..., p(\Delta)$ образуют класс Δ-эквивалентности. Для любых l, q из

$\{1,...,m\}$ и для всякого $A = (a_1,...,a_{p(\Delta)})$ из Δ/R_l существует $A' = (a'_1,...,a'_{p(\Delta)})$ из Δ/R_q такое, что $\pi(a_1,a'_1) \le 1,...,\pi(a_{p(\Delta)},a'_{p(\Delta)}) \le 1$; при $h \le p(\Delta)$ для всякого $i \in \{1,...,p(\Delta)\}$ $a_i = a'_i$ $a_i \ne E_k^{t_i}(R_q)$.

Пусть $\{R_1,...,R_m\} \in Q(m,h)$. Нетрудно видеть, что $E_k^{t_1}(R_1) \ne \emptyset$. Более того, $E_k^{t_1}(R_1) = \cdots = E_k^{t_1}(R_m)$. Пусть $a \in E_k^{t_1}(R_1)$. Через $R_l^{(a)}$ $(1 \le l \le m)$ обозначим подмножество Δ такое,что $A = (a_1,...,a_{p(\Delta)})$ принадлежит $\Delta/R_l^{(a)}$ тогда и только тогда, когда $A \in \Delta/R_l$ и $\pi(a,a_1) \le 1$. Очевидно, $R_l \subseteq R_l^{(a)}$,и $R_l^{(a)}$ — отношение из $G_k(\tau)$. На множестве $\{R_1,...,R_m\}$ введем отношение $\delta(a)$-*эквиваленостu*: R_l и R_q $\delta(a)$-эквивалентны, если $R_l^{(a)} = R_q^{(a)}$.Таким образом, множество $\{R_1,...,R_m\}$ разбивается на классы $\delta(a)$-эквивалентности.Пусть $m(a)$ — число таких классов. Через $R_{i_1^*},..,R_{i_{m(a)}^*}$ обозначим произвольную совокупность отношенийиз $\{R_1,...,R_m\}$, принадлежащих попарно различным классам $\delta(a)$-эквивалентности.

Пусть $\widetilde{Q}(m,h)$ — подмножество $Q(m,h)$, состоящее из всех систем $\{R_1,...,R_m\}$ таких, что для некоторого $a \in E_k^{t_1}(R_1)$, $m(a) = m$.

Семейство $H_k(\tau)$. ($H_k(\tau) \ne 0$ при любых $k \ge 9$, $\tau \ge 1$.) Отношение $R \in H_k(\tau)$ тогда и только тогда, когда $R = \bigcap_{l=1}^m R_l$, где $\{R_1,...,R_m\} \in \widetilde{Q}(m,h)$, $h \ge 3$, $m \ge 2$ и для любых $a \in E_k^{t_1}(R_1)$, $A_1,...,A_{m(a)}$ из $\Delta/R_{i_1^{(a)}}^{(a)},...,\Delta/R_{i_{m(a)}^*}^{(a)}$ соответственно, $j_1,...,j_{m(a)}$ из $\{1,...,h\}$ $\mathcal{E}_{j_1} \cap \cdots \cap \mathcal{E}_{j_{m(a)}}(A_{m(a)}) \ne \emptyset$.

Заметим, что каждое $R \in H_k(1)$ суть отношение из семейства отношений, гомоморфных »элементарным« [1,4,5]. При этом $h^m \le k$, $m > 1$.

Семейство $D_k(\tau)$. ($D_k(\tau) \ne \emptyset$ при любых $k \ge 3$, $\tau \ge 1$.) Отношение $R \in D_k(\tau)$ тогда и только тогда, когда $R = \bigcap_{l=1}^m R_l$, где $\{R_1,...,R_m\} \in \widetilde{Q}(m,2)$, $m \le k$ при $\tau \ge 2$, $m < k$ при $\tau = 1$, и для любых $a \in E_k^{t_1}(R_1)$, $A_1,...,A_{m(a)}$ из $\Delta/R_{i_1^*}^{(a)},...,\Delta/R_{i_{m(a)}^*}^{(a)}$ соответствено существуют $j_1,...,j_{m(a)}$ из $\{1,2\}$ такие, что $\mathcal{E}_{j_1}(A_1),...,\mathcal{E}_{j_{m(a)}}(A)$ попарно не пересекаются и $\mathcal{E}_{j_1}(A) \cup \cdots \cup \mathcal{E}_{j_{m(a)}}(A_{m(a)}) = E_k$.

Заметим, что каждое $R \in D_k(1)$ суть бинарное отрношение, определяющее нетривиальное разбиение E_k классы эквивалентости [1,4,5].

Пусть $t \ge 1$. Через Δ_h^t $(2 \le h \le k)$ обозначим π-множество такое, что $T(\Delta_h^t) = (\underbrace{t,...,t}_{h})$ и для любых i,j из $\{1,...,p(\Delta_h^t)\}$, $i \ne j$

$\pi_{\Delta_h^t}(i,j) = 1$.

Семейство $S_k(\tau)$. ($S_k(\tau) \neq 0$ при любых $k \geq 2$, $\tau \geq 1$.) Отношение $R \in S_k(\tau)$ тогда и только тогда, когда $R \subset \Delta_2^t$, $t \leq \tau$ и существует подстановка ϕ, определенная на E_k^t, которая разлагается в произведение циклов одинаковой простой длины $p \geq 2$ и такая, что для любого $a \in E_k^t(a, \phi(a)) \in R$ и, если $(a_1, a_2) \in R$, то $a_2 = \phi(a_1)$.

Семейство $M_k(\tau)$. ($M_k(\tau) \neq \emptyset$ при любых $k \geq 2$, $\tau \geq 1$.) Отношение $R \in M_k(\tau)$ тогда и только тогда, когда $R \subset \Delta_2^t$, $t \leq \tau$ и для некоторого отношения частичного порядка »$\overset{R}{\leq}$« определенного на E_k^t и имеющего в точности k^{t-1} максимальных и k^{t-1} минимальных элементов, справедливо следующее: если $a_1 \overset{R}{\leq} a_2$, то $(a_1, a_2) \in R$ и наоборот: если $(a_1', a_2') \in R$ то $a_1' \overset{R}{\leq} a_2'$.

Пусть $t \geq 1$. Пусть Φ_t — совокупность всех отображений множества E_k^t в множество подстановок, определеных на E_k. Подстановку, которую отображение $\phi \in \Phi_t$ ставит в соответствие элементу $a \in E_k^t$ обозначим через ϕ_a. Пусть $\widetilde{\Phi}_t$ — подмножество Φ_t, состоющее из всех отображений ϕ таких, что для любых a и a' из E_k^t ϕ_a совпадает с $\phi_{a'}$, если $\pi(a, a') \leq 1$. Пусть $k = p^m$, где p — простое число. Пуст $G = \langle E_k, + \rangle$ — абелева группа, в которой каждый ненулевой элемент имеет порядок p (элементарная p-группа).Если $p \neq 2$, то через l_p обозначим число из E_p такое, что $2 \cdot l_p \equiv 1 \pmod p$.

Семейство $L_k(\tau)$. ($L_k(\tau) \neq 0$ для любого $\tau \geq 1$, если $k = p^m$, где p — простое число, $m \geq 1$ при $k \geq 3$, $m > 1$ при $k = 2$.) Отношение $R \in L_k(\tau)$ тогда и только тогда, когда для некоторого ϕ из $\widetilde{Q}_t$ $t \leq \tau$ справедливо следующее:

а) Пусть $k = p^m, p > 2$. Тогда $R \subset \Delta_3^t$ элемент (a_1, a_2, a_3) из Δ_3^t принадлежит R, если $\phi_{a_1}(a_1(t)) = l_p(\phi_{a_1}(a_3(t)) + \phi_{a_1}(a_2(t))$ и не принадлежит R в противном случае;

б) Пусть $k = 2^n$. Тогда $R \subset \Delta_4^t$ элемент (a_1, a_2, a_3, a_4) из Δ_4^t принадлежит R, если $\phi_{a_1}(a_1(t)) + \phi_{a_1}(a_2(t)) = \phi_{a_1}(a_3(t)) + \phi_{a_1}(a_4(t))$ и не принадлежитR в противном случае.

Заметим, что отношения из $S_k(1)$, $M_k(1)$, $L_k(1)$ совпадают соответственно с отношениями, задающими классы самодвойственных, монотонных и квазилинейных функций k-значной логики [1,3,4,5].

Пусть $t \geq 2$ и Δ_t — бинарное π-подмножество такое, что $T(\Delta_t) = (t, t)$, $\pi_{\Delta_t}(1, 2) = 2$.

Семейство $V_k(\tau)$. ($V_k(\tau) \neq 0$ при любых $k \geq 2$, $\tau \geq 2$.) Отношение $R \in V_k(\tau)$ тогда и только тогда, когда $R \subset \Delta_t$, $t \leq \tau$ и справедливо следующее: (a_1, a_2) из $\Delta_t^{(M)}$ принадлежит R, если $a_1(t) = a_2(t)$ и не принадлежит R в противном случае; существует $\phi \in \widetilde{\Phi}_t$ такое,что (a_1, a_2) из $\Delta_t^{(m)}$ принадлежит R, если для некоторого $\alpha \in$

E_k $a_1(t) = \phi_{a_1}(\alpha)$, $a_2(t) = \phi_{a_2}(\alpha)$ и не принадлежит R в противном случае.

Пусть $W_k(\tau) = T_k(\tau) \cup G_k(\tau) \cup H_k(\tau) \cup D_k(\tau) \cup S_k(\tau) \cup M_k(\tau) \cup L_k(\tau) \cup V_k(\tau)$.

Теорема 1. *Пусть $k \geq 2$, $\tau \geq 1$. Множество $\mathcal{M} \subseteq P_g^k$ τ-полно тогда и только тогда, когда для любого $R \in W_k(\tau)$ в $\mathcal{M}$ существует функция f такая, что $f \notin \cup(R)$.*

Теорема 2. *Пусть $k \geq 2$, $\tau \geq 1$. Множество $\mathcal{N} \subset P_g^k$является τ-предполным классом тогда и только тогда, когда $\mathcal{N} = \cup(R)$ для некоторого R из $W_k(\tau)$.*

Теорема 3. *Пусть $k \geq 2$, $\tau \geq 1$.Пусть R_1 и R_2 принадлежат $W_k(\tau)$ и $\cup(R_1) = \cup(R_2)$. Тогда R_1 и R_2 имеют одинаковую арность $p \geq 1$, одновременно принадлежает либо $T_k(\tau)$ $G_k(\tau)$ либо $H_k(\tau)$ либо $D_k(\tau)$ либо $M_k(\tau)$ и существует подстановка γ чисел $1, ..., p$ такая, что $(a_{\gamma(1)}, ..., a_{\gamma(p)}) \in R_2$, если $(a_1, ..., a_p) \in R_1$ и $(a'_{\gamma^{-1}_{(1)}}, ..., a'_{\gamma^{-1}_{(p)}})) \in R_1$, если $(a'_1, ..., a'_p) \in R_2$.*

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Московский государственный униврзитет.

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V. A. Buevič
Moscow State University,
Moscow, Russia

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CE-MAPS OF NON-METRIZABLE COMPACTA

A. G. CHIGOGIDZE

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Abstract: We generalize the well-known result of G. Kozlowski which states that every CE-map onto a finite-dimensional compactum is a hereditary shape equivalence.

A map $f : X \to Y$ is said to be CE-map if for each point $y \in Y$ the fiber $f^{-1}(y)$ has trivial shape. A map $f : X \to Y$ is said to be a hereditary shape equivalence if for each closed subset $F \subseteq Y$ the restriction $f/f^{-1}(F) : f^{-1}(F) \to F$ is a shape equivalence (i.e. $f/f^{-1}(F)$ induces an isomorphism in the *shape* category [2]). It is well known [2] that a CE-map $f : X \to Y$ between metrizable compacta with $\dim Y < \infty$ is a hereditary shape equivalence (of course, the converse is true without any restriction). Our goal is to generalize the above result to the case of maps between non-metrizable compacta.

A necessary information concerning inverse spectra is contained in [3]. We note here only that $\exp_\omega A$ denotes a directed set of all countable subsets of the set A. All inverse spectra $S = \{X_\alpha, p_\alpha^\beta, A\}$ have surjective projections. For each $\alpha \in A$ we will denote by $p_\alpha : \lim S \to X_\alpha$ the corresponding limit projection. A map $f : X \to Y$ is said to be functionaly closed if for each closed and G_δ-subset F of X it's image $f(F)$ is a closed and G_δ-subset of Y as well. Evidently any map between metrizable compacta is functionaly closed.

The following proposition gives a spectral characterization of CE-maps.

Proposition. Let $f : X \to Y$ be a functionally closed surjection between compacta.Then the following conditions are equivalent:

(i) f is a CE-map;

(ii) there exists two ω-spectra S_X, S_Y and a strictly commuting morphism $\{f_\alpha\} : S_X \to S_Y$ consisting of CE-maps such that $X = \lim S_X$, $Y = \lim S_Y$ and $f = \lim\{f_\alpha\}$.

Proof. Embed Y into the product Q^τ (Q denotes the Hilbert cube) and X into the product $Q^\tau \times Q^\tau$ in such a way that $f = \pi_1/X$, where $\pi_1 : Q^\tau \times Q^\tau \to Q^\tau$denotes the natural projection onto the first factor (here τ is a suitable uncountable cardinal number). For each $\alpha, \beta \in \exp_\omega \tau$ with $\alpha \leq \beta$ let $Y_\alpha = \pi_\alpha(Y)$, $X_\alpha = (\pi_\alpha \times \pi_\alpha)(X)$, $q_\alpha = \pi_\alpha/Y$, $p_\alpha = (\pi_\alpha \times$

$\pi_\alpha)/X$, $q_\alpha^\beta = \pi_\alpha^\beta/Y_\beta$, $p_\alpha^\beta = (\pi_\alpha^\beta \times \pi_\alpha^\beta)/X_\beta$ and $f_\alpha = \pi_1^\alpha/X_\alpha$, where $\pi_1^\alpha : Q^\alpha \times Q^\alpha \to Q^\alpha$ denotes the projections onto the first factor and $\pi_\alpha : Q^\tau \to Q^\alpha$, $\pi_\alpha^\beta : Q^\beta \to Q^\tau$ denote the projection onto the corresponding subproducts. Evidently the limit spaces of ω-spectra $S_X = \{X_\alpha, p_\alpha^\beta, \exp_\omega \tau\}$ and $S_Y = \{Y_\alpha, p_\alpha^\beta, \exp_\omega \tau\}$ coincide with X and Y, respectively. Note also that $f = \lim\{f_\alpha\}$. By functionally closedness of f and by [3],we can suppose without loss of generality that the morphism $\{f_\alpha\} : S_X \to S_Y$ strictly commutes.

Let $\alpha \in \exp_\alpha \tau$ and $\mathcal{U}_\alpha = \{U_n^\alpha : n \in \omega\}$ be an open basis of Q^α. Then we put $A(\alpha, n, m) = \{y_\alpha \in Y_\alpha : f^{-1}(y_\alpha) \subseteq U_n^\alpha \times U_m^\alpha\}$.

Claim. *Let $\alpha \in \exp_\omega \tau, \mathcal{U}_\alpha = \{U_n^\alpha : n \in \omega\}$ be a countable open basis of Q^α containing unions and intersections of its finite subfamilies and let $A(\alpha, n, m) \neq \emptyset$. Then there exists an index $\beta = \beta(\alpha, n, m) \in \exp_\omega \tau$ such that $\beta \geq \alpha$ and for each point $y_\beta \in (q_\alpha^\beta)^{-1}(A(\alpha, n, m))$ there exists an open neighborhood G_{Y_β} of the fiber $f_\beta^{-1}(y_\beta)$ in $Q^\beta \times Q^\beta$ such that $G_{y_\beta} \subseteq (\pi_\alpha^\beta \times \pi_\alpha^\beta)^{-1}(U_n^\alpha \times U_m^\alpha)$ and the inclusion map*

$$G_{y_\beta} \hookrightarrow (\pi_\alpha^\beta \times \pi_\alpha^\beta)^{-1}(U_n^\alpha \times U_m^\alpha)$$

is null-homotopic (in $\pi_\alpha^\beta \times \pi_\alpha^\beta)^{-1})(U_n^\alpha \times U_m^\alpha)$).

Proof. Let $y \in q_\alpha^{-1}(A(\alpha, n, m))$. Then $f_\alpha^{-1}(q_\alpha(y)) \subseteq U_n^\alpha \times U_m^\alpha$. Consequently

$$f^{-1}(y) \subseteq p_\alpha^{-1}(f_\alpha^{-1}(q_\alpha(y))) \subseteq (\pi_\alpha \times \pi_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha).$$

Since the fiber $f^{-1}(y)$ has a trivial shape and $Q^\tau \times Q^\tau$ is an AR-compactum there exists an open F_σ neighbourhood G_y of $f^{-1}(y)$ in $Q^\tau \times Q^\tau$ such that $G_y \subseteq (\pi_\alpha \times \pi_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)$ and the inclusion map $G_y \hookrightarrow (\pi_\alpha \times \pi_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)$ is null-homotopic. Evidently

$$f^{-1}(q_\alpha^{-1}(A(\alpha, n, m))) \subseteq \bigcup\{G_y : y \in q_\alpha^{-1}(A(\alpha, n, m))\}.$$

Since the space $f^{-1}(q_\alpha^{-1}(A(\alpha, n, m)))$ is Lindelöff we can choose a countable subfamily $\{G_k : k \in \omega\}$ of $\{G_y : y \in q_\alpha^{-1}(A(\alpha, n, m))\}$ such that $f^{-1}(q_\alpha^{-1}(A(\alpha, n, m))) \subset \cup\{G_k : k \in \omega\}$. By [3],there exist an index $\beta \geq \alpha$ and open subsets G_k^β of $Q^\beta \times Q^\alpha$ such that $G_k = (\pi_\beta \times \pi_\beta)^{-1}(G_k^\beta)$, $k \in \omega$. Fix for each $k \in \omega$ a map $i_k : G_k^\beta \to G_k$ such that $(\pi_\beta \times \pi_\beta)i_k = id_{G_k^\beta}$ and a homotopy $H_k : G_k \times I \to (\pi_\alpha \times \pi_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)$ connecting the inclusion and the constant maps. Then the composition $(\pi_\beta \times \pi_\beta)H_k(i_k \times id_I)$ is a homotopy showing that the inclusion map

$$G_k^\beta \hookrightarrow (\pi_\alpha^\beta \times \pi_\alpha^\beta)^{-1}(U_n^\alpha \times U_m^\alpha)$$

is null-homotopic in $(\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m)$. It only remains to show that if $y_\beta \in (q^\beta_\alpha)^{-1}(A(\alpha,n,m))$, then $f^{-1}_\beta(y_\beta) \subseteq G^\beta_k$ for some $k \in \omega$. Indeed, fix a point $y \in q^{-1}_\alpha(A(\alpha,n,m))$ such that $q_\beta(y) = y_\beta$. Then $f^{-1}(y) \subseteq G_k$ for some $k \in \omega$. Since the morphism $\{f_\alpha\}$ is strictly commuting it follows easily that

$$f^{-1}_\beta(y_\beta) = p_\beta(f^{-1}(y)) \subseteq (\pi_\beta \times \pi_\beta)((\pi_\beta \times \pi_\beta)^{-1}(G^\beta_k)) = G^\beta_k\,.$$

The claim is proved.

We return to the proof of the proposition. We wish to use the proposition 1.3 from [3] with respect to the following relation $L \subseteq (\exp_\omega \tau)^2$:
$L = \{(\alpha,\beta) \in (\exp_\omega \tau)^2 : \alpha \leq \beta$ and there exists a countable open basis $\mathcal{U}_\alpha = \{U^\alpha_n : n \in \omega\}$ of Q^α containing unions and intersections of its finite subfamilies and satisfying the following condition: for each $(n,m) \in \omega^2$ with $A(\alpha,n,m) \neq \emptyset$ and for each point $y_\beta \in (q^\beta_\alpha)^{-1}(A(\alpha,n,m))$ there exists an open neighbourhood $G_{y_\beta}$of the fiber $f^{-1}_\beta(y_\beta)$ in $Q^\beta \times Q^\beta$ such that

$$G_{y_\beta} \subseteq (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m)$$

and the inclusion map

$$G_{y_\beta} \hookrightarrow (\pi^\beta_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m)$$

is null-homotopic in $(\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m)\}$.

Let us show that for each $\alpha \in \exp_\omega \tau$ there exists an index $\beta \in \exp_\omega \tau$ such that $(\alpha,\beta) \in L$ (the first condition of proposition (1.3) from [3]). Let $\mathcal{U}_\alpha = \{U^\alpha_n : n \in \omega\}$ be any countable open basis of Q^α containing unions and intersections of its finite subfamilies. For each pair $(n,m) \in \omega^2$ with $A(\alpha,n,m) \neq \emptyset$ fix an index $\beta(n,m) = \beta(\alpha,n,m) \geq \alpha$ satisfying the above claim. By [3], there exists an index $\beta \in \exp_\omega \tau$ such that $\beta \geq \beta(n,m)$ for each $(n,m) \in \omega^2$ with $A(\alpha,n,m) \neq \emptyset$. Let us show that $(\alpha,\beta) \in L$. For consider a pair $(n,m) \in \omega^2$ such that $A(\alpha,n,m) \neq \emptyset$ and let y_β be an arbitrary point of $(q^\beta_\alpha)^{-1}(A(\alpha,n,m))$. Clearly,

$$y = q^\beta_{\beta(n,m)}(y_\beta) \in (q^{\beta(n,m)}_\alpha)^{-1}(A(\alpha,n,m))\,.$$

By the choice of $\beta(n,m)$, there exists an open neighbourhood G of the fiber $f^{-1}_{\beta(n,m)}(y)$ in $Q^{\beta(n,m)} \times Q^{\beta(n,m)}$ such that the inclusion map

$$G \hookrightarrow (\pi^{\beta(n,m)}_\alpha \times \pi^{\beta(n,m)}_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m)$$

is null-homotopic. Let $W = (\pi^{\beta}_{\beta(n,m)} \times \pi^{\beta}_{\beta(n,m)})^{-1}(G)$. Then

$$\begin{aligned} f_\beta^{-1}(y_\beta) &\subseteq f_\beta^{-1}(q^{\beta}_{\beta(n,m)})^{-1}(q^{\beta}_{\beta(n,m)}(y_\beta))) = \\ &= (p^{\beta}_{\beta(n,m)})^{-1}(f^{-1}_{\beta(n,m)}(y))(\pi^{\beta}_{\beta(n,m)} \times \pi^{\beta}_{\beta(n,m)})^{-1}(f^{-1}_{\beta(n,m)}(y)) \\ &\subseteq (\pi^{\beta}_{\beta(n,m)} \times \pi^{\beta}_{\beta(n,m)})^{-1}(G) = W\,. \end{aligned}$$

At the same time

$$\begin{aligned} W &= (\pi^{\beta}_{\beta(n,m)} \times \pi^{\beta}_{\beta(n,m)})^{-1}(G) \subseteq (\pi^{\beta}_{\beta(n,m)} \times \pi^{\beta}_{\beta(n,m)})^{-1} \\ &((\pi_\alpha^{\beta(n,m)} \times \pi_\alpha^{\beta(n,m)})^{-1}(U_n^\alpha \times U_m^\alpha)) = \\ &= (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)\,. \end{aligned}$$

Since the restriction

$$\begin{aligned} &(\pi^{\beta}_{\beta(n,m)} \times \pi^{\beta}_{\beta(n,m)}) \,/\, (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha) : \\ &(\pi_\alpha \times \pi_\beta)^{-1}(U_n^\alpha \times U_m^\alpha) \to (\pi_\alpha^{\beta(n,m)} \times \pi_\alpha^{\beta(n,m)})^{-1})(U_n^\alpha \times U_m^\alpha) \end{aligned}$$

is a soft map (see [3]) it follows from the choice of G that the inclusion map $W \hookrightarrow (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)$ is null-homotopic in $(\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)$. Consequently, $(\alpha, \beta) \in L$ and the first condition of the proposition (1.3) from [3] is verified.

Now suppose that $(\alpha, \beta) \in L$ and $\gamma \geq \beta$. Let us show that $(\alpha, \gamma) \in L$ (the second condition of proposition 1.3 from [3]). Let $y_\gamma \in (q^\gamma_\alpha)^{-1}(A(\alpha, n, m))$ and $y_\beta = q^\gamma_\beta(y_\gamma)$. Clearly, $y_\beta \in (q^\beta_\alpha)^{-1}(A(\alpha, n, m))$. Since $(\alpha, \beta) \in L$ it follows that there exists an open neighbourhood G_β of the fiber $f_\beta^{-1}(y_\beta)$ in $Q^\beta \times Q^\beta$ such that inclusion map $G_\beta \hookrightarrow (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)$ is null-homotopic in $(\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)$. Let $G_\gamma = (\pi^\gamma_\alpha \times \pi^\gamma_\beta)^{-1}(G_\beta)$. Then

$$\begin{aligned} f_\gamma^{-1}(y_\gamma) &\subseteq f_\gamma^{-1}((q^\gamma_\beta)^{-1}(q^\gamma_\beta(y_\gamma))) = (p^\gamma_\beta)^{-1}(f_\beta^{-1}(y_\beta)) \subseteq \\ &\subseteq (\pi^\gamma_\beta \times \pi^\gamma_\beta)^{-1}(f_\beta^{-1}(y_\beta)) \subseteq (\pi^\gamma_\beta \times \pi^\gamma_\beta)^{-1}(G_\beta) = G_\gamma\,. \end{aligned}$$

At the same time

$$\begin{aligned} G_\gamma &= (\pi^\gamma_\beta \times \pi^\gamma_\beta)^{-1}(G_\beta) \subseteq (\pi^\gamma_\beta \times \pi^\gamma_\beta)^{-1}((\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)) = \\ &= (\pi^\gamma_\alpha \times \pi^\gamma_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha)\,. \end{aligned}$$

Since the restriction

$$\begin{aligned} &(\pi^\gamma_\beta \times \pi^\gamma_\beta) \,/\, (\pi^\gamma_\alpha \times \pi^\gamma_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha) : \\ &(\pi^\gamma_\alpha \times \pi^\gamma_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha) \to (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U_n^\alpha \times U_m^\alpha) \end{aligned}$$

is a soft map it follows from the choice of G_β that the conclusion map $G_\gamma \hookrightarrow (\pi^\gamma_\alpha \times \pi^\gamma_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m)$ is null-homotopic in $(\pi^\gamma_\alpha \times \pi^\gamma_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m)$. Hence $(\alpha, \gamma) \in L$ and the second condition of the proposition (1.3) from [3] is verified.

Now let $\{\alpha_k : k \in \omega\}$ be a countable chain in $\exp_\omega \tau$ and $(\alpha_k, \beta) \in L$ for each $k \in \omega$ and for some $\beta \in \exp_\omega \tau$. Let us show that $(\alpha, \beta) \in L$, where $\alpha = \sup\{\alpha_k : k \in \omega\}$ (the third condition of the proposition (1.3) of [3]).

Since $(\alpha, \beta) \in L$ we can fix a countable open basis $\{\mathcal{U}_{\alpha_k} = U^{\alpha_k}_n : n \in \omega\}$ of Q^{α_k} containing unions and intersections of it's finite subfamilies and satisfying the corresponding properties from the definition of L. We denote by $\mathcal{U}_\alpha$ the collection of unions and intersections of all finite subfamilies of the collection

$$\bigcup\{(\pi^\alpha_{\alpha_k})^{-1}(\mathcal{U}_{\alpha_k}) : k \in \omega\}.$$

Since $\alpha = \sup\{\alpha_k : k \in \omega\}$ it follows from the definition of an ω-spectrum [3] that Q^α is naturally homeomorphic to the limit space of the inverse sequence $\{Q^{\alpha_k}, \pi^{\alpha_{k+1}}_{\alpha_k}, \omega\}$. Consequently, $\mathcal{U}_\alpha$ is a countable open basis of Q^α containing unions and intersections of its finite subfamilies. Let $(n, m) \in \omega^2$, $A(\alpha, n, m) \neq \emptyset$ and $y_\beta \in (q^\beta_\alpha)^{-1}(A(\alpha, n, m))$. This means, by above definitions, that $f^{-1}_\alpha(y_\alpha) \subset U^\alpha_n \times U^\alpha_m$, where $y_\alpha = q^\beta_\alpha(y_\beta)$. It is easy to see that there exist an integer $k \in \omega$ and elements $U^{\alpha_k}_r, U^{\alpha_k}_s \in \mathcal{U}_{\alpha_k}$ such that $f^{-1}_\alpha(y_\alpha) \subseteq (\pi^\alpha_{\alpha_k} \times \pi^\alpha_{\alpha_k})^{-1}(U^{\alpha_k}_r \times U^{\alpha_k}_s) \subseteq U^\alpha_n \times U^\alpha_m$. Since the morphism $\{f_\alpha\}$ is strictly commuting it follows that

$$\begin{aligned} f^{-1}_{\alpha_k}(q^\alpha_{\alpha_k}(y_\alpha)) = p^\alpha_{\alpha_k}(f^{-1}_\alpha(y_\alpha)) = (\pi^\alpha_{\alpha_k} \times \pi^\alpha_{\alpha_k})(f^{-1}_\alpha(y_\alpha)) \subseteq \\ \subseteq (\pi^\alpha_{\alpha_k} \times \pi^\alpha_{\alpha_k})((\pi^\alpha_{\alpha_k} \times \pi^\alpha_{\alpha_k})^{-1}(U^{\alpha_k}_r \times U^{\alpha_k}_s)) = U^{\alpha_k}_r \times U^{\alpha_k}_s. \end{aligned}$$

Consequently, $q^\beta_{\alpha_k}(y_\beta) = q^\alpha_{\alpha_k}(y_\alpha) \in A(\alpha_k, r, s)$ and $y_\beta \in (q^\beta_{\alpha_k})^{-1}(A(\alpha_k, r, s))$. Since $(\alpha_k, \beta) \in L$ there exists an open neighbourhood G_β of $f^{-1}_\beta(y_\beta)$ in $Q^\beta \times Q^\beta$ such that

$$G_\beta \subseteq (\pi^\beta_{\alpha_k} \times \pi^\beta_{\alpha_k})^{-1}(U^{\alpha_k}_r \times U^{\alpha_k}_s)$$

and the inclusion map

$$G_\beta \hookrightarrow (\pi^\beta_{\alpha_k} \times \pi^\beta_{\alpha_k})^{-1} U^{\alpha_k}_r \times U^{\alpha_k}_s$$

is null-homotopic. It only remains to note that

$$\begin{aligned} G_\beta \subseteq & (\pi^\beta_{\alpha_k} \times \pi^\beta_{\alpha_k})^{-1}(U^{\alpha_k}_r \times U^{\alpha_k}_s) = \\ = & (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}((\pi^\alpha_{\alpha_k} \times \pi^\alpha_{\alpha_k})^{-1}(U^{\alpha_k}_r \times U^{\alpha_k}_s)) \subseteq (\pi^\beta_\alpha \times \pi^\beta_\alpha)^{-1}(U^\alpha_n \times U^\alpha_m). \end{aligned}$$

The third condition of the proposition (1.3) from [3] is verified as well.

By proposition (1.3) from [3], the set A of L-reflexive indexes is closed and cofinal in $\exp_\omega \tau$. It only remains to show that for each $\alpha \in A$ the map

$f_\alpha : X_\alpha \to Y_\alpha$ is a CE-map. Indeed, let W be any open neighbourhood of the fiber $f_\alpha^{-1}(y_\alpha)$ in $Q^\alpha \times Q^\alpha$, where y_α is an arbitrary point of Y_α. Since $\mathcal{U}_\alpha$ contains unions and intersections of its finite subfamilies, there exist $U_n^\alpha, U_m^\alpha \in \mathcal{U}_\alpha$ such that $f_\alpha^{-1}(y_\alpha) \subseteq U_n^\alpha \times U_m^\alpha \subseteq W$. Consequently, by the definition of L and by the condition $(\alpha, \alpha) \in L$, there exists an open neighbourhood G of $f_\alpha^{-1}(y_\alpha)$ in $Q^\alpha \times Q^\alpha$ such that $G \subseteq U_n^\alpha \times U_m^\alpha$ and the inclusion map $G \hookrightarrow U_n^\alpha \times U_m^\alpha$ is null-homotopic in $U_n^\alpha \times U_m^\alpha$. Since $Q^\alpha \times Q^\alpha$ is an AR-compactum it follows that $f_\alpha^{-1}(y_\alpha)$ has a trivial shape. This finishes the proof of implication (i)→(ii). An inverse implication is trivial. Proposition is proved.

Theorem. *Let $f : X \to Y$ be functionally closed surjection between compacta and $\dim Y < \infty$. Then the following conditions are equivalent:*

(i) *f is a CE-map;*

(ii) *f is a hereditary shape equivalnece.*

Proof. (i)→(ii). It suffices to show that f is a shape equivalence. Let $S_X = \{X_\alpha, p_\alpha^\beta, A\}$ and $S_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ be two ω-spectra such that $\lim S_X = X$ and $\lim S_Y = Y$. Fix also a strictly commuting morphism $\{f_\alpha\} : S_X \to S_Y$ such that $f = \lim\{f_\alpha\}$. By proposition, we can assume without loss of generality that each map $f_\alpha : X_\alpha \to Y_\alpha$ is a CE-map. By [1], we can assume additionally that $\dim Y_\alpha < \infty$ for each $\alpha \in A$. Then, by [2], each f_α is a shape equivalence.

Let us show that in this situation f is a shape equivalence as well. For it is sufficient to show that a natural correspondence $[Y, P] \to [X, P]$ induced by f is bijective for each finite polyhedron P.

Let $\phi : X \to P$ be any map. Then, by [3], there exists an index $\alpha \in A$ and a map $\phi_\alpha : X_\alpha \to P$ such that $\phi = \phi_\alpha p_\alpha$. Since f_α is a shape equivalence there is a map $\psi_\alpha : Y_\alpha \to P$ such that $\phi_\alpha \simeq \psi_\alpha f_\alpha$. Then $\phi = = \phi_\alpha p_\alpha \simeq \psi_\alpha f_\alpha p_\alpha = \psi_\alpha q_\alpha f$. Consequently, $\phi \simeq \psi f$, where $\psi = \psi_\alpha q_\alpha$. This shows that the above correspondence is a surjection.

Suppose now that we have two maps $\psi_1, \psi_2 : Y \to P$ such that $\psi_1 f \simeq \psi_2 f$. Fix an index $\alpha \in A$ and two maps $\psi_1^\alpha, \psi_2^\alpha : Y_\alpha \to P$ such that $\psi_k = = \psi_k^\alpha q_\alpha$ $(k = 1, 2)$. Clearly, $\psi_1^\alpha f_\alpha p_\alpha \simeq \psi_2^\alpha f_\alpha p_\alpha$. Then there is an index $\beta \in A$ such that $\beta \geq \alpha$ and $\psi_1^\alpha f_\alpha p_\alpha^\beta \simeq \psi_2^\alpha f_\alpha p_\alpha^\beta$. Consequently, $\psi_1^\alpha q_\alpha^\beta f_\beta = \psi_1^\alpha f_\alpha p_\alpha^\beta \simeq \psi_2^\alpha f_\alpha p_\alpha^\beta = \psi_2^\alpha q_\alpha^\beta f_\beta$. Since f_β is a shape equivalence we conclude that $\psi_1^\alpha q_\alpha^\beta \simeq \psi_2^\alpha q_\alpha^\beta$. It only remains to note that $\psi_1 = \psi_1^\alpha q_\alpha^\beta q_\beta \simeq \psi_2^\alpha q_\alpha^\beta q_\beta = = \psi_2$. Hence the above correspondence is injective. This finishes the proof of the implication (i)→(ii).

The other implication is trivial.

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A. G. Chigogidze
Department of Geometry and Topology
Mathematical Institute
Georgian Academy of Science
Tbilisi, Georgia

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ALTERNATIVE CONSTRUCTION OF COMPACTA WITH DIFFERENT DIMENSIONS

A. N. DRANISHNIKOV

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Key words: cohomological dimension, infinite dimensional compacta

Abstract: We present an alternative method of constructing compacta with different covering and cohomological dimensions (over arbitrary group G).

In [D-W1] there is a unified approach to constructions of compacta with different cohomological and also covering dimensions which were constructed in [Dr1-6] and [D-W2]. It is based on Edward-Walsh modifications of polyhedra. Here we give an alternative approach. The Edward-Walsh modification exists only with Eilenberg-MacLane space $K(G,n)$ where G is a ring with unit. This new approach is valid for arbitrary group G. Unfortunatelly it is not so geometric. I am very thankful to V. Uspenskii who recalled me the idea of that alternative approach.

Definition. *A map $f : L \to K$ between polyhedra is called combinatorial with respect to some triangulations λ and $\varkappa$ provided $f^{-1}(A)$ is a subpolyhedron of λ for every simplex $A \subset K$. It means that preimage of any polyhedron A with respect to $\varkappa$ is a polyhedron with respect to λ.*

By $\mathcal{P}(L,\lambda,k)$ we denote the set of all pairs (A,α) where A is a subpolyhedron of L with respect to triangulation λ and $\alpha : A \to K$ is a continuous map. Every combinatorial mapping $f : (L,\lambda) \to (K,\varkappa)$ induces an inclusion $\mathcal{P}(f) : \mathcal{P}(K,\varkappa,Y) \to \mathcal{P}(L,\lambda,Y)$ for any space Y.

Suppose that $A \subset L$ is a closed subset and $\alpha : A \to K$ is a continuous map. By K-resolution of L along (A,α) we call a projection ξ in the following pull-back diagram

$$\begin{array}{ccc} \hat{L} & \xrightarrow{\mu} & K \times I \\ \xi \downarrow & & \downarrow \pi \\ L & \xrightarrow{\beta} & \operatorname{con} K = K \times I / K \times \{1\} \end{array}$$

where π is natural projection of $K \times [0,1]$ onto the cone of K and β is an extension of $\alpha : A \to K = K \times \{0\}$. It is easy to see that the restriction $\alpha \circ \xi_{|\xi^{-1}(A)}$ has the extension $\omega \circ \mu$ where $\omega : K \times I \to K$ is the projection.

A generalized cohomology theory h^* is called continuous if for every countable CW-complex W with compact stratification $W_1 \subset W_2 \subset \ldots \subset W_n \subset \ldots$ there is an equality $h^*(W) = \varprojlim h^*(W_i)$.

Lemma. *For any continuous cohomology h^* and any h-acyclic countable CW-complex K for every compact polyhedron L and $\gamma \in h^*(L)$ there exist inverse system of compact polyhedra*

$$L = L_1 \xleftarrow{g_1^2} L_2 \longleftarrow \dots \longleftarrow L_i \xleftarrow{g_i^{i+1}} \dots$$

supplied with triangulations λ_i and with combinatorial projections and direct system of inclusions

$$C_1 \xrightarrow{g_2^1} C_2 \longrightarrow \dots \longrightarrow C_i \xrightarrow{e_{i+1}^i} \dots$$

where C_i is countable dense subset in $\mathcal{P}(L_i, \lambda_i, K)$ and e_{i+1}^i is the restriction of $\mathcal{P}(g_i^{i+1})$ and a map $\chi : \mathbf{N} \to C = \varinjlim C_i$ with the properties

1) *for every l* $\lim_{k\to\infty} \operatorname{mesh} g_l^k(\lambda_k) = 0$

   2) for all $n, \chi(n) \in \operatorname{im}(e_\infty^n)$ and the map $\alpha = (e_\infty^n)^{-1}(\chi(n)) : A \to K$ has the property

   ($*$) the restriction $\alpha \circ g_n^{n+1}{}_{|(g_n^{n+1})^{-1}(A)}$ has an extension to a map of L_{n+1}

3) *$\#\chi^{-1}(c) = \infty$ for each $c \in C$*

4) *$(G_1^k)^*(\gamma \neq 0$ for all k.*

Proof. Fix an epimorphism $\psi = (\psi_1, \psi_2) : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$. Apply induction to construct $L_n, \lambda_n, g_{n-1}^n, C_n, e_{n-1}^n$ and $\chi_n : \psi_1(n) \times \mathbf{N} \to C_n$ with the properties 1) $\operatorname{mesh} g_i^n(\lambda_n) < 1/n$, 2) the map $\chi_{n-1}(\psi(n-1))$ has the property ($*$) 3) $\#\chi_n^{-1}(c) = \infty$ for all $c \in \operatorname{im}\chi_n$ and 4) $(g_1^n)^*(\gamma) \neq 0$.

We may assume that g_0^1 is constant map and easily obtain all formalities for $n = 1$. Assume that it is done for n. First consider a K-resolution $\xi : \hat{L}_n \to L_n$ of L_n along $(A, \alpha) = \chi_n(\psi(n))$. Since $\xi^{-1}(x)$ is homeomorphic to a point or to K then by virtue of assumption of Lemma and Vietoris-Begle theorem [D-K] we have $\xi^*(g_1^n(\gamma)) \neq 0$. Without loss of generality we may assume that $\hat{L}_n$ is a polyhedron $L \subset \hat{L}_n$ such that $(\xi_{|L})^*(g_1^n(\gamma)) \neq 0$. Define $L_{n+1} = L$ and $g_n^{n+1} = \xi_{|L}$. Choose λ_{n+1} to satisfy 1). Since K-resolution ξ admits the property ($*$) for α then a restriction of ξ has the same property. Hence 2) holds.

Choose a countable dense subset C_{n+1} in $\mathcal{P}(_{n+1}, \lambda, K)$ such that

$$\mathcal{P}(g_n^{n+1})(\mathcal{P}_n) \subset C_{n+1}$$

and define e_{n+1}^n as a restriction of $\mathcal{P}(g_{n+1}^n)$. If $\psi_1(n+1) = \psi_1(k)$ for some $k \leq n$ then define $\chi_{n+1} = e_{n+1}^k \circ \chi_k$ otherwise define χ_{n+1} arbitrarily with the property $\#\chi_{n+1}^{-1}(c) = \infty$ for all $c \in C_{n+1}$. So, all the properties 1)–4) are satisfied.

We define $\chi = (\cup e_\infty^n \circ \chi_n) \circ \psi$. It is easy to verify that the properties 1)–4) hold.

Recall that $X\tau K$ denotes the property: for each closed subset $A \subset X$ and for every map $f : A \to K$ there is an extension $\hat{f} : X \to K$.

Theorem. *Let K be a countable CW-complex and h be a continuous cohomology theory with $h^*(K) = 0$. Then for every n there is a compactum X of the dimension $\dim X \geq n$ with the property $X\tau K$.*

Proof. Since h is not trivial then $h^*(S^n) \neq 0$. Let γ be nontrivial element of $h^*(S_n)$. Apply Lemma to obtain X as a limit of inverse system $\{L_k, g_k^{k+1}\}$. The condition 4) of Lemma implies that the projection $g_1^\infty : X \to S^n$ is an essential and hence $\dim X \geq n$.

Let us verify the property $X\tau K$. Consider arbitrary map $f : A \to K$ where $A \subset X$. From condition 1) of Lemma and compactness of X it follows that there exist a number K and a mapping $f_k : A_k \to K$ of star neighbourhood $A_k = \mathrm{St}(g_k^\infty(A), \lambda_k)$ of $g_k^\infty(A)$ with respect to triangulation λ_k such that $f_k \circ g_k^\infty$ is homotopic to f. Since the set C_k is dense in $\mathcal{P}(L_k, \lambda_k, K) = \coprod_A C(A, K)$ then there exists a map $\beta : A_k \to K$ homotopic to f_k and $\beta \in C_k$. By the property 3) of Lemma $\#\chi^{-1}(c) = \infty$ where $c = e_\infty^k((A_b, \beta))$. Choose $m \geq k$ and $m \in \chi^{-1}(c)$. Since $e_m^k((A_k, \beta)) = (e_\infty^m)^{-1} \circ \chi(m)$ then due to $(*)$ of 2) there is an extension η of the restriction $(\beta \circ g_k^m) \circ g_m^{m+1}|_{(g_k^{m+1})^{-1}(A_k)}$. Homotopy extension theorem implies that there is an extension $\zeta : L_{m+1} \to K$ of the map $f_k \circ g_k^{m+1}|_{(g_k^{m+1})^{-1}(A_k)}$. The restriction of $\zeta \circ g_{m+1}^\infty$ onto A is homotopic to f. Therefore there is an extension for f.

Proposition. $X\tau(Y \vee Z) \Rightarrow X\tau Y$.

By using this Proposition it is possible to generalize Theorem to the following

Theorem 1. *Let $\{K_i\}$ be a family of countable CW-complexes acyclic with respect to continuous cohomology theory h. Then for every n there exists a compactum X with covering dimension $\geq n$ having the property $X\tau K_i$ for all i.*

Remark. Theorem 1 is still valid for any truncated cohomology which are continuous and satisfy Vietoris-Begle theorem.

All familiar to me examples in cohomological dimension theory can be constructed by using Theorem 1 and some famous facts from algebraic topology such as

$$\tilde{H}^*(K(\mathbf{Z}[\frac{1}{p}], 1); \mathbf{Z}_p) \cong 0, \qquad \tilde{H}^*(K(\mathbf{Z}_p, 1); \mathbf{Q}) \cong 0,$$
$$\tilde{K}_c^*(K(\mathbf{Z}, 3); \mathbf{Z}_p) \cong 0, \text{ and } [K(\mathbf{Z}, 2)\Omega^* S^3]\hat{} \cong 0\,.$$

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A. N. Dranishnikov
Steklov Mathematical Institute,
Moscow, Russia

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and Its Applications,
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МАТЕМАТИЧЕСКИЕ МОДЕЛИ И СЛОЖНОСТЬ ИНФОРМАЦИОННОГО ПОИСКА

Э. Э. ГАСАНОВ

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Abstract. In the article author gives an overview of mathematical models of information query in data bases. The focus of the article is on special classes of algorithms.

В настоящее время одним из актуальнейших направлений развития современной математической науки являются вопросы проектирования автоматизированных систем. Причем широкое распространение получила концепция баз данних, согласно которой ядром информационной системы становятся данные, определенным образом организованные. Структуры организации данных при этом выбираются в соответствии со многими критериями, одним из основных среди которых являются время поиска информации. И поэтомы вполне понятно то внимание в литературе, которое проявляются к проблемам информационнога поиска [1–7]. Среди этих хотелось бы выделить те, которые связаны с иселедованием вычислительной сложности алгоритмов поиска информации. Основная масса работ в этом направлении связана с разработкой новых еффективных алгоритмов поиска, находнящих многочисленные приложения в различных областях, таких, как машинное проектирование, машинная графика, библиотечно-информационные системы, робототехника, системы искусственного интеллекта и многих других [1,5,7–10]. В этих работах оценивается сложность предлагаемых алгоритмов (чаще всего порядок сложности) и сравнивается со сложностю ранее разработанных алгоритмов. В ряде работ исповедуется другой подход, связанный с введением математических моделей вычислений, используемых главным образом для получения нижних оценок сложности вычислений [5,11–13]. Среди этих моделей наиболее известонй являяется, так называмое алгебраическое дерево вычислений Бен-Ора [11]. Как разновидность алгебраического дерева вычислений можно рассматривать алгебраическое дерево решений порядка d [12]. В случае когда d равно 1, получается линейное дерево решений, с использованием которого получены доказательства ряда нижних оценок сложности [13–16].

В данной статье предлагается обзор работ автора, посвященых математическому моделированию информационног апоиска. Если указанные выше математическе модели вычислений преднозначались для испльозования в широком круге вычислительных процессов (см. [7]), то автор предпочитает специализироватся на моделях, предназначенниых для исследования некоторых алгоритмов поиска информации. Такая »узкая специализация« предлагаемых модлей позволяет надеяться на полуычение более интересных резултатов, в частности более точных оценок сложности.

При разработке этих моделей используются принципы и аппарат теории управляющих систем, и вводятся новые классы управляющих систем.

Сначала формализуем понятие задачи информационного поиска (ЗИП). В работах [2,4–7] вводились различние формализации ЗИП, но мы дадим собственную формализацию, более удобную нам для использования в дальнейшем.

Пусть нам даны два множества Y и X. Первое множество Y является множеством объектов поиска. Из элементов этого множества составляются информационные массивы, в которых производится поиск нужных объектов. Элементы множества Y, будем называть, записями. Второе множество X назовем множеством запросов, а его элементы — запросами. Пусть на декартовом произведении $X \times Y$ задано бинарное отношение ρ, т.е. задано некое подмножество $R \subseteq X \times Y$ и $x \;\rho\; y$, если $(x, y) \in \mathbf{R}$. Отношение ρ будем называть отношением поиска. В содержательном смысле ρ описывает критерий семантического соответствия записи запросу, и мы будем говорить, что запись $y \in Y$ удовлетворяет запросу $x \in X$, если $x \;\rho\; y$.

Под задачей информационнога поиска (ЗИП) будем понимать тройки $I = \langle X, V, \rho \rangle$, где X — множество запросов; V — некоторое конечное подмножество множества Y, которое в дальнейшем будем называть библиотекой; ρ — отношение поиска, заданное на $X \times Y$; и будем считать, что задача $I = \langle X, V, \rho \rangle$ состоит в перечислании для произвольно взятого $x \in X$ всех тех и только тех записей из V, которые удовлетворяют запросу x.

Хронологически исследуемые автором модели эволюционировали в стороны все большего обобщения и охвата все более широкого класса алгоритмов. Но в этой работе пойдем в обратной хронологической последовательности и введем сначала самую общую модель, а именно класс управляющих систем називанных автором информационными сетями с переключателями.

Введем понятие информационной сети с переключателями (ИСП). Пусть нам даны множества X и Y, описанные ранее, множество F символов одноместних предикатов, определенных на множестве X, которое назовем базовым множеством предикатов, и множесво G символов одноместных переключателей, определенных на множестве X. Под переключателями будем понимать функции, областью значений которых являются конечные подмножества натурального ряда. Пары $\mathcal{F} = \langle F, G \rangle$ назовем базовым множеством.

Определение понятя ИСП можно празбить на два этапа. На первом этапе раскрывается структурная (схемная) часть этого понятия, на втором — функциональная.

1. этап. Определение ИСП с точки зрения ее структуры.

Пусть нам дана ориентирована многополюсная сеть.

Выделим в ней один полюс и назовем его корнем, а остальные полюса назовем листьями.

Выделим в сети некоторие вершины и назовем их точками переключения (полюса могут быть точками переключения).

Если β вершина сети, то через ψ_β обозначим полустепень исхода вершины β.

Каждой точке переключения β сопоставим некий символ из G, такой, что максимальное значение переключателя, соответствующего этому символу, не превышает ψ_β. Это соответствие назовем нагрузкой точек переключения.

Для каждой точки переключения β ребрам, из нее изходящим, поставим во взаимооднозначное соответсвие числа из множества $\{\overline{1,\psi_\beta}\}$. Эти ребра назовем переключательными, а это соответствие — ангрузкой переключательных ребер.

Ребра, не являющиеся переключательным, назовем предикатными.

Каждому предикатному ребру сети сопоставим некоторый символ из множества F. Это соответствие назовем нагрузкой ребер.

Сопоставим каждому листу сети некоторую запись из множества Y. Это соответствие назовем нагрузкой листьев.

Полученную нагруженную сеть назовем информационной сетью с переключателями над базовым множеством $\mathcal{F} = \langle F, G \rangle$.

2. этап. Определение функционирования ИСП.

Пусть нам дана ИСП U.

Последовательность ориентированных ребер сети (α_1, α_2), (α_2, α_3), ..., $(\alpha_{m-1}\alpha_m)$ назовем ориентированной цепью от вершины α_1 к вершине α_m.

Если $f(x)$ предикат, то $N_f = \{x \in X : f(x) = 1\}$.

Если n — натуралное число, а $g(x)$ — некий переключатель, то через $\xi_g^n(x)$ обозначим предикат, определенный на X, такой, что

$$N_{\xi_g^n} = \{x \in X : g(x) = n\}.$$

Обозначим

$$\hat{G} = \{\xi_g^n : g \in G, n \in \mathbf{N}\}.$$

Если c ребро сети, то через $[c]$ обозначим его нагрузку.

Проводимостью ребра (α,β) назовем предикат, равный

а) $[(\alpha,\beta)]$, если ребро — предикатное;

б) $\xi_g^{[(\alpha,\beta)]}$, если ребро — переключательное, где g — переключиватель, соответсвующий вершине α.

Проводимостью ориентированноий цепи назовем конъюнкцию проводимостей ребер цепи.

В ИСП по аналогии с контактными схемами введем для каждой пары вершин α и β функцию проводимости $f_{\alpha\beta}$ от вершины α к вершине β следующим образом:

1) если $\alpha = \beta$, то $f_{\alpha\beta}(x) \equiv 1 \quad (x \in X)$;
2) если $\alpha \neq \beta$ и не существует в ИПС ориенторованных цепей от α к β, то $f_{\alpha\beta}(x) \equiv 0$;
3) если $\alpha \neq \beta$ и множество ориентированных цепей от α к β не пусто, то $f_{\alpha\beta}(x)$ равно дизъюкции проводимостей всех ориентированных цепей от α к β.

Функцию проводимости от корня ИСП к некоторой вершине β ИСП назовем функцией фильтра вершины β и обозначим $\phi_\beta(x)$.

Определим для ИСП U функцию $\Im : X \to 2^Y$, которую назовем функцией ответа сети U, следующим образом. Возьмем произвольный запрос $x \in X$. Для каждого листа ИСП U вычислим функцию его фильтра на запросе x. Определим множество J, которое составим из записей, соответствующих листьям, функции фильтров которых оказались равным 1 на запросе x. Это множество записей назовем ответом на запрос x. Теперь объявим множество J значением функции ответа $\Im$ на запросе x, т.е. $\Im(x) = J$.

Будем говорить, что ИСП U реализует определенную выше функцию ответа $\Im$.

Тем самым мы описали функциониривание ИСП и полностью определили понятие информационной сети.

Пусть $\mathcal{U}$ — некоторая подсеть (т.е. произвольное подмножество вершин в ребер) ИПС U. Через $\langle\mathcal{U}\rangle$ обзначим множество записей, соответствующих листьям этой подсети.

В частности, если α — некоторый лист сети U, то $\langle\alpha\rangle$ — есть множество, состоящее из одного элемента — записи, соответсвтующей листу α, и поэтомы под $\langle\alpha\rangle$ мы будем понимать запись, соответствующую листу α.

Тогда скажем, что ИСП U разрешает ЗИП $I = \langle X, V, \rho\rangle$, если для любого запроса $x \in X$ ответ на этот запрос содержит все те и только те записи из V, которые удовлетворяют запросу x, т.е.

$$\forall x(\Im(x) = \{y \in V : x \,\rho\, y\}).$$

Введем следующие обозначения.

Пусть U — ИСП.

Через $\mathcal{R}(U), \mathcal{P}(U), \mathcal{L}(U)$ (или просто $\mathcal{R}, \mathcal{P}, \mathcal{L}$) обозначим множества вершин, точек переключения и листьев сети U соответственно.

Используя эти обозначения, функцию ответа можно представить следующим образом:

$$\Im(x) = \langle\{\alpha \in \mathcal{L}(U) : \phi_\alpha(x) = 1\}\rangle$$

Введем еще одно обозначение.

Пусть y произвольная запись из Y. Обозначим через $O(y,\rho) = \{x \in X : x\,\rho\,y\}$.

Функцию $\chi_{y,\rho}$, такую, что $N_{\chi_{y,\rho}} = O(y,\rho)$, будем называть характеристической функцией записи y.

Через $L_U(y)$ обозначим множество листьев сети U, которым соответствует запись y.

Справедлива следующая теорема.

Теорема 1. *ИСП U разрешает ЗИП $I = \langle X, V, \rho\rangle$ тогда и только тогда, когда для любой записи $y \in V$, такой, что*

а) $O(y,\rho) = \emptyset$, либо $L_U(y) = \emptyset$, либо $\bigvee_{\alpha \in L_U(y)} \phi(x) \equiv 0$;

б) $O(y,\rho) \neq \emptyset$ справедливо $L_U(y) \neq \emptyset$ и $\bigvee_{\alpha \in L_U(y)} \phi(x) = \chi_{y,\rho}(x)$.

Пусть нам даны множество запросов X, множество записей Y, отношение поиска ρ на $X \times Y$, и базовое множество $\mathcal{F} = \langle F, G\rangle$.

Скажем, что базовое множество предикатов $\mathcal{F}$ полно для отношения поиска ρ, если для любой ЗИП $I = \langle X, V, \rho\rangle$, где $V \subseteq Y$, существует ИСП U над базовым множеством $\mathcal{F}$, разрешающая ЗИП I.

Докажем следующий результат, относящийся к проблеме полноты для ИСП.

Теорема 2. *Пусть заданы множества запросов X, записей Y и отношение поиска ρ на $X \times Y$. Тогда базовое множество $\mathcal{F} = \langle F, G\rangle$ будет полным для отношения ρ тогда и только тогда, когда для любой записи $y \in Y$ функцию $X_{y,\rho}(x)$ можно представить формулой вида*

$$\chi_{y\rho}(X) \bigvee_{i=1}^{n} \mathop{\&}_{j=1}^{m_i} f_{ij}(x),$$

где $f_{ij} \in F \cup \hat{G}$.

Каждой ИСП U можно сопоставить некий алгоритм. Предполагается, что этот алгоритм хранит в своей (внешеней) памяти структуру ИСП U. Входними данными алгоритма является запрос. Выходным данными является множество записей.

Опишем этот алгоритм.

Пусть на вход алгоритма поступил запрос X. Работу алгоритма начинаем из корня сети U, объявляя это текущей вершиной первого шага. Если текущая вершина есть точка переключения, то вычисляем на запросе x переключатель, соответствующий данной вершине и объявляем конец ребра, исходящего из текущей вершины нагрузка которого равна значению переключителя, текущей вершиной следующего шага, если только эта вершина не была текущей на предыдущих шагах. Если текущая вершина не являетс-я точкой переключения, то просматриваем по очереди исходящие из нее ребра и вычисляем значения предикатов, приписанных этим ребрам, на запросе x. Концы ребер, которым соответствуют предикаты со значениями, равнами 1, объявляем текущими вершинами следующего шага, если только на предыдущих шагах эти вершины не объявлялись текущими. Затем переходим к следующему шагу, на котором данная процедура повторяется для всех текущих вершин очередного шага. Через некоторое количество шагов, мы попадем во все вершины, функции фильтров которых равны 1 на запросе x. Если среди этих вершин есть листья, то записи, соответствующие этих листьям включаем в выходные данные алгоритма. Остается заметить, что если ИСП разрешает задачу I, то множество, полученное на выходе алгоритма, будет содержать все те и только те записи библиотеки $\langle U\rangle$, которые удовлетворяют запросу x. Т.е. полученный алгоритм решает ЗИП $I = \langle X, V, \rho\rangle$, где $V = \langle U\rangle$, и значит является алгоритмом поиска.

Таким образом ИСП, как управляющая система, может рассматриваться, как модель алгоритма поиска, работающего над данными, организованными в структуру, определяемую структурой ИСП.

Введем теперь понятие сложности ИСП. Но сначала определим понятие сложности ИСП на запросе.

Будем считать, что время вычисления любого переключителя из G примерно одинаково и характеризуется числом a, а время вычисления любого предиката из F — числом b.

Пусть нам дана некая ИСП U и произвольно взятый запрос $x \in X$. Пусть A — определенный ранее алгоритм, сопоставленый ИСП U. Сложностью ИСП U на запросе x назовем число $T(U, x)$, равное количеству переключателей, вычисленных алгоритмом A при подаче на его вход запроса x, умноженное на a, плюс количество вычисленных предикатов, умноженное на b, т.е.

$$T(U,x) = b \sum_{\beta \in \mathcal{R}(U) \setminus \mathcal{P}(U)} \phi_\beta(x) \cdot \psi_\beta + a \cdot \sum_{\beta \in \mathcal{P}(U)} \phi_\beta(x) \, .$$

Величина $T(U, x)$ характеризует время работы алгоритма A при подаче на его вход запроса x.

Сложность ИСП можно вводить по разному, например как максималную сложность на запросе, как обычно и делеатся, но в рамках рассматриваемой модели оказывается удобным исследовать среднее значение сложности на запросе, поэтому мы введем понятие сложности ИСП как среднее значение сложности ИСП на запросе, взятое по множеству всех запросов, для этого, введем вероятностное пространство над множеством запросов X, под которым будем понимать тройку $\langle X, \sigma, P\rangle$, где σ — некоторая алгебра подмножеств множества X, P — вероятностная мера на σ, т.е. аддитивная мера, такая, что $P(X) = 1$.

Справедлива следующая лемма.

Лемма 1. *Если алгебра σ содержит все множества N_f, где $f \in F \cup \hat{G}$, то для любой ИСП U над базовым множеством $\mathcal{F} = \langle F, G\rangle$ функция $T(U, x)$, как функция от x, является случайной величиной.*

Теперь мы можем определить сложность $T(U)$ сети U, как математическое ожидание случайной величины $T(U, x)$:

$$T(U) = MT(U,x) = \int_X T(U,x)\,P(dx) = \int_X \Big(b \cdot \sum_{\beta\in\mathcal{R}(U)\setminus\mathcal{P}(U)} \phi_\beta(x)\cdot\psi_\beta +$$
$$+a \cdot \sum_{\beta\in\mathcal{P}(U)} \phi_\beta(x)\Big)\,P(dx) = b\cdot\sum_{\beta\in\mathcal{R}\setminus\mathcal{P}} \psi_\beta P(N_{\phi_\beta}) + a\cdot\sum_{\beta\in\mathcal{P}} P(N_{\phi_\beta})\,.$$

Если (β, α) — ребро, то сложностью этого ребра назовем число

а) $P(N_{\phi_\beta})$ — если (β, α) — предикатное ребро;

б) $P(N_{\phi_\beta})/\psi_\beta$ - елси это ребро — переключательное.

Тогда в этих терминах сложность ИСП равна сумме сложностей ребер ИСП.

Пусть нам дана некая ЗИП I. Сложностью задачи I при базовом множестве $\mathcal{F}$ назовем число

$$T(I,\mathcal{F}) = \inf\{T(U) : U \in \mathcal{U}(I,\mathcal{F})\}\,,$$

где через $\mathcal{U}(I,\mathcal{F})$ обозначено множество всех ИСП над базовым множеством $\mathcal{F}$, разрешающих задачу I.

Класс ИСП — это совсем новый класс и ранее в печати не появлялся.

В случае, когда базовое множество переключателей G пусто, т.е. в сетях нет переключателей, то ИСП называется информационными сетями с дублированием листьев (ИСД). Класс ИСД также сравнительно новый и вводился в [28]. В классе ИСД будем считать, что число b, характеризующее время вычисления предикатов из F, равно 1.

В классе ИСД справедлива следующая нижняя оценка.

Теорема 3. *Пусть* $I = \langle X, V, \rho \rangle$ — *ЗИП,* $\mathcal{F} = \langle F, G \rangle$— *базовое множество, полное для отношения* ρ *и удоблетворяющее условию леммы1, тогда*

$$T(I, \mathcal{F}) \geq \sum_{y \in V} P(O(y, \rho)) .$$

Суть теоремы заключается в том, что время писка не меньше времени, требуемого на перечисление ответа.

Такого результата стоит ожидать и в классе ИСП, но этот результат еще не опубликован.

ИСД, различным листьям которой соответствуют различные записи, называется информационной сетью (ИС). Это понятие впервые введено в [24]. Более доступными изданиями являются [25,26].

ИС, граф которой является деревом, а листья совпадаютс висячими вершинами дерева, назовем информационным деревом (ИД).

Впервые понятие ИД было опубликовано в работах [19–21]. Класс ИД исследовался также в кандидатской диссертации автора.

ИД удобны и интересны тем, что структуры данных, им соответствующие, практичны и их гораздо проще реализовать на ЭВМ. Тогда как ИС обладают большими возможностями и охватывают более широкий класс алгоритмов. Поэтому представляет интерес выявление классов задач информационного поиска, для которых оптимальные (т.е. с минимальной сложностью) ИС находятся в классе ИД.

Один из таких классов приводится в [27]. Опишем его и дадим основные резултаты [27].

Скажем, что ИСД обладает A-свойством, если корень сети имеет полустепень захода O, каждый лист сети имеет полустепень исхода O, и сеть состоит только из вершин и ребер, принадлежащих хотя бы одной цепи, ведущей из корня в какой-либо лист.

Скажем, что вершина α сети достижима из вержины β, если из β в α существует ориентированная цепь.

Пусть β — вершна некоторой ИСД. Обозначим через V_β множество записей, соответствующих листьям, достижимым из вершины β.

Скажем, что ИС, разрешающая ЗИП $I = \langle X, V, \rho \rangle$,обладает B_I-свойством,если для любой вершины β сети, за исключением корня

$$N_{\phi_\beta} = \bigcup_{y \in V_\beta} O(y, \rho) \quad \text{или} \quad \phi_\beta = \bigvee_{y \in V_\beta} \chi_{y,\rho} .$$

Обозначим

$$F_o^I = \{ \bigvee_{j=1}^{m} \chi_{y_{i_j}, \rho} : m = \overline{1,k},\ 1 \leq i_1 < i_2 < \ldots < i_m \leq k \} ,$$

где $I = \langle X, V, \rho \rangle$ — ЗИП, $\{y_1, y_2, ..., y_k\} = V$.

Скажем, что базовое множество $\mathcal{F} = \langle F, \emptyset \rangle$ обладает C_I-свойством, если для $\forall f \in F$ $N_f \in \sigma$, и множество $\mathcal{U}(I, \mathcal{F})$ не пусто (здесь $I = \langle X, V, \rho \rangle$ — ЗИП, σ — алгебра подмножеств X).

Скажем, что ИД над базовым множеством F_O^I обладает D_I-свойством, если оно обладает A-свойством, и B_I-свойством.

Обозначим через $\mathcal{D}^I$ множество всех ИД, обладающих D_I-свойством.

Скажем, что ЗИП $I = \langle X, V, \rho \rangle$ обладает E-свойством, если

а) для любой записи $y \in V$ $O(y, \rho) \in \sigma$ и $P(O(y, \rho)) \neq 0$;

б) для любых y, $y^{\check{}} \in V$, таких, что $y \neq y^{\check{}}$

$$P(O(y, \rho) \cap O(Y^{\check{}}, \rho)) = 0\,.$$

Справедлива следующая теорема.

Теорема 4. *Пусть $I = \langle X, V, \rho \rangle$ — ЗИП, обладающая E-свойством, $\mathcal{F} = \langle F, \emptyset \rangle$ — произвольное базовое множество, обладающее C_I-свойством, U — произвольная ИСП над базовым множеством $\mathcal{F}$, разрешающая ЗИП I. Тогда существуем ИД $D \in \mathcal{D}^I$, такое что*

$$T(D) \leq T(U)\,.$$

Скажем,что ЗИП $I = \langle X, V, \sigma \rangle$ обладает F-свойством, если она обладает E-свойством и для любых y, $y^{\check{}} \in V$ $P(O(y, \rho)) = P(O(y^{\check{}}, \rho))$.

Обозначим

$$R(k) = 3 \cdot k[\log_3 K] + 4 \cdot 4(k - 3^{[\log_3 k]}) + \max(O, k - 2 \cdot 3^{[\log_3 k]})\,.$$

Справедлива следующая теорема.

Теорема 5. *Если $I = \langle X, V, \rho \rangle$ — ЗИП, обладающая F-свойством, $\mathcal{F} = \langle F, \emptyset \rangle$ — базовое множество, обладающее C_I-свойством, U — ИСД, разрешующая СИП I, то*

$$T(U) \geq P(O(y, \rho)) \cdot R(k)\,,$$

где $y \in V$, $k = |V|$ — мощность библиотеки V.

Для сравнения отметим, что нижняя оценка сложности ЗИП, получаемя с теоремы 3 для задач, обладающих F-свойством, равна константе, не превышающей 1.

В работе [28] исследуется следующая задача поиска. Дано конечное множество точек из отрезка [0,1]. Запрос задает некий отрезок $[a, b] \subseteq [0, 1]$. Надо перечислить все точки из множества которые попадают в отрезок $[a,b]$. Это известная задача, являющаяся одной из базисных в геометрических задачах поиска [7], получивших распространение в звязи с развитием компьюторной графики и других компьюторнтых дисциплин. В данной работе не столько приводятся алгоритмы решения этой задачи (хотя последний из приведенных алгоритмов интересен и сам по себе, и в среднем требует помимо времени, хеобходимого на перечисление ответа, лишь константное время), сколько исследуется какие алгоритмы возникают, если ограничивать набор доступных средств,или, более формално, при различных базовых множествах. Получены также некоторые нижние оценки, с помощью которых показывается, что соответствующие полученные алгоритмы не могут быть существенно улубены при данных ограничениях на набор доступных средств.

В ЗИП из [28] множество записей Y есть отрезок $[0,1]$, множество запросов X есть множество отрезков $[u,v] \subseteq [0,1]$, или множество пар точек (u,v), таких,что $0 \leq u \leq v \leq 1$, т.е. $X = \{x = (u,v) : 0 \leq \leq u \leq v \leq 1\}$.

На X задано вероятностное пространство $\langle X, \rho, P\rangle$,где σ— алгебра подмножеств множества X, содержающя все прямоугольники со сторонами параллельным осям координат и прямоыгольные равнобедренные треугольники с катетами также параллельными осям координат, P — вероятностная мера на σ. Будем считать, что мера P определяется функцией плотности распределения вероятностей $p(u,v)$, т.е. для $\forall B \in \sigma$ $P(B) = \int_B p(u,v)\, du\, dv$. Причем для удобства договоримся считать, что $p(u,v)$ определена на всем квадрате $[0,1]\times[0,1]$, но при $(u,v) \notin X$ $p(u,v) = 0$.

Отношение поиска, которое будем обозначить через $\rho_{и}$, определяется соотношением

$$(u,v)\rho_{и} y \iff u \leq y \leq v\,,$$

где $(u,v) \in X$, $y \in Y$.

Обозначим

$$M_{a,b} = \{x = (u,v) \in X : u \leq b,\ v \geq a\}\,.$$

Рассмотрим случай, когда базовое можество предикатов равно

$$F_1 = \{f_{a,b} : (a,b) \in X\}\,,$$

где $N_{f_{a,b}} = M_{a,b}$, а базовое множество $\mathcal{F}_1 = \langle F_1, \emptyset\rangle$.

Отметим, что для $\forall f \in F_1$ $N_f \in \sigma$, а для $\forall y \in Y$ $O(y, \rho_{и}) = M_{y,y}$, а для произвольной ЗИП $I = \langle X, V, \rho_{и}\rangle$ $\mathcal{U}(I, \mathcal{F}_1) \neq \emptyset$.

Справедливы следующие теоремы.

Теорема 6. *Если функция плотности распределения вероятностей $p(u,v)$, определяющая меру P вероятностного пространства* *над множеством запросов X, ограничена, то для произвольной ЗИП $I = \langle X, V, \rho_и \rangle$*

$$\sum_{y \in V} P(O(y, \rho_и)) \leq T(I, \mathcal{F}_1) \leq \sum_{y \in V} P(O(y, \rho_и)) + \xi(k),$$

где $k = |V|$, $\xi(k) = \underline{O}(\sqrt{k})$ при $k \to \infty$.

Теорема 7. *Существуем такая функция плотности распределния вероятностней $p(u, v)$, что если с помощью нее определить меру p вероятностного пространства над множеством запросов X, по существуем такая ЗИП $I = \langle X, V, \rho_и \rangle$, что*

$$T(I, \mathcal{F}_1) = \sum_{y \in V} P(O(y, \rho_и)) + \xi(k),$$

где $k = |V|$, $\xi = \underline{O}(\sqrt{k})$ при $k \to \infty$.

Базовое множество $\mathcal{F}_1$ настолько узкое, что при нем ИС не имеют никакого преимущества перед ИД, поэтому можно считать оценки теорем 6 и 7, являются оценками, которые можно получит в классе ИД.

Возьмем теперь в качестве базового множества предикатов следующе множество

$$F_2 = P_1 \cup \{\bar{f}_{0,a} : a \in [0, 1]\},$$

где $\overline{}$ — символ логического отрицания, и примем $\mathcal{F}_2 = \langle F_2, \emptyset \rangle$.

Очевидно, что для $\forall f \in F_2$ $N_f \in \sigma$.

Спреведливы следующие теоремы.

Теорема 8. *Пусть $I = \langle X, V, \rho_и \rangle$ — ЗИП, где $V = \{y_1, ..., y_k\}$, причем $0 \leq y_1 \leq ... \leq y_k \leq 1$. Тогда*

$$T(I, \mathcal{F}_2) \leq \sum_{i=1}^{k-1} P(O(y_i, \rho_и)) + 2\,]\log_2 k[\,.$$

Эта оценка достигается в классе ИС.

Теорема 9. *Для любого $k \in \mathbb{N}$ существуем такая функция плотности распределния вероятностней $p^k(u, v)$, определяющая меру вероятностного пространства над X, и такая библиотека V_k мощности k, определяющая вместе с $p^k(u, v)$ ЗИП $I_k = \langle X, V_k, \rho_и \rangle$, что для* *любого базового множества $\mathcal{F} = \langle F, \emptyset \rangle$, такого, что $\mathcal{U}(I_k, \mathcal{F}) \neq \emptyset$, и для $\forall f \in F$ $N_f \in \sigma$*

$$T(I_k, \mathcal{F}) \geq \sum_{y \in V_k} P(O(y, \rho_{\text{и}})) + (3 \cdot \log_3 k - 1) \cdot k/(2 \cdot k + 1) \geq$$
$$\geq \sum_{y \in V_k} P(O(y, \rho_{\text{и}})) + c \cdot \log_2 k\,,$$

где c — константа, в качестве которой можно взять, например, $c = = 2 \cdot (3 \cdot \log_3 2 - 1)/5$.

Обозначим

$$A_a = \{(u, v) \in X : u \leq v < u + a\}$$

Рассмотрим случай, когда базовое множество предикатов равно

$$F_3 = F_2 \cup \{f_a \to N_{f_a} = A_a,\quad a \in [0,1]\} \cup \{\overline{f}_a : a \in [0,1]\}\,.$$

Понятно, что для $\forall f \in F_3$ $N_f \in \sigma$.

Пусть $\mathcal{F}_3 = \langle F_3, \emptyset \rangle$.

Справедлива следующая теорема.

Теорема 10. *Пусть функция плотности распределения вероятностей $p(u, v)$, определяющая меру P вероятностного пространства над множеством запросов X, такая, что $p(u, v) \leq c$. Пусть $I = \langle X, V, \rho_{\text{и}} \rangle$ — произвольная ЗИП, такая, что $|V| = k$. Тогда*

$$T(I, \mathcal{F}_3) \leq \sum_{y \in V} P(O(y, \rho_{\text{и}})) + 2 \cdot \log_2 \log_2 k + 6 + 2 \cdot c\,.$$

Эта оценка достигается в классе ИСД.

Введем следующие переключатели

$$g_a^1(x) = \begin{cases} 1, & \text{если } x \in A_a \\ 2, & \text{если } x \notin A_a. \end{cases}$$

Если $x = (u, v)$, то $g_m^2(x) = \max(1,]u \cdot (m+1)[)$.

Обозначим

$$G_1 = \{g_a^1(x) : a \in [0,1]\} \cup \{g_m^2(x) : m \in \mathbb{N}\}\,,$$
$$\mathcal{F}_4 = \langle F_2, G_1 \rangle\,.$$

Будем считать ,что число a, характеризирующее время вычисления переключителей из G_1 равна 1. Тогда справедлива следующая теорема.

Теорема 11. *Пусть функция плотности распределения вероятностей $p(u,v)$, определяющая меру P вероятностного пространства над множеством запросов X, ограничена. Пусть $I = \langle X, V, \rho_{\text{и}} \rangle$ — произвольная ЗИП. Тигда*

$$T(I, \mathcal{F}_4) \leq \sum_{y \in V} P(O(y, \rho_{\text{и}})) + 3\,.$$

Опишем алгоритм, на котором достигается последняя оценка.

Упорядочим записи в библиотеке $V = \{y_1, y_2, ..., y_k\}$ так, что $y_1 \leq \leq y_2 \leq ... \leq y_k$.

Пусть $S = \{s_1, ..., s_m\}$, где $s_i = i/(m+1)$, $i = \overline{1,m}$. Для каждого s_i $(i = \overline{1,m})$ найдем два целых числа l_i и r_i, первое из которых является номером ближайшей к s_i записи из V, меньшей, чем s_i, а второе — номером ближайщей к s_i записи из V, не меньшей, чем s_i.

Пусть нам дан некий запрос $x = (u,v) \in X$. Поиск по этому запросу будем осуществять следующим образом.

Сначала вычислим длину интервала x.

Если $v - u < 1/(m+1)$, т.е. если $g^1_{1/(m+1)}(x) = 1$, то с помощью дитохомического поиска за $\log_2$ к шагов находим самую любого запись, находящуюся не левее левого конца запроса. Затем слева направо, начиная с найденной записи, просматриваем записи, сравнивая их с правым концом запроса, и если оказывается, что очередная запись не больше правого конца, то эту запись включаем в ответ, а если больше, то поиск прекращаем.

Если $v - u \geq 1/(m+1)$, мы с помощью переключателя g^2_m найдем в множестве S самую точку s_i, попадающую в интервал x (такая точка обязательно существует), затем по ссылке l_i идем в ближайщую слева к s_i запись библиотеки V и проверяем попадет ли она в x, если попадает, то справа на лево просматриваем следующие записи и проверяем на попадание в x. Затем идем по ссылке r_i и, начиная с записи, в которую ведет эта ссылка, просматриваем слева направо записи с проверкой на попадание в x.

Таким образом, в первом случае помимо перечисления ответа мы тратим время $\underline{O}(\log k)$, а в втором — время, необходнимое на вычисленије двух переключателей $g^1_{1/(m+1)}(x)$ и $g^2_m(x)$. Выбрав подходящим образом параметр m (например, взяв его равным $[2 \cdot \cdot \log_2 k + 1) \cdot c]$, где c такое, что $p(u,v) \leq c$) можно добиться того, чтобы первый случай происходил настолько редко, что сложность, им даваемая в среднем, не превосходила 1, из чего и получается оценка теоремы 11.

В [25] в классе ИС исследовался случай, когда отношение поиска является отношением линейного квазипорядка.

Под отношением линейного квазипорядка будем понимать бинарное отношение $\preceq$, определенное на $X \times X$ и для любых $x, y, z \in X$, удовлетворющее условиям:
а) рефлексивности $X \preceq x$;
б) транзитивности $(x \preceq y) \& (y \preceq z) \Rightarrow (x \preceq z)$;
в) связности $(x \preceq y) \vee (y \preceq x)$.

Обозначим:
$F_0 = \{f_y(x) : N_{f_y} = O(y, \preceq), y \in X\}$ — множество предикатов;
$\mathcal{U}^C(I, \mathcal{F})$ — множество ИС над базовым множеством $\mathcal{F}$, разрешающих ЗИП I;
$T^C(I, \mathcal{F}) = \inf\{T(U) : U \in \mathcal{U}^C(I, \mathcal{F})\}$.
В работе [25] доказана следующая теорема.

Теорема 12. *Если множества запросов и записей совпадают и обозначены X, отношение поиска $\preceq$ на $X \times X$ является отношением линейного квазипорядка, вероятностное пространство над X $\langle X, \sigma, P\rangle$, такое, что алгебра σ содержит все множества $O(y, \preceq)$, где $y \in Y$, базовое множество $\mathcal{F} = \langle F, \emptyset\rangle$, тако, что $F \supseteq F_0$, то для любой ЗИП $I = \langle X, I, \preceq\rangle$*

$$T^C(I, \mathcal{F}) = 1 + \sum_{y \in V} P(O(y, \preceq)) - \min_{y \in V} P(O(y, \preceq)) .$$

В работах [19–23] исследовался класс ИД для следующих отношений поиска:
а) $X = Y = B_2^n = \{\alpha = (\alpha_1, ..., \alpha_n) : \alpha_i \in \{0,1\}, i = \overline{1,n}\}$, отношение поиска ρ_1 — »не меньше по-компонентно« — определяется соотношением:

$$(x_1, ..., x_n)\rho_1(y_1, ..., y_n) \iff x_i \geq y_i, i = \overline{1,n} ;$$

б) $X = Y = B_2^n$, отношение поиска σ_2 — »расстояние по Хэммингу не превосходит 1« — определяется соотношением:

$$x \,\rho_2\, y \iff \rho(x, y) \leq 1 ,$$

где $\rho(x, y)$ — количество компонент, по которым вектора x и y не совпадают;
в) $X = Y = B_2^n$, отношение поиска ρ_3 — »не больше по норме« — определяется соотношением:

$$x \,\rho_3\, y \iff \|x\| \leq \|y\|, \quad \text{где} \quad \|(x_1, ..., x_n)\| = \sum_{i=1}^{n} 2^{n-i} x_i + 1 ;$$

г) $X = B_3^n = \{\alpha = (\alpha_1, ..., \alpha_n) : \alpha_i \in \{0,1,2\},\ i = \overline{1,n}\}$, $Y = B_2^n$, отношение поиска ρ_4 — »идентичность выделенных компонент« — определяется соотношением:

$$(x_1, ..., x_1)\, \rho_4 (y_1, ..., y_n) \iff \text{ если } x_i \neq 2, \text{ то } x_i = y_i,\ i = \overline{1,n}.$$

Поскольку во всех этих случаях множества запросов являются дискретными множествами, то вероятностное пространство определим равномерным расределеним вероятностей, т.е. для любого запроса $x \in X$ $P(x) = 1/|X|$, где $|X|$ — мочность множества X.

В отличии от предидущих случаев для этих отношений исследовались не сложности ЗИП, а следующие функции, характеризующие сложность целого класса ЗИП:

$$1) T(k, n, \rho, F) = \max_{I \in \mathcal{T}_\rho^k} T^D(I, \mathcal{F}),$$

$$2) \bar{T}(k, n, \rho, F) = |\mathcal{T}_\rho^k|^{-1} \sum_{I \in \mathcal{T}_\rho^k} T^D(I, \mathcal{F}),$$

где n — параметр, характеризуюшщий размотреность множеств запросов и записей; $\rho \in \{\rho_1, \rho_2, \rho_3, \rho_4\}$; $\mathcal{F} = \langle F, \emptyset \rangle$; $\mathcal{T}_\rho^k = \{I = \langle X, V, \rho \rangle : |V| = k\}$; $T^D(I, \mathcal{F}) \min_{D \in \mathcal{D}(I, \mathcal{F})} T(D)$; $\mathcal{D}(I, \mathcal{F})$ — множество ИД над базовым множеством $\mathcal{F}$, разрешающих ЗИП I. Автором были получены асимпототическе оценки этих функций при $n \to \infty$.

В самых раниих работах автора [17,18] рассматривался следующий случай. Множества запросов и записей являются n-мерным единичним кубом. Отношение поиска есть отношение идентичности записи запросу. Базовое множество, такое, что базовое множество предикатов пусто, а базовое множество переключателей состоит из переключатетлей, принимаjищих 2 значения. Задача исследовалась на подклассе класса ИД, состоящем из бинарных ИД, т.е. деревьев, каждая внутреняя вершина которых имеет полустепень исхода 2. Сложность ИД бралась не как средняя сложность по запросам, а как максимальная сложность на запросе. Вводились функции Шеннона как максимальная сложност задач информационнога поиска с одинаковой мощностью библиотек, которые исследовались при разных базовых множествах. Исследовалась также функция, равная минимуму мощности базового множества переключателей, обеспечивающего логарифмический поиск, т.е. поиск, равный по времени логарифму от мощности библиотеки.

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E. E. Gasanov
Moscow State University,
Moscow, Russia

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RECONSTRUCTION OF MISSING VALUES IN DATA MATRICES

Ž. KNAP

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Abstract. In this article we describe a combinatorico-logical method for reconstructing missing values in data matrices, which is based upon the test approach.

In the article we examine the question of reconstruction of data matrices. Herein we propose a method for construction, which is based upon a method of pattern recognition. We will examine some logically possible cases, which arise given the hypothesis that either the columns or rows are properties, respectively, or that each could be either property or entity. This method can be generalised to include cases in which only some rows are properties and other rows are entities and also the case in which it is unknown which are entities and which are properties.

Given are two alphabets, $A = \{a_1, a_2, ..., a_n\}$ and $B = \{*\}$, and we are to examine the matrix $T = ||d_{ij}||$, whose elements are members of the union $A \cup B$; let us define $d = \{*\}$ as undetermined.

We will consider the following cases (problems):

(1) the columns of the matrix are properties or parameters and the rows represent entities;
(2) the columns of the matrix represent entities and the rows represent properties or parameters (problem (2) is a dual of the problem (1));
(3) both the rows and columns represent properties (or parameters);
(4) it is unknown whether the rows or columns represent properties;
(5) some of the columns and some of the rows represent properties.

We will be concern with reconstuction of matrices which represent cases (1) to (5). Hencefore we will refer to each of this cases as problem (1), problem (2), ... , problem (5).

Obviously problems (1) and (2) are mutually eqivalent. Meanwhile, problem (5) implies problems (1), (2) and (3). In particular, we examine problem (1) and problem (3) with a view to demonstrate how to solve problems (4) and (5) on the basis of these solutions.

In order to solve problem (1) we constuct in matrix T a submatrix T', which consists in successive rows and columns which do not contain any „*" as element. We will refer to such a submatrix as a block. We call block T' maximal block, if there does not exist any submatrix M which contains submatrix T' and is not equal to T'. We refer to the set of all maximal blocks T' of T as $\mathcal{B}(T)$. We select an arbitrary maximal block T' from $\mathcal{B}(T)$. For simplicity's sake we will asume that the submatrix T' lies in the upper left corner of matrix T and has t' rows and s' columns. Let us consider an arbitrary and not everywhere determined row α, bearing the number $l > t'$ of the submatrix T_1 which consists in the first s' columns of matrix T. We will show how to reconstuct the unknown elements „*" of α. We construct submatrix T'' from T' by selecting all those columns of row α which do not contain undetermined elements „*". In the case in which T'' is empty, we don't reconstruct any unknown elements at all. If T'' is not empty then we construct from it T''' which consists in all pairwise distinct rows of T'' such that each row of T'' has an equivalent representative row in T'''. On the bases of matrix T''', we compute the set of all tests which we designate as $\mathcal{T}(T''')$. It will be recalled that the test of an arbitrary matrix A is the set of all columns of A, which form such a submatrix of A, which consists only in pairwise distinct rows. We recomend a method for determining „similarity" of α to rows of T'''. On the bases of similarity of α to rows of T''' we determine similarity of α to equivalent rows of T'' and of T'. This in turn enables reconstruction of the unknown elements „*" of α and likewise of the whole matrix.

Let us examine problem (3). We treate this case in analogous fashion as problem (1), and in this sense we reconstruct all the missing elements „*" of T. Then we solve problem (1) with hypothesis that the rows represent properties and the columns entities (i.e. transposition of matrix in problem (1) proper) and so we reconstruct each missing element. In this solution to each missing element „*" of matrix T correspond to a unique mean code which we will refere to as virtual reconstruction element for the value of „*" according to the method of problems (1) and (2). We compute the average values of these codes, and, we construct the matrix TT^{**}. Matrix T^{**} constitutes the solution to the problem (3). Let us remark, in passing, that problem (3) admits approximate solution by implication from problem (1), which also admits approximate solutions. Now, let us examine problem (4). Here we must consider three cases; i.e. firstly, the columns represent properties and rows do not; secondly, the rows represent properties and the columns do not; thirdly, both, rows and columns represent properties. Therefore the following consequencies are obvious. In the first case we solve the problem (1), in the second case we solve problem (2) and in the third case we solve problem (3). By implication, the set of all three solutions constitutes the complete solution of problem (4). Accordingly again by implication we derive approximate solutions for problem (4) from the corresponding approximate solutions to problems (1), (2) and (3) respectively. By specifying the problem further a customer may select any one of solutions (1), (2) or (3) or any of their combinations.

Now we are in a position to examine problem (5), which we divide into several cases. First we consider the case in which a columns represent properties and rows do not. In this case, however, we construct a submatrix of T, which contains all the columns representing properties and for this case we solve problem (1), which in turn yields all possible missing values for elements „*" for this particular submatrix of T. Then we compute the transpose matrix of T and repeat the above procedure, which yields other set of reconstructed missing values. From the results of these two cases we compute a set of weights, which enable (in part) the reconstructin of the unknown elements „*" of the matrix in problem (5). There are cases in which this problem is not solvable in its entirety. As above, approximate solutions of problem (5) follow from the approximate solutions to each of problems (1), (2) and (3) respectively. All the procedures described above lend themselves to algorithmic formulation and indeed algorithms for these procedures exist and are described in [1],[2],[3].

Moreover, our approach affords generalisation to include cases in which rows and columns of matrix T are members of an equivalence classes (taxons). In this generalised form our problem approaches the problem of pattern recognition, which is the subject of a subsequente article.

The lecture derives for the most part from articles [4],[5].

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Ž. Knap
Institute for Social Sciences,
University of Ljubljana,
Ljubljana, Slovenia

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АВТОМАТИ В ГЕОМЕТРИЧЕСКИХ СРЕДИНАХ

В. Б. КУДРЯВЦЕВ

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Abstract. In the article the author gives an overview of various approaches, which were published in past 20 years, to the description of automata behaviour in defined geometrical environments. For each approach, the author describes the fundamental concepts, problems, methods of solution, unsolved problems and achievements.

В последние годи все большее внимание привлекает тематика, связаная с автоматным анализом изображений, графов, формальних языков и других дискретных систем. В общей сложности по этой тематике уже опубликованы свыще ста работ.

Повидимому, одной из первых статей этого направления следует считать работу К. Шеннона 1951 года [47], в которой фактически рассматривалась задача поиска автоматом-мышью определенной цели в лабиринте, что в значительной мере определило тематику направления на последующие годы. Другим источником направления можно считать рассмотрение вычислительных систем с внешеней памятью в виде плоскости или пространства [18], хотя здесь они сравнительно быство были вытеснены многоленточными вычислителями.

Работа К. Шеннона довольно долго не получала развития. Возможно, это было связано с тем что основное внимание специалистов по теории автоматов было связано с изучением возможностей автоматов при переработке слов, за которыми не скрывались интерпретации. Это было характерно для всех основных видов поведений автоматов и прежде всего таких, как автоматы-преобразаватели, автоматы-акцепторы и др. Различные вопросы, связанные с этими типами поведений попрежнему остаются главными в теории автоматов. Другой характерной особенностью здесь является то, что автомат по отношению к множеству входных слов, то-есть к »среде«, воздействующей на него, выступает лишь в роли »получателя«, никак не влияя на нее. Эти особенности отсутствуют в модели »автомат в лабиринте«, что сущаественно ограничивает перенос результатов для других типов поведений автоматов на эту модель.

Определенную активность в изучении поведения автоматов в лабиринтах и графах вызвала публикаця К. Дёппа [16,17]. В ней была формализована модель Шеннона и в качестве лабиринта рассмотрен шахматообразная связная конфугурация клеток на плоскости или аналогичных кубиков в пространстве (шахматные лабиринты), а в качестве автоматов — конечные автоматы, которые, обозревая некоторию окрестность клетки вкоторой находятся, могут перемещаться в соседную клетку в одном из координатных направлений. В работе получены некоторые результаты по задаче обхода таких лабиринтов автоматов и выделен как актуальный вопрос о существовaтии автомата, обходящего все такие лабиринты; приведены соображения в пользу того, что в случае пространственного лабиринта для автомата можно построить лабиринт-ловушку. X. Мюллер [33] для заданного автомата построил плоскую ловушку в виде 3-графа, а Л. Дудах — шахматную ловушку, однако его обоснование оказалось слушком громаздким. А. С. Подколзин [71,72] существенно упростил показательство этого факта. X. Антельман [2] оценил сложность такой ловушки по числу клеток в ней, а X. Мюллер [34] указал, что всегда в качестве нее можно выбрать трехсвязный лабиринт. Затем X. Антельман, П. Будак и X.А. Роллик [1] построили конечную ловушку для конечной системы автоматов и бесконечную ловушку сразу для всех допустимых автоматов. Ф. Хофман [24] дал характеризацию специальных типов графов, которая позволила затем существенно упростить все конструкции в упомянутых утверждениях, построить приводимую здесь ловушку для всех конечных систем автоматов, в которой автоматы остаются в ограниченном шаре.

Наряду с этими пезультатам, указыванющими на ограниченость вовзможностей автоматов, были построены примеры классов лабиринтов, которые обходятся одним автоматом. Эти результаты были обобщены А. Н. Зыричевым [62], который установил, что класс всех плоских шахматных лабиринтов, именющих дыры ограниченного диаметра, также обходястя одним автоматом. А. А. Золотык [61] расширил этот класс, показав, что можно расматривать ограниченность дыр лишь по фиксированному направлению. В этих паботах содержатсь также оценки времени обхода лабиринтов и числа состояний для автоматов. Анализу свойств нагруженных графов посвящена работа Е. К. Кудрявцева [73], котороя установливает, с какой сложностью может быть решена задача эквивалентности поведения автоматов в таких графах.

Невозможность обхода всех плоских шахмарных лабиринтов одним автоматом выдвинула вопрос об изучении возможных услиений модели автомата, уже решенющих задачу обхода.

Рассмотрены несколько вариантов такого усиления. Главным из них является система взаимодействующих автоматов, наизываемя коллективом. В отличие от системы независимых автоматов коллектив анализитует лабиринты с учетом положения его »членов« в лабиринте. Простейшим представителем коллектива является система автоматов с камнями; камни представляют собой автоматы без памяты и их перемещение определяется другими автоматами коллектива; таких образом камни играют роль огранниченой внешеней памяти. Установленно Ф. Хофманом [22,23], что коллектив из одного автомата и одного камна не может обойти все к.п.м. лабиринты; М. Блюм и Д. Козен [6] дали набросок обоснавания того, что коллектив из оного автомата и двух камней решает эту задачу, отметив при этом, что коллектив из двух автоматов должен решать ее тоже. В работе [65] приведено полное доказательство этих фактов. Наряду с этим в А. Хемерлинг [21], К. Кригел [30] показано, что класс указанных лабиринтов допускает естественное раслоение такое, что для любого слоя его наидутся коллектив из одного автомата с камнем, обходящий этот слой; в качестве параметра сдесь выступает число дыр в лабиринте.

Аналогичный вопрос для класса всех конечных и бесконечных п. м. лабиринтов исследуется в таботах М. Блюма, У. Сакоды [5], З. Хабасински, М. Карпински [19], А. Szeptiowski [49], Г. Килибарды [64], А. В. Анджанса [58]. В них установлены некоторые простейшие по числу автоматов и камней коллективы, обходящие все такие лабиринты; в работе [64] завершено описание всех таких коллективов, в ней же приведено решение указанной задачи для лабиринтов, несодержащих бесконечные дыры. Для специального случая лабиринта, именющего вид плоскости в Анджанс [58] указано два типа простейших коллективов (два автомата один камень и один автомат три камня, обходящий его).

Для лабиринтов более общего вида [5,67] показано наличие ловушки уже в трехмерном случае. Установлjено [67] наличие бесконечной трехмерной ловушки сразу для всех коллективов автоматов. При этом коллективы остаются в шаре ограниченного радиуса в этой ловушке. Подобные результаты оказыванются верными и в планарном случае, для лабиринтов имеющих вид кубического графа [38].

Специальным классами лабиринтов является так називаемые сигнатурный и π-лабиринты. Для первого вида в работах [74,66] получены описания простейших коллективов автоматов с камнями, находящих специальную вершину в этих лабиринтах. Для второго вида установлена редукция обхода их по специальным циклам к открытой проблеме совпадения языков расспознаваемых детерминированным и недетерминированными линейно ограниченными машинамиы тьюринга, что свидетельствует о больших потенциальных трудностях тематики.

Начато исследование задач о встрече коллективов автоматов в лабиринтах. Она состоит в установлении для заданой пары коллективов и лабиринтов встречаются ли они в нем или нет. Одним из возможных толкований этой задачи может быть описание для заданного класса лабиринтов всех тех пар коллективов, которые встречанются в любом лабиринте из этого класса. В Анджанс [58] рассмотрен специальный случаы всей одномерной и всей двумерной ленты и указаны простейшие тпиы коллективов автоматов решающих задачу о встрече в них.

Здесь приводится решение такой задачи для классов всех конечных п. м. лабиринтов, и также конечных и бесконечных п. м. лабиринтов путем указания всех простейших типов пар, коллективов автоматов, решающих задачу о встрече в них.

В. И. Грунской [60] рассмотрен вариант задачи о всртечи двух автоматов, хаходящихся в отношении »хищник-жертва«, где автомат-»хищник"« питается погнать жертву, а та — убежат от него; взаимодействие происходит в квадратном лабиринте. Приводятся условия, при которых указанная встреча происходит. В работах [14,12,58] рассматривались возможности более общих моделей автоматов в лабиринтах. В работах [14,12] показано, что автомат с магазинной памятью не может обойти все лабиринты, имеjищие вид 3-графов, а в [58] привидены примеры автоматов со счетчиками, со стеками и магазинами. Установлено, что автомат с магазином обходит все односвязные конечные плоские м. лабиринты и остановливается после обхода.

Задача анализа для автоматов и лабиринтов изучалась [4,26, 21,36,44,50]. Она состоит для заданного коллектива автоматов в описании всех лабиринтов, которые обходятся этим коллективом при возможных дополнительных соглашениях типа требования остановок после обхода. Попытки описать эти лабиринты в виде алгебры Клини встретили затрудения [26,27]; аналогичко обстоит дело с выяснением отношений между классами лабиринтов, представляющих прешение задачи анализа для заданных коллективов автоматов [26]. В работе [4] показано, что классы лабиринтов, обходимые автоматами с камнями, неограничено возрастают с увеличением числа камнй. В работе [44] привидены примеры классов лабиринтов, которые могыт быть решением задачи анализа. Этот доклад сделан В. Б. Кудрявцевим по совместной с Ш. Ушгумличем и Г. Килибардой работы, поэтому эта публикация может считатся совместной.

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V. B. Kudryavcev
Moscow State University
Moscow, Russia

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ON REPRESENTATIONS OF CARTAN ALGEBRAS

D. PAGON

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Key words: Lie algebras, classic simple Lie algebras, Lie algebras of Cartan type, the representations of Lie algebras, irreducible representations of simple modular Lie algebras.

Abstract: We give a description of simple modular Lie algebras and their irreducible modules. An attemp is made to reduce the study of modular Lie algebra modules to the well known classification of irreducible modules of simple Lie algebras over an algebraically closed field of characteristic zero.

1 Introduction

Definition 1. A **Lie algebra** is a vector space over some field F with an additional **antisymmetric and bilinear** binary operation (usually called the **bracket operation**) which must also satisfy the following condition

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0$$

known as *Jacobi identity*.

Definition 2. The Lie algebra L is called **simple** if it contains no nontrivial subspaces closed under the bracket operation when being multiplied with all elements from L (such subspaces are called **ideals**).

Example. Let C^3 be the 3-dimensional complex vector space and $\times\ : C^3\times C^3 \longrightarrow C^3$ the usual vector product. Then $(C^3,+,\times)$ will be a Lie algebra and this algebra is simple.

Lets denote by $B=\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ an orthonormed basis of C^3. If a subspace K contains a nonzero vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then multiplying it twice by the right element from B we'll get one of the basical vectors and multiplying it with the others we'll obtain all elements from B. So the only nonzero ideal is the whole algebra and this means that L is simple. ∎

2 Classical simple Lie algebras

The complete description of simple finite dimenional complex Lie algebras was given by E. Cartan [1] and H. Coxeter [2]. Such algebras form four infinite series which can be represented in the following matrix form:

$A_n=\{T\in M_{n+1}(C)|trT=0\}$ (**general Lie algebras**),

$B_n = \{T \in M_{2n+1}(C) | f_1(Tx, y) = -f_1(x, Ty)\}$ and

$D_n = \{T \in M_{2n}(C) | f_2(Tx, y) = -f_2(x, Ty)\}$ **(orthogonal algebras)**,

$C_n = \{T \in M_{2n}(C) | f_3(Tx, y) = -f_3(x, Ty)\}$ **(symplectic algebras)**,

where f_1, f_2, f_3 are bilinear forms, given by matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \text{ respectively.}$$

Further more, five exceptional finite dimensional simple Lie algebras exist, which are usually denoted as E_6, E_7, E_8, F_4 and G_2.

Definition 3. A linear mapping $\rho : L \longrightarrow \mathrm{Lin}(V)$ from a Lie algebra L to the space of homomorphisms of some vector space V over the same field F is called the **representation** of Lie algebra L if and only if for any two elements $x, y \in L$ $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$. The vector space V is in this case called L-**module**.

Definition 4. The representation ρ (module V) is called **irreducible** if no nontivial subspace $U \subset V$ is mapped into itself by all homomorphisms $\rho(x)$, $x \in L$. A **completely reducible** representation (module) is the direct sum of irreducible subrepresentations (submodules).

H. Weyl [9] proved the complete reducibility of finite dimensional representation of simple Lie algebras over an algebraically closed field of characteristic zero. At the end of this introduction we are giving a brief description of irreducible modules in this case.

Let L be a finite dimensional simple complex Lie algebra and H its abelian subalgebra, containing all elements, which action on L by bracket operation is semissimple (**the Cartan subalgebra**). Then we can write down the **root decomposition** of $L = H \bigotimes \sum_\alpha L_\alpha$.

Also, H acts diagonally on any L-module V : $hv = \lambda(h)v$, which is a direct sum of **weight spaces**, corresponding to **weights** λ. The vector $v \in V_\lambda$ is called **maximal**, if it is annihilated by all subspaces of algebra L, corresponding to positive roots. The L-module generated by maximal vector v of weight λ is called **standard cyclic module** (of weight λ).

It turns out that for each element λ of the dual space H^* a unique (up to isomorphism) irreducible standard cyclic L-module of highest weight λ exists, which is finite dimensional precisely in the case when all values $\lambda(h_i)$ (h_i belongs to the basis of H) are nonnegative integers.

All details can be found in Jacobson's monography [4].

3 Differential extensions of linear Lie algebras

Let $\mathcal{A}$ be a commutative associative algebra over a field F, and $\{D_i : i = = 1, 2, ..., n\}$ a set of commutating derivations of algebra $\mathcal{A}$. Then the space $\mathcal{D} = \{\sum_{i=1}^{n} a_i D_i : a_i \in \mathcal{A}\}$ with the Poisson bracket operation

$$[\sum_{i=1}^{n} a_i D_i, \sum_{j=1}^{n} b_j D_j] = \sum_{i,j=1}^{n} (a_i D_i(b_j) - b_i D_i(a_j)) D_j$$

will be a Lie algebra.

The linearity and anticommutativity of the operation follow immediately from the definition. Also:

$$[[\sum_{i=1}^{n} a_i D_i, \sum_{j=1}^{n} b_j D_j], \sum_{k=1}^{n} c_k D_k] =$$

$$= \sum_{i,j,k=1}^{n} (a_i D_i(b_j) D_j(c_k) - b_i D_i(a_j) D_j(c_k) - c_j D_j(a_i) D_i(b_k) + c_j D_i(a_k) D_j(b_i) -$$

$$- a_i c_j D_i(D_j(b_k)) + b_i c_j D_i(D_j(a_k))) D_k.$$

Permutting twice cyclicaly all three elements and summing we get 0, so the Jacobi identity is also satisfied. ■

If $E_{i,j}$ is the usuall basis of the space of linear transformations of a vector space V over F, we define $[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j}$ and extend this operation by linearity. The obtained general linear Lie algebra we'll denote $M(n)$, where $n = \dim V$.

We can also introduce the structure of Lie algebra on the product $\underline{M}(n) = = \mathcal{A} \bigotimes_F M(n)$ defining $[a \otimes A, b \otimes B] = ab \otimes [A, B]$. Finally, for every linear transformation D, acting on $\mathcal{A}$, we denote by $\underline{D}$ the element $D \otimes I \in \mathrm{Lin}(\mathcal{A}) \bigotimes_F M(n)$ acting on elements from $\underline{M}(n)$ in the natural way: $\underline{D}(a \otimes A) = D(a) \otimes A$.

Theorem 1. For every Lie subalgebra $K \subset M(n)$ the set

$$\mathcal{K} = \{\sum_{i=1}^{n} a_i D_i : \sum_{i,j=1}^{n} D_i(a_j) \otimes E_{i,j} \in \mathcal{A} \otimes K\}$$

is a subalgebra of $\mathcal{D}$.

Proof. If $D = \sum_{i=1}^{n} a_i D_i$, $D' = \sum_{j=1}^{n} b_j D_j$ and $[D, D'] = \sum_{k=1}^{n} c_k D_k$, and we denote

$$\tilde{D} = \sum_{i,k=1}^{n} D_k(a_i) \otimes E_{k,i}, \quad \tilde{D}' = \sum_{j,k=1}^{n} D_k(b_j) \otimes E_{k,j},$$

by direct computation using our definitions we obtain

$$\sum_{i,j=1}^{n} D_i(c_j) \otimes E_{i,j} =$$

$$= \sum_{i,j=1}^{n} \big(\sum_{k=1}^{n} (D_i(a_k)D_k(b_j) - D_i(b_k)D_k(a_j) + a_k D_i(D_k(b_j)) -$$

$$-b_k D_i(D_k(a_j)))) \otimes E_{i,j} =$$

$$= [\tilde{D}, \tilde{D'}] + \underline{D}(\tilde{D'}) - \underline{D'}(\tilde{D}) \in \mathcal{A} \otimes K, \text{ yielding } [D, D'] \in \mathcal{K} . \quad \blacksquare$$

We'll call the Lie algebra $\mathcal{K}$ the **extension** of algebra K.

4 Lie algebras of Cartan type

The four infinite series of simple modular Lie algebras known as Lie algebras of Cartan type were first introduced by Kostrikin and Shafarevich [6], using the derivations annihilating or acting invariantly on some differential forms. We use the same representation of these algebras (the first three series), but give a slightly different definition.

Let $\mathcal{I}(n) = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) : \alpha_i \in \mathcal{N} \cup \{0\} \}$.

For a prime number $p = charF$ and a n-tuple $\underline{m} \in \mathcal{N}^n$ we define

$\mathcal{I}(n, \underline{m}) = \{\alpha \in \mathcal{I}(n) : \alpha_i < p^{m_i}\}$ and

$\mathcal{P}(n) = F[t_1, t_2, ..., t_n] = \langle t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} ... t_n^{\alpha_n} \rangle$

Definition 5. Let $O(n) = \mathcal{P}(n)^*$ be the dual space and $\{x^{(\alpha)} : \alpha \in \mathcal{I}(n)\}$ its basis, dual to all t^α.

Introducing the multiplication $x^{(\alpha)}\ x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}$, where $\binom{\alpha+\beta}{\alpha} = \prod_{i=1}^{n} \binom{\alpha_i+\beta_i}{\alpha_i}$, we obtain an associative algebra, called the **divided power algebra.**

This algebra has a natural gradation: $O(n) = \bigoplus_k O_k(n)$, where

$$O_k(n) = \langle x^{(\alpha)} : |\alpha| = \sum_{i=1}^{n} \alpha_i = k \rangle.$$

We'll concentrate our attention on its subalgebra $O(n, \underline{m}) = \langle x^{(\alpha)} : \alpha \in \mathcal{I}(n, \underline{m}) \rangle$.

Definition 6. The linear mappings $D_i : x^{(\alpha)} \longmapsto x^{(\alpha')}$, where $\alpha'_j = \alpha_j - \delta_{i,j}$, called **special derivations** of devided power algebra, with Poisson multiplication span **general Lie algebras of Cartan type** $W(n, \underline{m}) = \{ \sum_{i=1}^{n} a_i D_i : a_i \in O(n, \underline{m})\}$.

These algebras are simple, excluding the case $p = 2,\ n = 1,\ \underline{m} = 1$.

Definition 7. Extending the special linear algebra A_n, containing all matrices with trace 0, we obtain algebras $\tilde{S}(n, \underline{m}) = \{\sum_{i=1}^{n} a_i D_i : \sum_{i=1}^{n} D_i(a_i) = = 0\}$.

Their commutators $S(n, \underline{m}) = \tilde{S}(n, \underline{m})'$ are simple algebras called **special algebras of Cartan type.**

Definition 8. Similarly, starting with the symplectic algebra C_n containig all matrices of the form $\binom{A\ B}{C\ D}$ where B, C are simmetric matrices of dimension n and $D = -A^T$ and taking the second commutator of the extension, we obtain **Hamiltonian algebras of Cartan type** $H(2n, \underline{m})$, which are simple in the case of $Char\ F > 2$.

5 The mixed product of representations and graded representations of graded Lie algebras

In the following we'll denote by $L = L(n, \underline{m})$ any of the introduced above finite dimensional Lie algebras of Cartan type and by $\underline{L}$ the direct sum $L \bigoplus O(n, \underline{m})$. As in the above constructions elements of L are represented as derivations of $O(n, \underline{m})$, this representation is called the **identity** or the **derivation representation** of L and is usually denoted by δ.

Theorem 2. Let K be a subalgebra of $M(n)$, $\mathcal{K}$ its extension and $\underline{\mathcal{K}} = \mathcal{K} \bigoplus O(n, \underline{m})$. If ρ is any representation of algebra K and σ is a representation of $\underline{\mathcal{K}}$, such that the restriction $\sigma|_{O(n,\underline{m})}$ is an associative algebra representation, then $\tau = \rho \underset{k}{\times} \sigma$ defined as

$$\tau(D + f) = \sigma(D + kf) \bigotimes I + (\sigma|_{O(n,\underline{m})} \bigotimes \rho)(\tilde{D}), \text{ where } k \in F,$$

is a representation of algebra $\underline{\mathcal{K}}$, called the **mixed product of representations** σ, ρ.

Proof. Let $D = \sum_{i=1}^{n} a_i D_i$, $D' = \sum_{j=1}^{n} b_j D_j$, $f \in O(n, \underline{m})$ and $\underline{\sigma}(D) = = \sigma(D) \bigotimes I$. Then $[\underline{\sigma}(D), \underline{\sigma}(D')] = \underline{\sigma}([D, D'])$ and $[\underline{\sigma}(D), \underline{\sigma}(f)] = \underline{\sigma}(D(f))$.

Let us denote $\underline{\rho} = \sigma|_{O(n,\underline{m})} \bigotimes \rho$ and for any $f_i, g_j \in O(n, \underline{m})$, $T_i \in K$

$$\tilde{D} = \sum_{i=1}^{n} f_i \bigotimes T_i,\ \tilde{D}' = \sum_{j=1}^{n} g_j \bigotimes T_j.$$

Then $\underline{\rho}(\tilde{D})\underline{\sigma}(D') = (\sum_{i=1}^{n} \sigma(f_i) \bigotimes \rho(T_i))(\sigma(D') \bigotimes I) = \sum_{i=1}^{n} \sigma(f_i)\sigma(D') \bigotimes \rho(T_i)$, while

$$\underline{\sigma}(D')\underline{\rho}(\tilde{D}) = (\sigma(D') \bigotimes I)(\sum_{i=1}^{n} \sigma(f_i) \bigotimes \rho(T_i)) = \sum_{i=1}^{n} \sigma(D')\sigma(f_i) \bigotimes \rho(T_i).$$

So we obtain the following commutators:

$$[\underline{\rho}(\tilde{D}), \underline{\sigma}(D')] = -\underline{\rho}(\underline{D'}(\tilde{D})), \quad [\underline{\rho}(\tilde{D'}), \underline{\sigma}(D)] = -\underline{\rho}(\underline{D}(\tilde{D'})).$$

We already know (proof of theorem 1) that $\widetilde{[D, D']} = [\tilde{D}, \tilde{D'}] + \underline{D}(\tilde{D'}) - \underline{D'}(\tilde{D})$, so applying $\underline{\rho}$ to this identity we obtain $\underline{\rho}(\widetilde{[D, D']}) = [\underline{\rho}(\tilde{D}, \underline{\rho}(\tilde{D'})] + + \underline{\rho}(\underline{D}(\tilde{D'}) - \underline{\rho}(\underline{D'}(\tilde{D})$.

Then $[\tau(D), \tau(D')] = [\underline{\rho}(\tilde{D}) + \underline{\sigma}(D), \underline{\rho}(\tilde{D'}) + \underline{\sigma}(D')] =$

$$= [\underline{\rho}(\tilde{D}), \underline{\rho}(\tilde{D'})] + [\underline{\rho}(\tilde{D}), \underline{\sigma}(D')] +$$

$$+ [\underline{\sigma}(D), \underline{\rho}(\tilde{D'})] + [\underline{\sigma}(D), \underline{\sigma}(D')] =$$

$$= \underline{\rho}([\tilde{D}, \tilde{D'}]) - \underline{D'}(\tilde{D}) + \underline{D}(\tilde{D'}) + \underline{\sigma}([D, D']) =$$

$$= \underline{\rho}(\widetilde{[D, D']}) + \underline{\sigma}([D, D']) = \tau([D, D']).$$

Finally, from $\tau(f) = k\sigma(f) \otimes I$ for $f \in O(n, \underline{m})$, we obtain

$[\tau(f), \tau(g)] = 0 = \tau([f, g])$ and

$[\tau(D), \tau(f)] =$

$$(\ \sigma(D) \otimes I + \sum_{i=1}^{n} \sigma(f_i) \otimes \rho(T_i)\ ) (k\sigma(f) \otimes \rho I) - (k\sigma(f) \otimes I) (\ \sigma(D) \otimes I +$$

$$+ \sum_{i=1}^{n} \sigma(f_i) \otimes \rho(T_i)\ ) =$$

$$k[\sigma(D), \sigma(f)] \otimes I = k\sigma([D, f]) \otimes I = k\sigma(D(f)) \otimes I =$$

$$= \tau(D(f)) = \tau([D, f]).$$

So, τ indeed is a representation of algebra $\underline{\mathcal{K}}$. ∎

All Cartan algebras, mentioned in Section 3, are graded of depth 1:

$$\underline{L} = \bigoplus_{i \geq -1} \underline{L}_i \text{ and } [\underline{L}_i, \underline{L}_j] \subset \underline{L}_{i+j},$$

where $\underline{L}_i = \{\sum_{j=1}^{n} a_j D_j : deg\ a_j = i + 1\} \bigoplus \langle x^{(\alpha)} : |\alpha| = i \rangle$.

So we can study special representations (modules) of this algebras, connected with their gradation.

Definition 9. The representation $\rho : L \longrightarrow \mathrm{Lin}(V)$ (L- module V) is graded if $V = \bigoplus_{j \geq 0} V_j$ and $\rho(L_i)V_j \subset V_{i+j}$.

Without loosing generality we can assume, that the subspace V_0 is nonzero and call it the **base space** of our representation. Then the following connections between ρ and its base space can be obtained.

Theorem 3. A graded representation $\rho : L \longrightarrow \mathrm{Lin}(V)$ of a graded Lie algebra L is irreducible if and only if its restriction $\rho_0 = \rho|_{L_0} : L_0 \longrightarrow \mathrm{Lin}(V_0)$ is irreducible, $\rho(L)V_0 = V$ and the only vectors annihilated by $\rho(L_{-1})$ are elements of V_0.

Proof. Let $\mathcal{U}(L)$ be the universal enveloping algebra of algebra L and $V = \bigoplus_{i \geq 0} V_i$ an irreducible graded L-module.

If W is a proper submodule of V_0, by straightforward verification we obtain that $\mathcal{U}(L)W$ will be a proper L-submodule of V.

On the other hand, let us denote by $N_i = \langle v \in V_i : \mathcal{U}(\bigoplus_{k>0} L_k)_i v = 0\rangle$ and $N_V = \bigoplus_{i>0} N_i$.

Then by induction on $i + j$ we can prove that

$$\forall v_i \in N_i \, \mathcal{U}(\bigoplus_{k<0} L_k)_{-i-j} \mathcal{U}(L)_j v_i = 0 \,.$$

Also V/N_V is graded and taking any its element $\overline{v_i} = v_i + N_V \in (V_i + + N_V)/N_V$, $i > 0$ such that

$\mathcal{U}(\bigoplus_{k<0} L_k)\overline{v_i} = 0$, we obtain $\mathcal{U}(\bigoplus_{k<0} L_k)v_i \in N_V$ and finally $\mathcal{U}(\bigoplus_{k<0} L_k)_{-i}v_i \in N_V \bigcap V_0 = 0$. ■

Theorem 4. For every irreducible representation $\rho_0 : L_0 \longrightarrow \mathrm{Lin}(V_0)$ there exists exactly one irreducible graded representation of graded Lie algebra $L = \bigoplus_{i \geq -1} L_i$ with the base space V_0.

Proof. Let V be the irreducible graded L-module $\mathcal{U}(L) \bigotimes_{\mathcal{U}(L_{-1}+L_0)} V_0$ and $W = V_0 \bigoplus W^+$ any other graded L-module with the same base space. Then W/N_W will be irreducible and, as $N_W \subset W^+$, isomorphic to V_0. So the linear mapping $\varphi : x \bigotimes v \mapsto xv$, where $x \in \mathcal{U}(L)$, $v \in V_0$, is a module homomorphism and its kernel must be the unique homogeneous maximal submodule $N_W \subset W$. Finally $V \cong W/N_W$. ■

Theorem 5. An irreducible representation $\rho : L \longrightarrow \mathrm{Lin}(V)$ is equivalent to a graded representation of graded Lie algebra L if and only if $\rho|_{L_0}$ contains a nontrivial irreducible representation: $\rho_0(L_0)V_0' \subset V_0' \subset V_0$ and $\mathcal{I}_L V_0 \bigcap V_0 = 0$, where $\mathcal{I}_L$ is the ideal spanned by $\bigoplus_{k>0} L_k$ in $\mathcal{U}(L)$.

Proof. If V is a graded irreducible module, than the first condition is obvious and the second one follows from more general inclusion $\mathcal{I}_L V_0 \subset V^+$.

On the other hand, suppose that both conditions, mentioned above, are satisfied. Then $V = \mathcal{U}(L)V_0$ and φ defined by $u \otimes v \mapsto uv$ will be a homomorphism of L-module $W = \mathcal{U}(L) \underset{\mathcal{U}(L_{-1}+L_0)}{\otimes} V_0$ onto V. Lets denote by $W_0 = Ker\ \varphi$ the maximal proper submodule of W. If we assume, that V is not graded, W_0 will not coincide with N_W and W will be the sum of this two submodules. For each $v \in V_0$ $1 \otimes v = w + n$, $w \in W_0, n = \sum u_i \otimes v_i \in N_W$. As the linear space W is isomorphic to $\mathcal{U}(\bigoplus_{k>0} L_k) \underset{F}{\otimes} V_0$ and $N_W \subset \bigoplus_{j>0} W_j$, this yields $1v = \varphi(w+n) = \varphi(n) = \sum u_i v_i \in \mathcal{I}_L V_0$, contradictory to our second presumption. ∎

6 Irreducible representations of simple modular Lie algebras

Now, let's return to the extended derivation representation of our Cartan algebras. If ρ is an irreducible representation of L_0 and $k \in F \setminus \{0\}$, then $\delta \underset{k}{\times} \rho$ will be an irreducible representation of $\underline{L}$. As $\underline{L_0} = L_0 \bigoplus F$, its irreducible representations are of the form $\underline{\rho} = (\rho, k)$, where ρ is an irreducible representation of L_0 and $k\ I = \underline{\rho}(1)$.

Then, from theorem 4 it immediately follows that for any irreducible representation
$\rho_0 : L_0 \longrightarrow \mathrm{Lin}(V_0)$ and $k \in F \setminus \{0\}$ the unique irreducible graded representation of $\underline{L_0}$, determined by (ρ_0, k) is $\tilde{\rho} : \underline{L_0} \longrightarrow \mathrm{Lin}(\tilde{V_0})$, where $\tilde{V_0}$, is the mixed product of modules $O(n, \underline{m})$ and V.

Theorem 6. If $\rho : L \longrightarrow \mathrm{Lin}(V)$ is a graded representation with base space V_0 and $\forall v \in \bigoplus_{i>0} V_i$ $\rho(L_{-1})v \neq 0$, then there exists a graded monomorphism
$V \longrightarrow \tilde{V_0}$, which extends the natural isomorphism $V_0 \longrightarrow 1 \otimes V_0$.

Proof. Suppose that the vector v belongs to the i-th grade of $V = \bigoplus_{k \geq 0} V_k$ and $\alpha \in \mathcal{I}(n, \underline{m})$. Then the the linear operator $\varphi : V \longrightarrow \tilde{V_0}$, defined by

$$\varphi(v) = \sum_{|\alpha|=i} x^{(\alpha)} \otimes D_1^{\alpha_1} \ldots D_n^{\alpha_n} v,$$

will be a graded monomorphism $V \longrightarrow \tilde{V_0}$, and its restriction on V_0 maps each $w \in V_0$ into $1 \otimes w$. So φ is the required monomorphism. ∎

Finally, we can formulate the statement, describing the close connection between the irreducible graded representations of simple modular Lie algebras of Cartan type and the representations of classical simple Lie algebras, Cartan algebras are obtained from as differential extensions.

Theorem 7. For any zero grade representation of a Lie algebra from the first three Cartan series $\rho_0 : L_0 \longrightarrow \mathrm{Lin}(V)$, $dimV < \infty$ and the subrepresentation $\sigma : \underline{L} \longrightarrow \mathrm{Lin}(W)$, such that $\tilde{V}_1 \subset W$ in the case of general Cartan algebra and $\tilde{V}_2 \subset W$ if $L = S(n, \underline{m})$, $H(n, \underline{m})$, the modules W and $\tilde{V}$ coincide ($\sigma = \rho$).

Proof. If V is irreducible, we take for any $k \in \mathcal{Z}^+$ such $\underline{m}$ that p^{m_i} is greater than k. As $\tilde{V} \bigcap W$ is a submodule of $\tilde{V}$, which contains $\tilde{V}_1$ or $\tilde{V}_2$, we have $\tilde{V}_k \subset W$.

In the case of reducible representation ρ_0 we can take its decomposition and obtain the same result using the following property of the mixed product of representations:

$$(U \underset{k}{\times} V)/(U \underset{k}{\times} W) \cong U \underset{k}{\times} (V/W), \text{ for any submodule } W \subset V.$$

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D. Pagon
Institute for Mathematics,
Physics and Mechanics,
University of Ljubljana,
Ljubljana, Slovenia

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and Its Applications,
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POISSON POINT PROCESSES WITH AN APPLICATION TO COMBINATORICS

M. PERMAN

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Key words: Poisson point processes, Lévy processes, subordinators, order statistics, limit laws for functionals on symmetric groups

Abstract. Poisson point processes are widely used in probability theory and mathematical statistics. In this paper these processes are introduced in a simple setting when the space where these processes take values is just $(0,\infty)$. The two theorems proved in the paper treat the case when the Poisson point processes are renormalised to be probability measures. The distribution of the largest atom in this renormalised process is described and formulae for the joint distribution of the n largest atoms are given. These results are used to rederive some formulae for the limiting distribution of the longest cycle in a random permutation.

1. Introduction.

The theory of Poisson point processes has experienced rapid growth in the last two decades and has found wide application in probability, particularly excursion theory, and mathematical statistics. Informally, a Poisson point process is a random scatter of points in an arbitrary space, such that the number of points in disjoint sets are independent Poisson random variables.

In order to define a Poisson point process formally, let $(S,\mathcal{S})$ be a measurable space and Λ a σ-finite measure defined on this space. A Poisson point process associated with the measure Λ is a random measure N taking values in the space S^* of all discrete measures on the space S endowed with the σ-field generated by all random variables $\{N(A) : A \in \mathcal{S}\}$, and such that for any finite collection of disjoint measurable sets $\{A_i \in \mathcal{S} : 1 \leq i \leq n\}$ we have

$$P(N(A_1)=k_1,\ldots,N(A_n)=k_n)=\prod_{i=1}^{n}\frac{e^{-\Lambda(A_i)}[\Lambda(A_i)]^{k_i}}{k_i!} \tag{1.1}$$

for any integers $k_i \geq 0, 1 \leq i \leq n$. In other words, for the above collection of disjoint sets the random variables $\{N(A_i) : 1 \leq i \leq n\}$ are independent of each other and are distributed according to the Poisson law with mean $\Lambda(A_i)$. Note that for any Borel set $A \in \mathcal{S}$

$$EN(A)=\Lambda(A). \tag{1.2}$$

The number $\Lambda(A)$ is thus the expected mass of the Borel set A and the measure Λ will therefore be referred to as the mean measure of the Poisson point process. The notation $PPP(\Lambda)$ will be used to denote a Poisson point process with mean measure Λ.

Example: The simplest example of a Poisson point process is obtained by taking $S = (0, \infty)$ endowed with the usual Borel σ-field, and, by taking the Lebesgue measure as the mean measure Λ. An alternative description of the random measure N in this case is the following: Let $\xi_1, \xi_2, \xi_3, \ldots$; be a sequence of independent, identically distributed random variables taking values in $(0, \infty)$, each having the exponential distribution with parameter 1. In other words

$$P(x \leq \xi_1 \leq y) = \int_x^y e^{-s} ds, \quad 0 \leq x \leq y.$$

Let the sequence $(X_i : i \geq 1)$ denote the partial sums of the original sequence:

$$X_i = \sum_{j=1}^{i} \xi_j ,$$

and finally define the random measure N on $(0, \infty)$ by

$$N(A) = |\{i : X_i \in A\}|$$

for any Borel set $A \subset (0, \infty)$ where $|\cdot|$ denotes the cardinality of a set. For a proof that the random measure N defined this way is a Poisson point process see for example Billingsley, pp. 260–265.

This paper will be concerned with Poisson point processes on $(0, \infty)$ whose mean measure satisfies some additional hypotheses. We will require Λ to be a Lévy measure which means that

$$\Lambda(\epsilon, \infty) < \infty \quad \text{for all } \epsilon > 0, \tag{3.a}$$

$$\Lambda(0, \infty) = \infty \quad \text{and} \tag{3.b}$$

$$\int_0^1 s\Lambda(ds) < \infty. \tag{3.c}$$

Let $V_1 \geq V_2 \geq V_3 \ldots$ denote the positions of the atoms of the $PPP(\Lambda)$ ranked by size where atoms of mass k are counted k times. By condition (3.a) such a decreasing arrangement is always possible. The main consequence of the conditions (3.a). (3.b) and (3.c) is

Lemma 1.1: *If $V_1 \geq V_2 \geq V_3 \ldots$ are the sizes of atoms of a $PPP(\Lambda)$ process arranged in decreasing order then*

$$P(0 < \sum_{i=1}^{\infty} V_i < \infty) = 1. \tag{4}$$

Proof: That the sum is almost surely bigger than 0 is a trivial consequence of (3.b). To prove that the sum is, with probability 1, finite, observe that the number of V_i's bigger than 1 is finite with probability 1. The expectation of the sum of the V_i's that are smaller than 1 is computed by

$$E \sum_{V_i \leq 1} V_i = \int_0^1 s\Lambda(ds) < \infty.$$

Since this last integral is finite by (3.c) the expectation on the left is finite, and hence the sum is finite with probability 1. ■

2. Order statistics.

Let N be a Poisson point process with Lévy mean measure Λ and let again $V_1 \geq V_2 \geq V_3 \dots$ be the positions of the atoms of N ranked by size. By Lemma 1.1 we know that $T = \sum_{i=1}^{\infty} V_i < \infty$ with probability 1. Define the infinite vector $\mathbf{D} = (D_1, D_2, \dots)$ by

$$D_i = V_i/T = V_i / \sum_{i=1}^{n} V_i, \; i \geq 1. \tag{2.1}$$

The random vector $\mathbf{D}$ is taking values in the infinite simplex of sequences of non-negative numbers adding up to 1.

This section will be concerned with finding finite dimensional distributions of the vector $\mathbf{D}$. The key observation is the following:

Lemma 2.1: *Let $V_1 \geq V_2 \geq \dots$ be the positions of the atoms of a $PPP(\Lambda)$. Conditionally on $(V_1, V_2, \dots, V_n)$, the remaining $(V_i)_{i \geq n+1}$ are distributed as the positions of atoms of a $PPP(\Lambda^{V_n})$ ranked by size where Λ^{V_n} is the original mean measure Λ restricted to $[0, V_n]$.*

Proof: The assertion follows from the definition of a Poisson point process. Since the number of points in disjoint intervals are independent Poisson random variables we get for any $m > n$

$$\begin{aligned} P(V_1 \in dv_1, V_2 \in dv_2, \dots, V_m \in dv_m) = \\ \exp\{-\Lambda(v_1, \infty)\}\Lambda(dv_1)\exp\{\Lambda(v_2, v_1)\}\Lambda(dv_2)\dots \\ \dots\exp\{-\Lambda(v_m, v_{m-1})\}\Lambda(dv_m). \end{aligned}$$

From this the proof follows easily. ■

In the sequel we will assume that the mean measure Λ has a density.

Theorem 2.2: *Let the vector D be defined as $D_i = V_i/T$ where $(V_i)_{i\geq 1}$ are the positions of the atoms of a $PPP(\Lambda)$ and $T = \sum_{i=1}^{\infty} V_i$. Assume* *further that the Lévy mean measure Λ has the density h with respect to the Lebesgue measure.*

Then the vector $(T, D_1, D_2, \ldots, D_n)$ has a density p_n for all $n \geq 1$. The density p_1 satisfies the integral equation

$$p_1(t,y) = th(ty) \int_0^{1 \wedge \frac{y}{1-y}} p_1(t(1-y),u)du \tag{2.2a}$$

where $t > 0$ and $0 < y < 1$. For $n \geq 2$ the density p_n satisfies the identity

$$p_n(t, y_1, \ldots, y_n) = \frac{t^{n-1}h(ty_1)\ldots h(ty_{n-1})}{\bar{y}_n} p_1(t\bar{y}_n, \frac{y_n}{\bar{y}_n}) \tag{2.2b}$$

where $\bar{y}_n = 1 - y_1 - \ldots - y_{n-1}$ and $t > 0$, $y_1 \geq y_2 \geq \ldots \geq y_n > 0$ and $\sum_{i=1}^n y_i < 1$.

Proof: In order to prove that the random variable vectors have a density it must be shown that the random variable T has a density. It is known that if the Lévy mean measure is absolutely continuous with respect to the Lebesgue measure then T has a density. For a proof see for example Brockett and Hudson (1980).

Denote the density of T by ϕ. If we restrict the Lévy measure Λ to the interval $(0, s]$ and define the random variable T^s to be the sum of the positions of atoms of the resulting $PPP(\Lambda^s)$ where Λ^s is the restricted Lévy measure then by the same argument as in the previous paragraph T^s will also have a density. Denote this density by ϕ^s.

By Lemma 2.1 the conditional density of T, given $(V_1, V_2, \ldots, V_n)$, is just ϕ^{V_n}. In formulae this means

$$\begin{aligned} &P(T \in dt, V_1 \in dv_1, \ldots, V_n \in dv_n) = && \text{(2.3a)} \\ &\phi^{v_n}(t - v_1 - \ldots - v_n)dtP(V_1 \in dv_1, \ldots, V_n \in dv_n) \quad \text{by Lemma 2.1} \\ &\phi^{v_n}(t - v_1 - \ldots - v_n)dth(v_1)dv_1 \ldots h(v_n)dv_n \exp\{-\Lambda(v_n, \infty)\}. \end{aligned}$$

for $t > v_1 + v_2 + \ldots + v_n$. By a change of variables to $ty_i = v_i, dv_i = tdy_i, 1 \leq i \leq n$ from (2.3a) one gets the formula for the density of $(T, D_1, D_, \ldots, D_n)$ as

$$\begin{aligned} &P(T \in dt, D_1 \in dy_1, \ldots, D_n \in dy_n) = && \text{(2.3b)} \\ &t^n \phi^{ty_n}(t)dt\; h(ty_1)h(ty_2)\ldots h(ty_n)dy_1 \ldots dy_n \end{aligned}$$

for $t > 0$, $y_1 \geq y_2 \geq \ldots y_n > 0$ and $\sum_{i=1}^n y_i < 1$. For $n = 1$ (2.3b) reads as

$$P(T \in dt, D_1 \in dy_1) = t\phi^{ty_1}(t(1-y_1))dth(ty_1)dy_1 \tag{2.3c}$$

for $t > 0$ and $0 < y_1 < 1$. Substituting $t\bar{y}_n$ for t and $y_n/\bar{y}_n$ for y_1 in this last density and comparing the result to (2.3b) yields (2.2b).

To derive the integral equation (2.2a) observe that the density of (T, D_1) is a marginal density of (T, D_1, D_2). Using the identity (2.2b) one gets by integration that

$$p_1(t, y_1) = \int_0^{1-y_1} p_2(t, y_1, y_2)dy_2 = \int_0^{1-y_1} th(ty_1)p_1(t(1-y_1), y_2/(1-y_1))dy_2 .$$

The equation (2.2a) follows by a simple change of variable in the integral on the right. ∎

Lemma 2.3: *The equation (2.2a) determines the density p_1 uniquely.*

Proof: For $y_1 > 1/2$ the integration on the right is over the interval $(0, 1)$ which simply gives the density ϕ of T. So for $y_1 > 1/2$ and $t > 0$

$$p_1(t, y_1) = th(ty_1)\phi(t(1 - y_1)).$$

Suppose we know the density for $t > 0$ and $y_1 > 1/n$. The integral on the right of (2.2a) for $y_1 \in (1/(n+1), 1/n)$ only involves p_1 in the domain where we suppose it is known. So p_1 is determined in the domain $t > 0$ and $y_1 >$ $> 1/(n + 1)$. The proof follows by induction. ∎

3. An application to permutations.

Any permutation can be uniquely decomposed into a product of cycles, for example

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 3\,5\,4\,1\,2\,6 \end{pmatrix} = (134)(25)(6) .$$

The cycles of a permutation σ of n objects are the subsets $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ such that $\sigma(i_j) = i_{j+1}$ for $1 \leq j \leq k - 1$ and $\sigma(i_k) = i_1$. The permutation given above has cycle lengths 3, 2 and 1. If we choose permutations of n objects at random, each with probability $1/n!$, the length of the longest cycle becomes a random variable. More generally, the lengths of cycles arranged in decreasing order are random variables. Consider these random variables as the beginning of an infinite random vector, padding it with zeros when running out of cycle lengths. Denote this vector by $\mathbf{D}^n =$ $= (D_1^n, D_2^n, \ldots)$. The random vector

$$\frac{1}{n}\mathbf{D}^n = (\frac{D_1^n}{n}, \frac{D_2^n}{n}, \ldots)$$

takes values in the infinite simplex

$$\Delta_\infty = \{(x_1, x_2, \ldots) : x_i \geq 0, i \geq 1, \sum_i x_i = 1\} .$$

In this section it will be shown that the random vector $(1/n)\mathbf{D}^n$ of relative cycle lengths has a limiting distribution as $n \to \infty$ and formulae for finite dimensional distributions will be given.

Fristedt (1986) has an elegant treatment of this problem which will be adapted in the sequel. The question was first treated by Goncharov (1944) who derived an integral equation for the density of the relative length of the longest cycle. Shepp and Lloyd (1965) derived moment transforms of the limiting distribution of the relative length of the r^{th} longest cycle in a random permutation. Vershik and Schmidt (1977) gave formulae for the joint limiting distribution for the first r longest cycles in a random permutation. All their formulae can be rederived from Theorem 2.2.

Fristedt (1986) constructs the sequence $(\mathbf{D}^n)$ on the same probability space in such a way that $(1/n)D_i^n$ converges almost surely to a random variable D_i, say. It will be shown that the limiting vector $\mathbf{D} = (D_1, D_2, \ldots)$ has the same distribution as the order statistics introduced in (2.1) for some $PPP(\Lambda)$. Once the Lévy measure Λ is identified, Theorem 2.2 will be used to find equations for the densities of the limiting distributions.

To prove that the distribution of $\mathbf{D}$ can be described in terms of order statistics introduce for each $p = (0,1)$ a geometrically distributed random variable L_p defined on the same probability as the sequence $(\mathbf{D}^n)$ but independent of it. Fristedt's construction provides for such random variables and moreover $L_p \to \infty$ with probability 1 as $p \downarrow 0$. Because of *a.s.* convergence of $(1/n)D^n$ the limit will be the same if we look at $(1/L_p)D^{L_p}$ and let $p \downarrow 0$. It will be shown that pD^{L_p} is a PPP process and the mean measure of the limit will be identified using standard convergence theorems from the theory of Poisson point processes.

More explicitly the random variable L_p has the distribution

$$P(L = k) = (1-p)^k p, \; k = 0, 1, 2, \ldots;$$

To derive the main result of this section the following combinatorial result will be needed. For fixed n let $(i_1, i_2, \ldots, i_n)$ be an n-tuple of non-negative integers such that $\sum_{j=1}^n i_j = n$, and define α_j^n to be the number of cycles of length exactly j in a random permutation of n objects. For the permutation given at the beginning of this section these random variables have values $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = \alpha_5 = \alpha_6 = 0$. It is known that

$$P(\alpha_1^n = i_1, \alpha_2^n = i_2, \ldots, \alpha_n^n = i_n) = \prod_{j=1}^n \frac{1}{j^{i_j} i_j!}. \tag{3.1}$$

For a proof see for example Riordan (1978). The key result is

Lemma 3.1: *Fix $p \in (0,1)$. The discrete random measure N_p having atoms in positions $pD_1^{L_p}, pD_2^{L_p}, \ldots$ is a $PPP(\Lambda_p)$ process on $(0,\infty)$ where the mean measure Λ_p is discrete and*

$$\Lambda_p(\{pj\}) = \frac{1}{j}(1-p)^j, \; j = 1, 2, \ldots; \tag{3.2}$$

Proof: Let $\alpha_j^{L_p}, j \geq 1$, be the number of cycles of length exactly j in the randomly chosen permutation of L_p objects. It suffices to prove that for any sequence of non-negative integers $(i_1, i_2, \ldots)$ such that only finitely many are different from 0 we have

$$P(N_p(\{pj\}) = i_j, j \geq 1) = \prod_{j=1}^{\infty} \frac{e^{-\Lambda_p(\{pj\})}[\Lambda_p(\{pj\})]^j}{j!}.$$

This identity is proved by conditioning on L_p and using formula (3.1). Let $\sum_{j=1}^{\infty} j i_j = k$. Then

$$\begin{aligned} P(pD_j^{L_p} = i_j, j \geq 1) &= P(L_p = k)P(pD_j^k = i_j, j \geq 1) \\ &= (1-p)^k p \prod_{j=1}^{\infty} \frac{1}{j^{i_j} i_j!} \text{ by (3.1)}. \\ &= p \prod_{j=1}^{\infty} \frac{\left[(1-p)^j/j\right]^{i_j}}{i_j!}. \end{aligned} \tag{3.2a}$$

Substituting the elementary identity

$$p = \prod_{j=1}^{\infty} \exp\{-(1-p)^j/j\}$$

into (3.2a) concludes the proof. ∎

The following lemma adapted from Fristedt (1986) will be used to identify the mean measures of the limiting PPP as $p \downarrow 0$.

Lemma 3.2: *Let $V_1^p \geq V_2^p \geq \ldots >$ be the positions ranked by size of atoms of a $PPP(\Lambda_p)$ where atoms of size k are counted k times, i.e. if the atom with the largest position has mass k then the first k largest V_i^p's are equal to this position. If for all $z > 0$*

$$\lim_{p \to 0} \int_{(0,z)} s\Lambda^p(ds) = \int_{(0,z)} s\Lambda(ds) \tag{3.3}$$

then the vector $(V_1, V_2, \ldots)$ converges weakly.

A simple computation gives that

$$\lim_{p \to 0} \int_{(0,z)} s\Lambda^p(ds) = \int_{(0,z)} s s^{-1} e^{-s} ds. \tag{3.4}$$

Since for any $p \in (0,1)$ the vector $(D_1^{L_p}, D_2^{L_p}, \ldots)$ is obtained from $(V_1^p, V_2^p, \ldots)$ as in (2.1), the limiting distribution will be obtained the same way from a $PPP(\Lambda)$ process where the mean measure Λ is given by $\Lambda(ds) = = s^{-1}e^{-s}, s > 0$. By a well known formula, see for example Rogers and Williams, pp. VI 2., the Laplace transform of the sum $T = \sum_{i=1}^{\infty} V_i$ where $V_i \geq V_2 \geq \ldots$ are the positions of the atoms of the $PPP(\Lambda)$ is given by

$$Ee^{-\lambda T} = \exp\{-\int_0^{\infty}(1-e^{-\lambda s})\Lambda(ds)\}.$$

An elementary calculation gives that for $\Lambda(ds) = s^{-1}e^{-s}$ the Laplace transform is $(1+\lambda)^{-1}$ which means that T has the exponential distribution with mean 1.

In this particular case the application of Theorem 2.2 is further simplified by

Lemma 3.3: *Let the mean measure of a Poisson point process Λ be given by $\Lambda(ds) = s^{-1}e^{-s}, s > 0$ and let the vector $(T, D_1, D_2, \ldots, D_n)$ be defined as in Theorem 2.2. Then the random variable T and the vector $(D_1, D_2, \ldots, D_n)$ are independent.*

Proof: The mean measure is obviously Lévy so T is well defined. Let $U_1, U_2, \ldots$ be a sequence of independent random variables distributed uniformly on $(0,1)$ and independent of $(T, V_1, V_2, \ldots)$ where the V_i's are the positions of atoms of the $PPP(\Lambda)$ ranked by size. Define a process $(Y_t : 0 \leq t \leq 1)$ by

$$Y_t = \sum_i V_i 1(U_i < t).$$

We get an increasing right continuous process and it is known, see for example Rogers and Williams (1987), pp.308–313, that (Y_t) has independent increments which are distributed according to *gamma* laws with the same shape parameter. An elementary result says that if $(Z_1, Z_2, \ldots, Z_n)$ are independent *gamma* random variables with the same shape parameter then, see for example Wilks (1964),

$$\sum_{i=1}^{n} Z_i \text{ and } (Z_1/\sum Z_i, \ldots, Z_n/\sum Z_i)$$

are independent. A consequence for the process (Y_t) is that the random variable Y_1 is independent of the „renormalised" process $(Y_t/Y_1 : 0 \leq t \leq 1)$. But $Y_1 = T$ and the vector $(D_1, D_2, \ldots, D_n)$ is a functional of the renormalised process $(Y_t/Y_1 : 0 \leq t \leq 1)$. This concludes the proof of the lemma. ∎

Let us denote by q_1 the density of the random variable D_1. Equation (2.2a) simplifies to give an integral equation for q_1:

$$yq_1(y) = \int_0^{1 \wedge \frac{y}{(1-y)}} q_1(s)ds, \; 0 < y < 1.$$

An elementary calculation gives

$$q_1(y) = \begin{cases} y^{-1} & \text{for } 1/2 \le y < 1 \\ y^{-1}(1 - \log((1-y)/y)) & \text{for } 1/3 \le y < 1/2. \end{cases}$$

As an example the asymptotic probability that the longest cycle in a random permutation will contain more than half of the permuted objects is $P(D_1 \ge 1/2) = \int_{1/2}^{1} q_1(s)ds = \log(2) = 0.6931$.

For the distribution q_n of $(D_1, D_2, \ldots, D_n)$ the formula (2.2b) simplifies to

$$q_n(y_1, y_2, \ldots, y_n) = \frac{1}{y_1 y_2 \cdots y_{n-1} \bar{y}_n} q_1\left(\frac{y_n}{\bar{y}_n}\right)$$

where $\bar{y}_n = 1 - y_1 - \ldots - y_{n-1}$, $y_1 > y_2 > \ldots > y_n > 0$ and $\sum_{i=1}^{n} y_i < 1$. This identity was derived by Vershik and Schmidt (1977) by solving a recursive system of integral equations. They used the results to prove central limit type theorems for the random variables D_i as $i \to \infty$.

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M. Perman
Institute for Mathematics,
Physics and Mechanics,
University of Ljubljana,
Ljubljana, Slovenia

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and Its Applications,
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FRACTALS FROM COUNTEREXAMPLES TO APPLICATIONS

P. PETEK

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Key words: fractal, Cantor set, Koch snowflake, Julia set, iteration

Abstract. Some classical counterexamples are recognized as fractals. Iteration theory provides analogous objects in a more natural setting. One aspect of applications is mentioned, the IFS of Barnsley.

Fractals have lived for almost a century in the mathematical conscience. But in the first years they were made up examples and counterexamples that defied mathematical common sense and advertised caution in jumping to hasty conclusions. Later, to no small surprise, variations of those very fractals popped up quite naturally in simple mathematical operations. It is not surprising that the explosion of interest in fractals coincides with the era of computers. To draw and see a fractal the computer screen is the most useful tool.

As geometric objects, fractals live in Euclidean spaces, but they really live as the geometry of one aspect of nonlinear dynamics, chaos. A most common situation represents dynamics in an open subset of Euclidean space; however dynamics is chaotic on a subspace and this subspace is fractal.

We shall consider three examples, the Cantor set, the Koch snowflake and the Weierstrass function.

The construction of Cantor is well known, as are the properties of the resulting set. It has zero Lebesgue measure, but continuum many points, it is totally disconnected and perfect. The construction of Cantor seems somewhat artificial and the result was at his time regarded as a pathological object, such that never appears in nature. However in the theory of iterations there is a very natural way to encounter the Cantor set. Suppose we iterate the function $f(x) = \mu x(1-x)$, $\mu > 4$. The set of points

$$\Lambda = \{x \mid f^n(x) \in [0,\ 1] \ \text{ for all } \ n \geq 0\}$$

is homeomorphic to the original Cantor middle third set.

Instead of the above quadratic function any unimodal function would do. And this is essentially the reason that the Cantor set appears as part of the structure in many „strange attractors". One such interesting example is the ring os Saturn. The Cassini gap has been known well over 200 years, but the photos the Voyager brought back from its mission showed a very good matching of gaps with the gaps in Cantor set.

The Koch snowflake is a geometrical object in the plane, having finite area and infinite perimeter, i.e. its boundary is a curve of infinite length. Moreover, as is the habit of most fractals, the Koch curve is self-similar: any smallest part of it, when sufficiently magnified, looks as the whole curve.

The Koch snowflake and curve were constructed to give a counterexample to the „common sense" belief that finite area should also be finitely bounded. And as with Cantor set, again the very same function, namely the quadratic one purchases through iteration an analogous example, albeit in the complex plane. The Julia set of the quadratic map $f(z) = z^2 - \frac{1}{2}$ is an honest Jordan curve, homeomorphic to the circle, but of infinite length, self-similar, fractal. The interior, the basin of attraction of $\frac{1-\sqrt{3}}{2}$ corresponds as a counterexample to the Koch snowflake.

Karl Weierstrass constructed a function

$$W(t) = \frac{1}{\sqrt{1-w^2}} \sum_{n=0}^{\infty} w^n e^{2\pi i b^n t}$$

where b is an odd integer, $w = b^{-H}$, $0 < H < 1$. The function is well defined by the Fourier series and has peculiar properties. Most functions that appear in „natural" problems are continous and differentiable, noncontinuity and nondifferentiability is more of an exception. But the real and imaginary components of Weierstrass' function are continous, yet nowhere differentiable functions. It is this last reqirement that made the construction difficult. Namely Riemann also gave his example

$$R(t) = \sum_{n=1}^{\infty} \frac{1}{n} \cos n^2 t$$

which proved to be continous, but there were some points of differentiability.

In the iteration theory examples like Weierstrass' are plenty. Just take the above mentioned function $f(z) = z^2 - \frac{1}{2}$ and the corresponding iteration sequence

$$z_{n+1} = z_n^2 - \frac{1}{2}.$$

Terms cs can be expressed as

$$z_n = \Phi(2^n \Phi^{-1}(z_0))$$

where Φ is the solution of the functional equation

$$\Phi(2w) = (\Phi(w))^2 - \frac{1}{2}$$

defined for $\mathrm{Re}(w) \geq 0$ as a series

$$\Phi(w) = \mathrm{e}^{w} + B_1 \mathrm{e}^{-w} + B_2 \mathrm{e}^{-3w} + \dots$$

whose coefficients can be determined recurrently.

For purely imaginary $w = it$ both the real and imaginary part of $\Phi(it) = x(t) + iy(t)$ furnish an example like the one of Weierstrass, and geometrically give a parametrization of the Julia set.

All the mentioned examples, the classical constructions as well as their iteration counterparts, are fractals. What is a good definition of a fractal is not yet resolved. Mandelbrot, the father and advocate of fractals, started with the reqirement that the Hausdorff dimension of an object X be noninteger ($D_H(X) \notin \mathbf{Z}$) to be declared a fractal. Later however some prominent examples appeared with all the properties of the known fractals and yet integer dimension. Maybe one should wait a few years for the theory to settle down before fixing a definition.

And while doing so let's see how Barnsley applied fractals. His definition is so generous to encompass all nonempty compact subsets of a Euclidean space, say the plane. A fractal then represents a black and white picture. Barnsley then shows that any picture can be with a desired accuracy reproduced if we know an IFS (iterated function system).

What is an IFS? We are given a set of mappings

$$w_i : \mathbf{R}^2 \to \mathbf{R}^2, \; 1 \leq i \leq r$$

where each w_i is an affine contraction. Starting with any point x_0 in the plane, we choose at each step an index $j(n)$ at random and plot the sequence of points by the rule

$$x_{n+1} = w_{j(n)}(x_n)\,.$$

With probability 1 the picture that appears is the attractor — a fractal — determined by the choice of the set of contractions.

This proves to be a very effective way of storing information on images, certainly much better than pixel by pixel storing. And not only can black and white pictures be coded that way, colours can be brought into picture with invariant measure.

Apart from the affine transformations w_i probabilities can be attached to each index $p_i = P(j(n) = i)$ and density of points on the attractor can be influenced this way. Density intervals then can be assigned different colours.

However there is still the problem of encoding a given picture. The theory gives the existence of an IFS that would with the required accuracy reproduce the picture. But there is not yet known any general algorithm that would work for all kinds of pictures. Some very specific types have so far been succesfully dealt with and nice pictures of flowers, ferns, trees appeared, yet there is still much work to be done.

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P. Petek
Institute for Mathematics,
Physics and Mechanics,
University of Ljubljana,
Ljubljana, Slovenia

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and Its Applications,
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MATHEMATICA AND GRAPH THEORY

M. PETKOVŠEK AND T. PISANSKI

AMS Subject Classification (1991): 68R10, 68Q40, 05C25

Key words: graph theory, automorphism groups, symbolic computing

Abstract. In the paper we present some applications of symbolic computing to graph theory.

Mathematica [2] is a sophisticated, powerful system for symbolic computation developed and distributed by Wolfram Research, Inc. Its power can be used in various branches of mathematical research. The lecture was intended to give an overview of *Mathematica* and some hints of possible applications.

Although powerful, *Mathematica* can be easily used by novices as well as by expert programmers. One of the main advantages of *Mathematica* is its extendibility. *Mathematica* has a great number of built-in operations and functions. New operations and functions can be readily added to this list if they are programmed and stored on files as packages and loaded when needed. In this way the programmed operations become indistinguishable from the built-in operations. Such extensions make *Mathematica* so useful.

One of the packages that comes with *Mathematica* version 2 is called `Combinatorica`. This package covers many areas of combinatorics and graph theory. It was written by Steve Skiena and is described in [1]. There are other packages available when purchasing *Mathematica*. One of them that we used in our lecture is called `Polyhedra` and gives some operations on polyhedra.

The lecture was composed of two parts. In the first part we presented some of the features of *Mathematica*. The major components of the *Mathematica* system are its symbolic, numerical, graphics, and programming capabilities.

As an illustration of symbolic computation, consider the problem of finding the order of local convergence for some numerical method for finding roots of equations, such as Newton's method.

```
Mathematica 2.0 for MS-DOS 386/7 (June 21, 1991)
Copyright 1988-91 Wolfram Research, Inc.
```

First we tell the system that $f(x)$ has a zero at $x = a$.

```
In[1]:= f[a] := 0
```

Now we let the system do the tedious Taylor-series expansion for us. We expand the error of the next iterate as a function of the previous one, x, around $x = a$, through terms of order 3.

```
In[2]:= Series[(x - f[x]/f'[x]) - a,  {x, a, 3}]

                         2              2     (3)
        f''[a] (-a + x)        -f''[a]       f    [a]          3             4
Out[2]= ---------------- + (-------- + -------) (-a + x)  + O[-a + x]
           2 f'[a]                  2     3 f'[a]
                               2 f'[a]
```

From this we conclude that Newton's method has (at least) quadratic convergence, provided that $f'(a)$ which appears in the denominators is nonzero. If furthermore $f''(a) = 0$ then convergence is at least cubic. Obviously, we can get as many terms of the error as we like.

But what if $f'(a) = 0$? Let's see:

```
In[3]:= f'[a] := 0

In[4]:= Series[(x - f[x]/f'[x]) - a, {x, a, 2}]

                   (3)            2
        -a + x    f    [a] (-a + x)             3
Out[4]= ------ + ----------------- + O[-a + x]
          2          12 f''[a]
```

Now convergence is only linear, and the error approximately halves at each step provided that $f''(a) \neq 0$. We can go on and set higher derivatives to zero as well:

```
In[5]:= f''[a] := 0

In[6]:= Series[(x - f[x]/f'[x]) - a, {x, a, 2}]

                        (4)            2
        2 (-a + x)     f    [a] (-a + x)             3
Out[6]= ---------- + ----------------- + O[-a + x]
            3               (3)
                       36 f    [a]
```

In the second part of the lecture we presented our original programming contributions. The implementations of some of these functions are listed in the Appendix. First we showed some new functions that we were using in combining the above-mentioned two packages. For example, we showed how one can take a polyhedron and then form its one-skeleton that can be later processed as any other graph.

In the following example we use the function `OneSkeleton` to convert the dodecahedron to its skeletal graph; see Figures 1 and 2.

```
In[1]:= gd = OneSkeleton[Dodecahedron];

In[2]:= Show[Polyhedron[Dodecahedron]];

In[3]:= ShowGraph[gd];
```

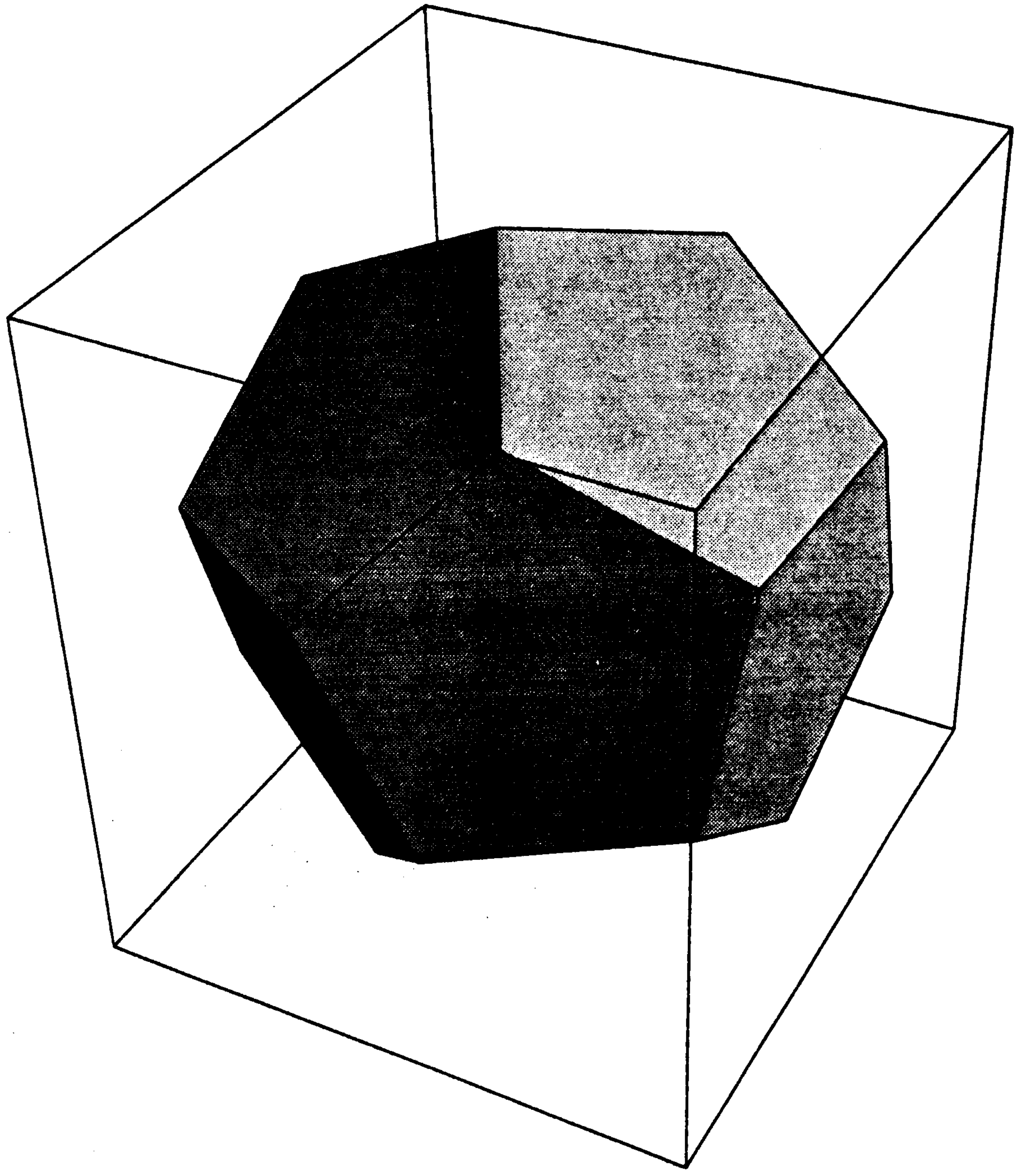

Figure 1. The dodecahedron.

Next we showed that for computation intensive calculations it is often better to write a programme in another programming language and then interface it with *Mathematica*. As an example we used the well-known Brendan McKay's *Nauty*. This is an excellent programme for calculating the automorphism group of a graph. For large graphs it was then evident that the straightforward algorithm from the package `Combinatorica` is much slower than *Nauty*, even though on one side the user cannot tell the difference between the two functions and on the other side the interface to *Nauty* spends some time as it stores intermediate data and results on disk.

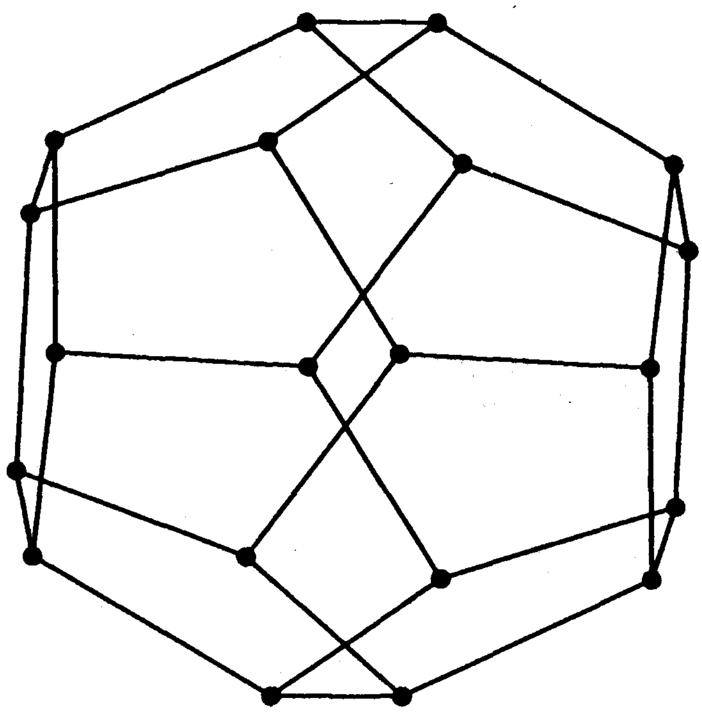

Figure 2. The one-skeleton of dodecahedron

Our version of the function which computes the automorphism group of a graph using *Nauty* is called `AutomorphismGroup`. The following example shows that the automorphism group of the one-skeleton of the dodecahedron has 120 elements.

```
In[4]:= AutomorphismGroup[gd];

In[5]:= Length[%]

Out[5]= 120
```

The one-skeleton of the tetrahedron is the complete graph K_4. Its automorphism group is listed below. It contains the 24 permutations on four letters.

```
In[6]:= AutomorphismGroup[OneSkeleton[Tetrahedron]]

Out[6]= {{1, 2, 3, 4}, {1, 2, 4, 3}, {1, 3, 2, 4}, {2, 1, 3, 4},
```

```
>    {1, 3, 4, 2}, {2, 1, 4, 3}, {1, 4, 2, 3}, {2, 3, 1, 4}, {3, 1, 2, 4},
>    {1, 4, 3, 2}, {2, 3, 4, 1}, {3, 1, 4, 2}, {2, 4, 1, 3}, {3, 2, 1, 4},
>    {4, 1, 2, 3}, {2, 4, 3, 1}, {3, 2, 4, 1}, {4, 1, 3, 2}, {3, 4, 1, 2},
>    {4, 2, 1, 3}, {3, 4, 2, 1}, {4, 2, 3, 1}, {4, 3, 1, 2}, {4, 3, 2, 1}}
```

As a final series of examples we showed a programme in *Mathematica* that calculates the index of the automorphism group of a polyhedron in the automorphism group of its one-skeleton. We also showed a programme for calculating the dual polyhedron of an arbitrary polyhedron. We were discussing stellation and truncation of polyhedra.

Let $A(P)$ denote the group of automorphisms of the one-skeleton of P. We have written a programme which selects those elements of $A(P)$ that preserve the faces of P. The selected automorphisms form a subgroup $S(P)$ of $A(P)$. The function `SymmetryIndex` computes the index of $S(P)$ in $A(P)$. For example, the symmetry index of the dodecahedron is 1.

```
In[7]:= SymmetryIndex[Dodecahedron]

Out[7]= 1
```

For the polyhedron `ProjPlane` depicted in Figure 3, the one-skeleton is the octahedral graph $K_{2,2,2}$ (see Figure 4).

```
In[8]:= Show[Polyhedron[ProjPlane]];
```

```
In[9]:= ShowGraph[ShakeGraph[OneSkeleton[ProjPlane],0.4]];
```

The polyhedron `ProjPlane` is self-intersecting. Topologically, it represents a map on the projective plane that is composed of three squares and four equilateral triangles. The symmetry index of `ProjPlane` is 2.

```
In[10]:= SymmetryIndex[ProjPlane]

Out[10]= 2
```

Appendix

The programmes in *Mathematica*.

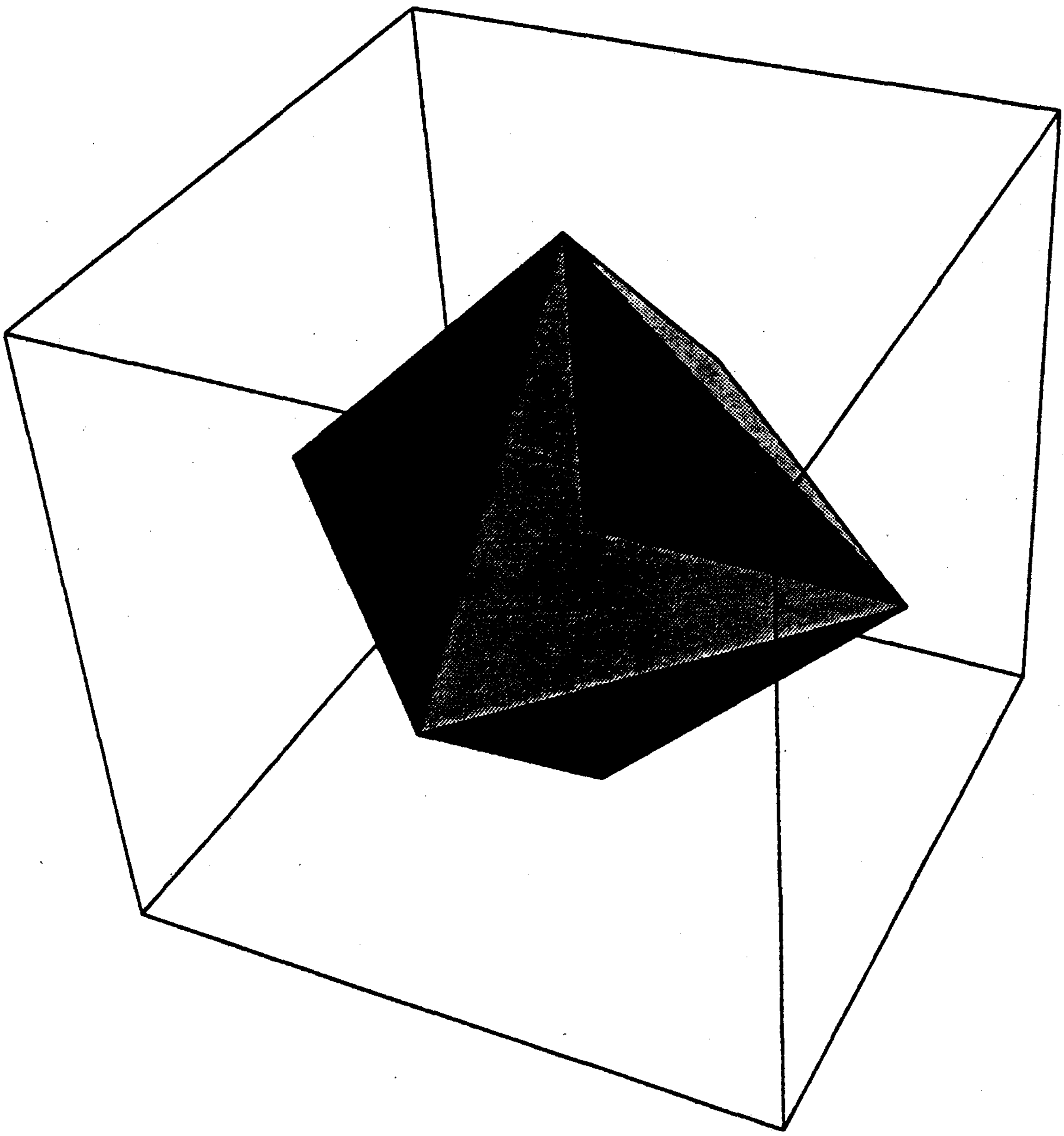

Figure 3. The polyhedron called ProjPlane.

```
SymmetryIndex::usage = "SymmetryIndex[poly_] returns the index of the
symmetry group of a polyhedron in the full automorphism group of its
one-skeleton."

SymmetryIndex/:
  SymmetryIndex[p_] :=
   Block[{autg = AutomorphismGroup[OneSkeleton[p]]},
    Length[autg]/Length[Select[autg, AutoQ[Faces[p], #1] & ]]]

SymmetryGroup::usage = "SymmetryGroup[poly] calculates the symmetry group
of a polyhedron poly."
```

```
SymmetryGroup/:
  SymmetryGroup[p_] :=
   Select[AutomorphismGroup[OneSkeleton[p]], AutoQ[Faces[p], #1] & ]

AutoQ::usage = "AutoQ[list,permutation] returns True if permutation
represents an automorphism of the list."

AutoQ/: AutoQ[l_, p_] := ekvl[l, transform[l, p]]

transform::usage = "transform[l,p] applies permutation p on a list l."

transform/: transform[l_, p_] := l /. Thread[Sort[p] -> p]
```

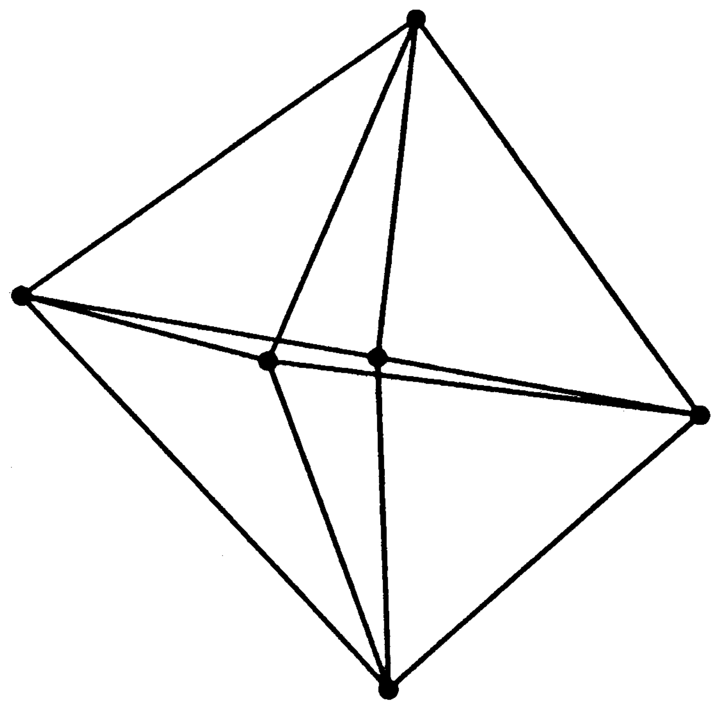

Figure 4. The one-skeleton of `ProjPlane`.

```
ekvl::usage = "ekvl[l1,l2] returns True if the two lists of polygons are
equivalent (not necessarily oriented)."

ekvl/: ekvl[{}, {}] = True

ekvl/: ekvl[l1_, l2_] :=
   Block[{e = Scan[If[Ekv[#1, l1[[1]]], Return[#1]] & , l2], pos},
    If[e == Null, Return[False]]; pos = Position[l2, e][[1,1]];
     Return[ekvl[Rest[l1], Drop[l2, {pos, pos}]]]]

Ekv::usage = "Ekv[p,q] returns True if the two polygons p and q are
equivalent (not necessarily oriented)."

Ekv/: Ekv[p_, q_] := ekv[p, q] || ekv[p, Reverse[q]]


ekv::usage = "ekv[p,q] returns True if the two oriented polygons p and q
```

```
are equivalent."

ekv/: ekv[p_, q_] :=
   Block[{pposq = Position[p, q[[1]]]},
    If[pposq == {}, Return[False]]; pposq = pposq[[1,1]];
     Return[Join[Drop[p, pposq - 1], Take[p, pposq - 1]] == q]]


AutomorphismGroup::usage = "AutomorphismGroup[g] finds the automorphism
                            group of a graph  g. The graph may reside on
                            a file."
AutomorphismGroup[g_] :=
            Block[{gener = Generators[g], order},
                (* Generators uses Nauty *)
                order := Length[gener[[1]]];
                (* Does not work for rigid graphs *)
                Group[gener,{Range[order]},PP]
                (* PP is permutation multiplication *)
            ]

Group::usage = "Group[g,h,op] multiplies elements of a list g with
elements of a list h (multiplication is given by operation op) and
recursively appends results to the list h."

Group/: Group[g_, h_:{1}, op_:NonCommutativeMultiply] :=
   Block[{hh = h, kk = 0, ii, t},
         While[kk < Length[hh],
               Increment[kk];
               Do[
                  If[!MemberQ[hh, t = op[ g[[ii]], hh[[kk]] ]],
                     AppendTo[hh, t]
                    ],
                  {ii, Length[g]}]];
   hh]
```

References

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M. Petkovšek, T. Pisanski
Institute for Mathematics,
Physics and Mechanics
University of Ljubljana,
Ljubljana, Slovenia

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A CRITERION FOR THE ENDPOINT COMPACTIFICATION OF AN OPEN 3-MANIFOLD WITH ONE END TO BE A GENERALIZED 3-MANIFOLD

D. REPOVŠ

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Secondary: 57P05, 57N99, 54D05

Key words and phrases: open 3-manifold, ends, endpoint compactification, generalized 3-manifold, pushing loops close to infinity, local simple connectivity

Abstract: We study the following problem: to find conditions, "checkable from within" an open 3-manifold M, which guarantee that the endpoint compactification $\hat{M}$ of M is a generalized 3-manifold. Our main result is: The endpoint compactification $\hat{M}$ of a 3-manifold M with one end is a generalized 3-manifold if and only if M satisfies the property that (1) given a neighbourhood U of ∞ there exists a neighbourhood $V \subset U$ of ∞ such that for every $k = 2,3$ and for every mapping $f : \partial B^k \to V$ and every neighbourhood $W \subset V$ of ∞, there exist pairwise disjoint k-cells $D_1, ..., D_t \subset \text{int}\, B^k$ and a mapping $F : D \to U$ such that $D = \overline{B^k - (D_1 \cup ... \cup D_t)}$, $F \mid \partial B^k = f$, and $F(\partial D_j) \subset W$ for every $j \in \{1, ..., t\}$; (2) $H_2(\_; \mathbf{Z})$ is stable at the end ε; and (3) $H_2(\varepsilon; \mathbf{Z}) \cong \mathbf{Z}$.

Assume throughout this paper that M is a topological 3-manifold with the following properties: (i) M is noncompact; (ii) ∂M is either compact or empty; (iii) M has one end; and (iv) M contains no fake 3-cells.

Recall the definition of an *end* of a locally compact space X: this is a collection E of open subsets of X satisfying the following properties: (1) Each element of E is open, connected, and nonempty; (2) Each element of E has compact frontier; (3) If $e_1, e_2 \in E$ then there is $e_3 \in E$ such that $e_3 \subset e_1 \cap e_2$; (4) $\cap\{\bar{e} \mid e \in E\} = \emptyset$; and (5) E is maximal with respect to properties (1)–(4).

A prime example is $W = N - C$, where N is a compact topological manifold with boundary and $C \subset \partial N$ is a boundary component. Then W has exactly one end [8] [9] [14].

Denote by $\hat{M}$ the endpoint (Freudenthal) compactification of M and let $\hat{M} - M = \{\infty\}$. The following theorem was first proved by C. H. Edwards, Jr. [7] and, independently, by C. T. C. Wall [16]:

Theorem 1. (C. H. Edwards and C. T. C. Wall) *$\hat{M}$ is a 3-manifold if and only if M is simply connected at ∞.*

A *neighbourhood of infinity* in a locally compact space X is an open set $U \subset X$ such that $X - U$ is compact. A locally compact space X is *simply connected at* ∞ if for every neighbourhood $U \subset X$ of ∞ there exists a neighbourhood $V \subset U$ of ∞ such that every loop in V is null-homotopic in U. Thus M is simply connected at ∞ if and only if $\{\infty\}$ is 1-LCC in $\hat{M}$.

Note the distinction between these properties and assertion that $\hat{M}$ is 1-LC at ∞: $\{\infty\}$ is 1-LCC in $\hat{M}$ if for every open set $\hat{U}$ in $\hat{M}$ there exists an open set $\hat{V} \subset \hat{U}$ such that the inclusion-induced homomorphism $\Pi_1(\hat{V} - - \{\infty\}) \to \Pi_1(\hat{U} - \{\infty\})$ is zero. On the other hand, $\hat{M}$ is 1-LC at ∞ if and only if for every open set $\hat{U} \subset \hat{M}$ there is an open set $\hat{V} \subset \hat{U}$ such that $\Pi_1(\hat{V}) \to \Pi_1(\hat{U})$ is zero.

The question which we wish to address here is: Are there conditions "checkable from within M" that are collectively equivalent to $\hat{M}$ being a *generalized 3-manifold*, i.e. a locally compact, finite-dimensional, separable metrizable ANR which is also a $\mathbf{Z}$-homology 3-manifold (i.e. for every $x \in \hat{M}$, $H_*(\hat{M}, \hat{M} - \{x\}; \mathbf{Z}) \cong H_*(\mathbf{R}^3, \mathbf{R}^3 - \{0\}; \mathbf{Z})$)?

One approach to this problem is to break the statement "$\hat{M}$ is a generalized 3-manifold" into simpler properties and search for solutions to the problem using these more basic properties. For example, $\hat{M}$ is clearly finite-dimensional, so $\hat{M}$ is an ANR if and only if $\hat{M}$ is locally contractible at ∞ [2]. Now, $\hat{M}$ is clearly 0-locally connected (0-LC) at ∞. Since $\hat{M}$ deforms to the one-point compactification of a locally finite 2-dimensional polyhedron (an unpublished result of G. Kozlowski), LC^2 implies LC^∞ [11]. Therefore, $\hat{M}$ is an ANR if and only if $\hat{M}$ is 1-LC and 2-LC at ∞. (Recall that X is k-LC at $x \in X$ if for every neighbourhood $U \subset X$ of x there is a neighbourhood $V \subset U$ of x such that $\Pi_k(V) \to \Pi_k(U)$ is zero, and LC^k means n-LC for all $n \leq k$.)

Furthermore, using the local version of the Hurewicz theorem [11], the property 2-LC may be substituted by its homological equivalent, 2-lc, if it is desirable. Similarly, it can be shown that $\hat{M}$ is a $\mathbf{Z}$-homology 3-manifold if and only if $H_q(\hat{M}, M; \mathbf{Z}) \cong H_q(\mathbf{R}^3, \mathbf{R}^3 - \{0\}; \mathbf{Z})$ for $1 \leq q \leq 3$. Can each of these more basic conditions be recognized from within M?

First, we consider such a criterion for the local k-connectedness of $\hat{M}$, due to J. Dydak [5] (see also [6]). It will be called the PS^kCI *property* (for "Pushing k-Spheres Close to Infinity"): M has the PS^kCI *property* if, given a neighbourhood $U \subset M$ of ∞ there exists a neighbourhood $V \subset U$ of ∞ such that for every mapping $f : \partial B^{k+1} \to V$ and every neighbourhood $W \subset V$ of ∞ there exist pairwise disjoint $(k+1)$-cells $D_1, ..., D_t \subset \operatorname{int} B^{k+1}$ and a mapping $F : D \to U$ such that $D = \overline{B^{k+1} - (D_1 \cup ... \cup D_t)}$, $F \mid \partial B^{k+1} = = f$, and $F(\partial D_j) \subset W$ for every $j = 1, ..., t$. For example, M has the PS^kCI property if and only if for every neighbourhood U of infinity there exists a neighbourhood of infinity $V \subset U$ such that for every neighbourhood $W \subset V$

of infinity, loops in V are freely homotopic within U to a product of loops in W.

Theorem 2. *Let M be a noncompact 3-manifold with ∂M either empty or compact with one end and let $k \in \{1,2\}$. Then the endpoint compactification $\hat{M}$ of M is k-LC at ∞ if and only if M has the property* $\mathrm{PS}^k\mathrm{CI}$.

Proof of Theorem 2. Theorem 2 follows immediately by [5; Lemma (3.2)]. Nevertheless, for the sake of exposition we present here a detailed proof of the $k = 1$ case. Suppose first that $\hat{M}$ is 1-LC at ∞. Given a neighbourhood U of infinity let $\hat{U} = U \cup \{\infty\}$. Since $\hat{M}$ is 1-LC at ∞, there exists a neighbourhood $\hat{V}$ of ∞ in $\hat{M}$ such that any loop in $\hat{V}$ is null-homotopic in $\hat{U}$. Let $V = \hat{V} - \{\infty\}$.

Let $f : \partial B^2 \to V$ be a mapping and W a neighbourhood of ∞. Let $F : B^2 \to \hat{U}$ be an extension of f. Choose a polyhedral manifold neighbourhood N of $F^{-1}(\infty)$ in B^2, small enough so that $N \subset F^{-1}(W)$. Let D be the component of $\overline{B^2 - N}$ containing ∂B^2 and define $G = F \mid D$. D is a disk-with-holes as in the definition of $\mathrm{PS}^1\mathrm{CI}$ and $G(\partial D - \partial B^2) \subset W$. Therefore M has the $\mathrm{PS}^1\mathrm{CI}$ property.

Suppose now that M has the $\mathrm{PS}^1\mathrm{CI}$ property and let U be a neighbourhood of ∞ in $\hat{M}$. Let $U = \hat{U} - \{\infty\}$, and let $V \subset U$ be a neighbourhood of ∞ as in the definition of $\mathrm{PS}^1\mathrm{CI}$. Finally, let $\hat{V} = V \cup \{\infty\}$. Clearly, $\hat{V}$ is a neighbourhood of ∞ in $\hat{M}$, and it remains to be shown that any mapping $f : \partial B^2 \to V$ can be extended to a mapping $F : B^2 \to U$.

As a special case, assume $f(\partial B^2) \subset V$. Let $U_0 = U$, $U_1 = V$, and in general, let U_{n+1} be a neighbourhood of ∞ such that the pair (U_{n+1}, U_n) satisfies the requirements for (U, V) in the definition of $\mathrm{PS}^1\mathrm{CI}$. Furthermore, construct the U_j's so that $\{\hat{U}_j\}_{j\in\mathrm{N}}$ is a neighbourhood basis for M at ∞. Extend f to a mapping $f_1 : D_1 \to U$, where D_1 is a disk-with-holes in B^2 and $f_1(\partial D_1 - \partial B^2) \subset U_2$. Inductively, extend f_n to a mapping $f_{n+1} : D_{n+1} \to U_1$, where D_{n+1} is a disk-with-holes in B^2, $(D_n - \partial B^2) \subset \operatorname{int} D_{n+1}$, $f_{n+1}(D_{n+1} - D_n) \subset U_{n-1}$ and $f_{n+1}(\partial D_{n+1} - \partial B^2) \subset U_{n+1}$. The disk-with-holes D_{n+1} should be constructed so that the components of $B^2 - D_{n+1}$ have diameters $< \frac{1}{n+1}$, so that $D_\infty = \bigcup_{n\geq 1} D_n$ is the complement of a 0-dimensional compactum in $\operatorname{int} B^2$. Define $f_\infty : D_\infty \to U$ by $f_\infty \mid D_n = f_n$, $n \in \mathbf{N}$. Then f_∞ is a proper mapping of D_∞ into U, with the ends of D_∞ all going to the end of U at ∞. Therefore $F \mid D_\infty = f$ and $F(B^2 - D_\infty) = \infty$, defines a mapping of B^2 into $\hat{U}$ that extends f.

Now let $f : \partial B^2 \to \hat{V}$ be an arbitrary mapping $K = f^{-1}(\infty)$. Let $V_1 \supset V_2 \supset \ldots$ be connected neighbourhoods of ∞, chosen so that $\{\hat{V}_j\}_{j\in\mathrm{N}}$ is a basis for $\hat{M}$ at ∞ and so that any loop in V_{j+1} is null-homotopic in $\hat{V}_j$, as in the Special case. Recall that any loop in V is null-homotopic in $\hat{U}$, so we may set $V_1 = V$ and $V_0 = U$.

The complement of K in B^2 may be written as the union of 2-cells $B_1 \subset B^2 \subset ...$, where $(B_j \cap \partial B^2) \cup f^{-1}(V_j) = \partial B^2$. Using connectivity of V_j, f may be extended over $\partial B_j - \partial B^2$ so that $f(\partial B_j - \partial B^2) \subset V_j$. Let $C_j = B_{j+1} - B_j$ and $C_0 = B_1$. Then C_j is a union of 2-cells and $f(\partial C_j) \subset V_j$ for each j. Applying the Special case to $f \mid \partial C_j$, we extend f to a mapping of C_j into $\hat{V}_{j-1}$ for each j, resulting in an extension of f to a mapping $F : B^2 \to \hat{V}_0$. ∎

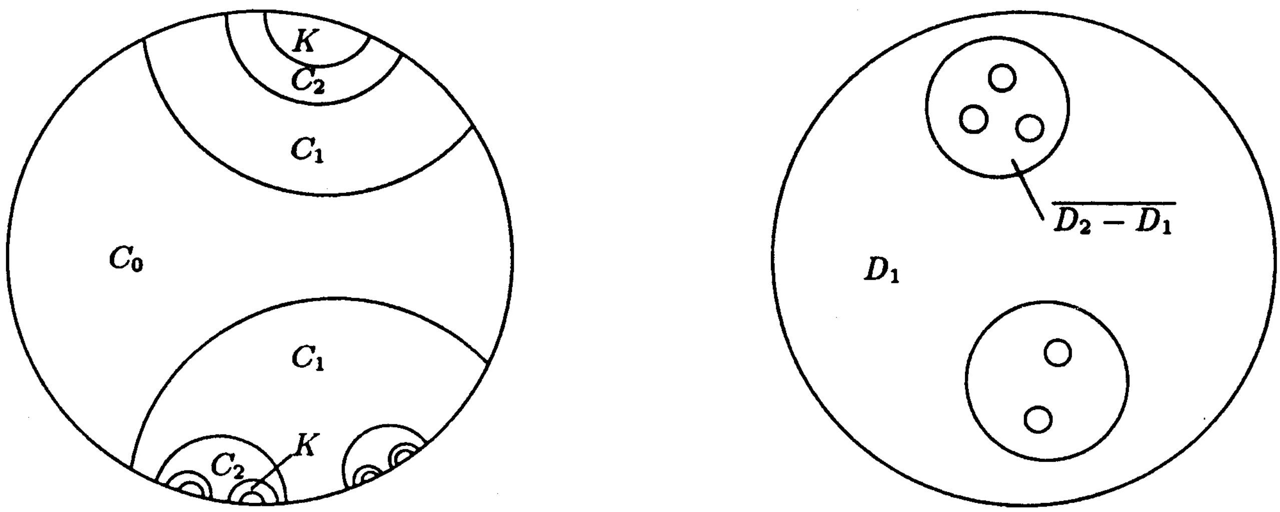

Figure 1

Before we continue we need to introduce a new concept — stability of homology groups at an end, and following that, homology groups of an end. So let X be a locally compact space with one end ε. Following [14], we shall define $H_*(\varepsilon; \mathbf{Z})$ to be the inverse limit of the inverse system $\{H_*(U_i; \mathbf{Z}); \alpha_{i,i+1}\}_{i \in \mathrm{N}}$, associated to a system of open neighbourhoods

$$U_1 \xleftarrow{\alpha_{1,2}} U_2 \xleftarrow{\alpha_{2,3}} U_3 \xleftarrow{\alpha_{3,4}} ...$$

of the end ε, which is *stable*, i.e. for some subsequence $\{H_*(U_{i_j}; \mathbf{Z}); \alpha_{i_j, i_j+1}\}_{j \in \mathrm{N}}$, the induced maps are isomorphisms:

$$\cdots \longleftarrow \operatorname{im} \alpha_{i_1, i_2} \underset{\cong}{\xleftarrow{\alpha_{i_1,i_2}|}} \operatorname{im} \alpha_{i_2, i_3} \underset{\cong}{\xleftarrow{\alpha_{i_2,i_3}|}} \cdots$$

It can be shown, using the same ideas as in [14] that $H_*(\varepsilon; \mathbf{Z})$ is well-defined.

We define two more properties. Let X be any space. Then X is said to satisfy the *Kneser finiteness* if no compact subset of X intersects more than a finite number of pairwise disjoint fake 3-cells. Next, X is said to have the *map separation property* if for every collection $f_1, ..., f_n : B^2 \to X$ of Dehn disks such that if $i \neq j$ then $f_i(B^2) \cap f_j(\operatorname{int} B^2) = \emptyset$ and for every neighbourhood $U \subset X$ of the set $\bigcup_{i=1}^{n} f_i(B^2)$ there exist maps $g_1, ..., g_n : B^2 \to U$ such that (i) for every i, $f_i \mid \partial B^2 = g_i \mid \partial B^2$; and (ii) for every $i \neq j$, $g_i(B^2) \cap g_j(B^2) = \emptyset$. Recall that a disk $f : B \to X$ is said to be *Dehn* if the closure of the set $\{x \in B^2 \mid f^{-1}(f(x)) \neq x\}$ misses ∂B^2. For more on these properties see [13].

We now come to the main result of the paper — an interior characterization of generalized 3-manifolds:

Theorem 3. *Let M be an open 3-manifold with one end ε. Then $\hat{M}$ is a generalized 3-manifold if and only if the following conditions hold:*

(i) *for every neighbourhood $U \subset M$ of ∞ there is a neighbourhood $V \subset U$ of ∞ such that for every map $f : \partial B^k \to V$, $k = 2, 3$, and every neighbourhood $W \subset M$ of ∞ there exist k-cells $B_1^k, \ldots, B_m^k \subset B^k$ and an extension*

$$f' : (B^k - \bigcup_{i=1}^{m} \operatorname{int} B_i^k) \to U$$

of f, such that $(\operatorname{int} B_i^k) \cap (\operatorname{int} B_j^k) = \emptyset$ for all $i \neq j$ and $f'(\partial B_i^k) \subset W$ for all $i \leq m$; and

(ii) *$H_2(\_;\mathbf{Z})$ is stable at ε and $H_2(\varepsilon;\mathbf{Z}) \cong \mathbf{Z}$.*

Proof. We only need to prove the sufficiency. Clearly, $\hat{M}$ is always finite-dimensional since such is already M, so $\hat{M}$ is an ENR if and only if $\hat{M}$ is LC^∞ at ∞ [2]. Now, $\hat{M}$ is always LC^0 at ∞ and since $\hat{M}$ deforms onto a one-point compactification of some locally finite 2-dimensional polyhedron (as observed by G. Kozlowski) it suffices to prove that $\hat{M}$ is LC^2 at ∞. The latter is by Theorem 2 precisely the condition (i) above.

Next, by the Hurewicz theorem, M is 1-lc ($\mathbf{Z}$) at ε. Let $\{\hat{U}_i\}$ be a neighbourhood base at ∞. Consider the long exact sequence for the Borel-Moore homology [1] with compact supports for the pair $(\hat{U}_i, \hat{U}_i - \{\infty\})$:

$$\cdots \to H_k^c(\hat{U}_i) \to H_k^c(\hat{U}_i, \hat{U}_i - \{\infty\}) \to H_{k-1}^c(\hat{U}_i - \{\infty\}) \to H_{k-1}^c(\hat{U}_i) \to \cdots$$

Then by the Sklyarenko theorem [15], $\overleftarrow{\lim_i} H_*^c(\hat{U}_i) \cong 0 \cong \overleftarrow{\lim_i}{}^1 H_k^c(\hat{U}_i)$. Now, by excision, $H_k^c(\hat{U}_i, \hat{U}_i - \{\infty\})$ doesn't depend on the choice of U_i. It therefore follows by the condition (ii) of the theorem that

$$H_3^c(\hat{M}, \hat{M} - \{\infty\}) \cong H_3^c(\hat{U}_i, \hat{U}_i - \{0\}) \cong \overleftarrow{\lim_i} H_2^c(\hat{U}_i - \{\infty\}) \cong H_2^c(\varepsilon) \cong \mathbf{Z}.$$

Similarly, for $k \leq 2$, $H_k^c(\hat{M}, \hat{M} - \{\infty\})$ belongs to the short exact sequence

$$0 \longrightarrow \overleftarrow{\lim_i}{}^1 \tilde{H}_{k-1}^c(\hat{U}_i - \{\infty\}) \longrightarrow H_k^c(\hat{M}, \hat{M} - \{\infty\}) \longrightarrow$$
$$\longrightarrow \overleftarrow{\lim_i} \tilde{H}_k^c(\hat{U}_i - \{\infty\}) \longrightarrow 0$$

and the condition (i) implies that $\{\tilde{H}_1^c(\hat{U}_i - \{\infty\})\}$ vanishes.

It follows that: $H_i(\hat{M}, \hat{M} - \{\omega\}; \mathbf{Z}) \cong H_i(\mathbf{R}^3, \mathbf{R}^3 - \{0\}; \mathbf{Z})$, $i = 1,2,3$. Thus $\hat{M}$ is also a $\mathbf{Z}$-homology 3-manifold hence a generalized 3-manifold. ∎

Remark. If the Poincaré conjecture is true then the one-point compactification of an open 3-manifold M with one end need not be a generalized 3-manifold *even* if M is contractible. Let M be Kister-McMillan's open 3-manifold [10]. Then M is contractible and has one end. Suppose $\hat{M}$ were a generalized 3-manifold. Then by M. G. Brin [3] $\hat{M}$ would have a resolution so by Brin-McMillan [4] M would embed in a compact 3-manifold. However, the latter is known to be false.

If we add a general position hypothesis to Theorem 3, we get the following recognition theorem for 3-manifolds, by invoking the main theorem of [12]:

Theorem 4. *Let M be an open 3-manifold with one end ε. Then $\hat{M}$ is a topological 3-manifold if and only if the following conditions are satisfied:*

(i) *$\hat{M}$ satisfies the Kneser finiteness;*

(II) *$\hat{M}$ possesses the map separation property;*

(iii) *M satisfies the PS^kCI property for $k = 1$ and 2; and*

(iv) *$H_2(\_; \mathbf{Z})$ is stable at ε and $H_2(\varepsilon; \mathbf{Z}) \cong \mathbf{Z}$.* ∎

We shall conclude with the following open problem. Let X be a connected ENR with one end. Let $U, W \subset X$ be open neighbourhoods of infinity such that W is connected and $W \subset U$. Let $x_0 \in U$ and $x_1, x_2 \in W$. Then there are paths γ_1 from x_1 to x_0, γ_2 from x_1 to x_0, and γ_0 from x_1 to x_2, and γ_0 lies in W. The inclusions induce isomorphisms $(i_k)_\# : \Pi_1(W, x_k) \to \Pi_1(U, x_k)$ for $k = 1,2$. The maps γ_k induce isomorphisms $(\gamma_j)_\# : \Pi_1(U, x_j) \to \Pi_1(U, x_0)$, $j = 1,2$ and $(\gamma_0)_\# : \Pi_1(W, x_1) \to \Pi_1(W, x_2)$. Let $\phi = (\gamma_2)_\#(\gamma_0)_\#(\gamma_1)_\#^{-1} : \Pi_1(U, x_0) \to \Pi_1(U, x_0)$. Then ϕ is an inner automorphism. Let $H_k = ((\gamma_k)_\#^{-1}(i_k)_\#)\Pi_1(W, x_n)$, $k = 1,2$ and let N_k be the normal closure of H_k in $\Pi_1(U, x_0)$. Then $N_1 = N_2$. Therefore, if we let G_W be the normal closure of $\mathrm{im}[\Pi_1(W, x_1) \to \Pi_1(U, x_0)]$, then G_W is well-defined and we may set $\Pi_1^\infty(U, x_0) = \bigcap\{G_W \mid W \text{ open, connected } U\}$. We define thad X has the property PS^{1}I (for "Pushing 1-Spheres to Infinity") if for every open neighbourhood $U \subset X$ of ∞ there is an open neighbourhood $V \subset U$ of ∞ such that $\mathrm{im}[\Pi_1(V) \to \Pi_1(U)] \subset \Pi_1^\infty(U)$ where we restrict to those W in the definition of $\Pi_1^\infty(U)$ which lie in V. Clearly, the PS^{1}CI property implies the PS^{1}I property. Does the converse also hold?

Acknowledgements. The preliminary version of this paper was written in 1979 (at which time the author was unaware of the related manuscript of J. Dydak [5] which was published in 1981). Taking into account the papers [5] and [6], the final revision was then made during the visit to the University of Nantes, France in February 1992. The author would like to acknowledge the kind hospitality of Professor H. Ibisch.

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D. Repovš
Institute for Mathematics,
Physics and Mechanics,
University of Ljubljana,
Ljubljana, Slovenia

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and Its Applications,
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ON CHARACTER RECOGNITION VIA CRITICAL POINTS

E. V. SHCHEPIN AND G. M. NEPOMNYASHCHII

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Key words: optical character recognition, critical point graph, Y-equivalence, cript-method

Abstract. A new approach to image coding, based on ordinate-preserving plane homeomorphism invariants, is presented. A critical point graph bearing full information on these invariants is outlined. Apropriate noise-smoothing techniques are developed. The proposed method has been implemented and tested with a new OCR-program CRIPT.

To know the boundary means to know the set. To recognize a symbol, it is sufficient to see its contour. For most of the English characters their contours represent a simple closed curve (the characters are realized as having non zero thickness — the typical thickness of a character produced by scanning with a resolution of 300 dpi is 4–5 pixels). For example, they include such characters as "c","s","v","x","y".

As far as "b","p","d","o","i" are concerned, they have two component contours.

Since all simple closed curves are topologically equivalent, topology seems to be of little service in recognizing thick characters. That is why when applying topology to character recognition, one usually makes use of a thinning procedure. The latter reduces a thick original to its skeleton, i.e. subset of one pixel thickness, which globally looks like the original.

A thin set can be interpreted as being 1-dimensional object, i.e. a graph. Topology of graphs is essentially richer then that of 1-dimensional manifolds (every contour is a 1-dimensional manifold that is a disjoint union of simple closed curves). In addition to homotopical invariants (homotopical type of a connected graph is uniquely defined by its Euler characteristics), such singular points as branching and ends are topologically invariant. In particular, thin "c" has two ends, thin "y" has three ends, hence, they are topologically different.

Different type graphs (e.g. that of neighbourhood, lines or region adjacency) turned out to be useful in character encoding and recognition without thinning (see [1,2] for example).

Searching for topological differences between letters is of crucial importance in handwrittten character recognition because it is the topological properties which are the most stable in infinite-dimensional variations of the different writtings of the same character.

Topology itself fails to distinguish "p" and "d", "b" and "d", since a simple rotation of the plane translates "p" onto "d", and a symmetry "b" onto "d". Hence, to recognize a general symbol, we should know the plane orientation. To distinguish "b" and "d" it is necessary to know where "left" is and where "right" is.

Our main idea is to involve *topology of a function* to provide an effective combination of topology and geometry to recognize both handwritten and typed symbols.

By *topological homeomorphism* of a function $f : A \to B$ we mean *a pair* of homeomorphisms $g : A \to A$, $h : B \to B$, such that $f \circ g = g \circ h$.

For our purposes we are mainly interested in function $Y : \mathbf{P} \to \mathbf{R}$, being a projection of the plane $\mathbf{P}$ onto vertical axis.

If a pair of orientation-preserving homeomorphisms g (of plane) and h (of line) produces a homeomorphism of Y, (i.e. $Y \circ g = h \circ Y$), then g has the following property of monotonicity.

Definition. *A mapping $f : A \to A$ is called monotonic with respect to a function $h : A \to \mathbf{R}$ or briefly h-monotonic, if for every pair of points $a, b \in A$ the inequality $h(a) > h(b)$ is equivalent to inequality $h(f(a)) > h(f(b))$.*

It should be noted that every Y-monotonic homeomorphism is able to be coupled with line homeomorphism to compose a homeomorphism of Y.

Now we can distinguish contours. Namely, we say that two contours are *Y-equivalent* if there exists an Y-monotonic orientation-preserving homeomorphism translating one contour onto another (in this situation the bodies of symbols, i.e. the regions of the plane bounded by contours also have to be translated one onto another).

There are three reasons why the group of Y-monotonic homeomorphisms seems to be a powerful tool in handwritten character recognition.

1. The Y-equivalence is sufficiently *fine* tool to distinguish most of the characters. Particularly, all letters of the folowing sequence represent different classes of Y-equivalence almost in any font:

"a", "b", "c", "e", "f", "g", "h", "i", "j", "k", "l", "m", "n", "o", "p", "s", "t", "v", "w", "x".

The characters usually being Y-equivalent are "A" and "R", "B" and "8", "V" and "Y". Depending on font "b" and "d", "p" and "q" may turn out to be either equivalent or not.

2. The Y-equivalence is a sufficiently *rough* tool, since for every handwritten letter there are only a few different Y-types arising in reasonably accurate writtings (3–5 character).

3. And the last but not the least: there exists a fast algorithm for verifying the Y-equivalence.

The topological type of a smooth function is known to be determined by its critical points, i.e. by points with a vanishing gradient. In particular, for 1-dimensional manifolds every Morse function (in our case this is simply a function with non-zero second derivates in critical points) has critical point only of two types, that is, local minima and maxima (see [3]).

But the notion of local extreme is naturally defined in discrete situation. Thus, for every discrete image of a symbol one effectively defines all critical points of Y using line adjacency (see [2]).

The number of all critical points as well as that of all maxima (or minima) are Y-invariants (i.e. they are invariants of Y-equivalence).

The main goal of this paper is to construct a full system of Y-invariants. To do this we introduce the concept of a critical points graph (CPG).

It is interesting to compare the method of critical points (*cript method*) with that of *crossing* (see [4]). The latter distinguishes characters using the dynamics and crossings. Namely, for every subset of the plane and every line the crossing number is defined as being the number of black-to-white subset transitions in a path through the line.

The vertical crossing number of a plane subset is defined as a sequence of crossing number of this subset with a variable horizontal lines moving from top to bottom. (One adds a new member to the sequence only in changing of crossing numbers thus, the sequence does not contain a pair of equal and subsequent numbers.) The following theorem clarifies the relationship between the cript- and crossing-methods.

Theorem 1. *Two plane regions have the same vertical crossing sequence if and only if there exists a bijective (but in general not continuous) Y-monotonic mapping of the plane translating one of them onto the other.*

Hence, Y-equivalence is *finer* than vertical crossing sequence equivalence (VCS-equivalence).

The insufficient distinguishing power of VCS-equivalence is usually compensated by coupling it with horisontal crossing sequence equivalence.

Similarly to the Y-equivalence, one is able to define *X-equivalence* generated by the projection $X : \mathbf{P} \to \mathbf{R}$ on horizontal axis.

The original OCR-program, based on the cript-method, made use of X-invariants as well. Having gained some experience, we rejected X-invariants Y-invariants turned out to be sufficient for our purpose.

In addition, there are theoretical arguments in favor of Y-invariants in contrast to X-invariants. Namely, incline variations which are rather large not only in handwritting but also in typed texts (let us recall italic) have a considerable effect on X-type and do not change Y-type.

There is another way to introducing Y-equivalence based on the lexicographical order. Two different lexicographical orders are on the plane: the first one for which the ordinate (i.e. Y) is more important than abscissa (i.e. X) and the second (being dual) with opposite priority of the coordinates.

A mapping F of the plane onto itself is referred to as *lex-monotonic* if it preserves the lexicographical order (i.e. $F(z) < F(z') \iff z < z'$).

This allows us to state teh following

Theorem 2. *Two regions are Y-equivalent iff there exists a continuous lex-monothonic (with Y being priority) bijection of the plane translating one of the region onto the other.*

Omitting the word "continuous" and substituting "VIS" for "Y" we obtain an alternative form of Theorem 1.

The proofs of both theorems are routine for topologist and needed only as a motivation of basic definitions and their better understanding.

The proposed approach based on criticalpoint analysis was implemented by the authors into OCR-program CRIPT (CRItical PoinTs). This program is in Turbo Pascal. It turned out to be a very compact (CRIPT may be used on PC XT with 300 K free memory) and gives good results in recognition of English, Russian and Armenian texts as well as for isolated handwritten letters.

The topological nature of the system implies a high flexibility of the program. After training, when using any (English or Russian) text, another text of similar but not identical type may be recognized. This property implies rapid training of the program with the new texts. As a rule we achieve good results after training with 5–10 text strings, it takes about 10–20 min to produce the new reference data base. CRIPT recognizes about 20 symbol/s (IBM PC AT/286, 8 MHz).

As to recognition rate, it is closely related to the text quality. For example, it achieves 99.5–99.9% in any laser-printed text recognition, is reduced to 99.0% in recognizing NLQ-printed texts and dramatically falls for texts with fractured characters or with numerous touching between symbols.

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E. V. Shchepin, G. M. Nepomnyashchii
Steklov Mathematical Institute,
Moscow, Russia

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and Its Applications,
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SELECTIONS OF MAPS WITH NONCONVEX VALUES

P. V. SEMENOV

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Keywords: selections, convexity, Caratheodory theorem, Michael Selection theorem

Abstract The purpose of this survey paper is to present the technique by means of which one can eliminate the convexity hypothesis from some important topological theorems on selections of manyvalued maps.

0.

This lecture I begin form two elementary formulated problems about graphs of continuous functions of one real variable. Then the solutions of these problems will be given.

After two theorems will be stated which are accordingly the generalizations of the above problems for the case of functions of many real variables. Finally, it will be said about the applications in the so called theory of singlevalued selections of manyvalued maps. Briefly, the principal aim of this lecture is to demonstrate the technique which allow us to reject the convexity in some important topological theorems I suppose that such controlled refusal from a convexity may be useful at other domains of mathematics.

1.

Problem 1. *Let A and B be points onto a graph G_f of a monotone continuous function f defined on an interval. Let the distance AB be equal to $2R$. Then the middle point O of the segment $[A, B]$ is situated from G_f at the distance equal or less than $(\sqrt{2}/2)R$:* $\operatorname{dist}(O, G_f) \leq (\sqrt{2}/2)R$.

Problem 2. *Let A, B, C be points onto a graph G_f of a continuous function f defined on an interval. Let D be a point in the triangle ΔABC. Then the point D lies in some segment $[E, F]$, where:*

- *$E \in G_f$ and $F \in G_f$;*
- *$EF \leq \max\{AB, BC, AC\}$.*

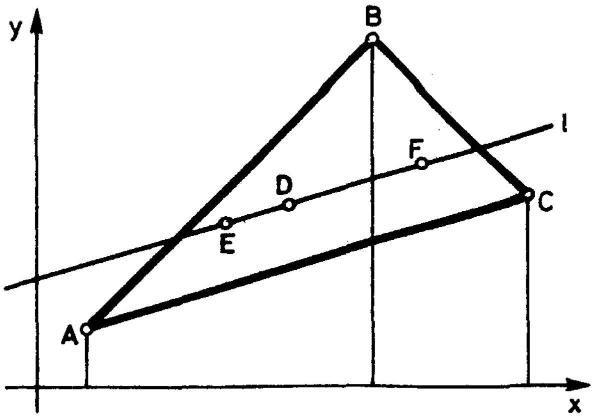

Fig. 2

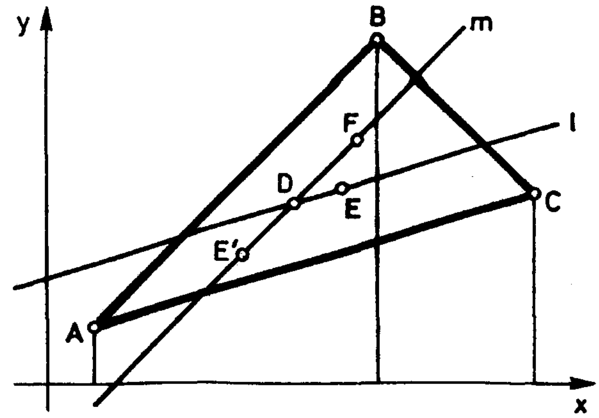

Fig. 3

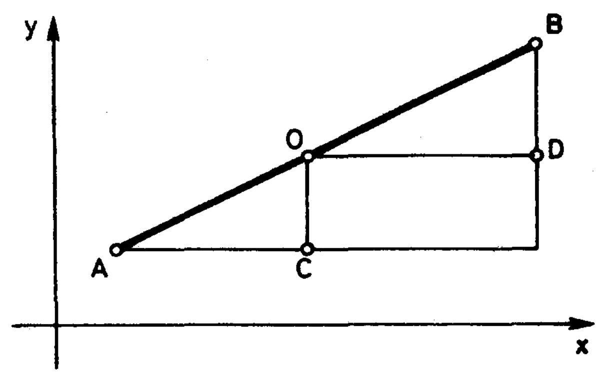

Fig. 1

Solution of the problem 1. Let f be an increasing function and let the graph G_f intersect the vertical line through the point O below this point (see fig.1).

Then the graph G_f intersects a side OC of the triangle ΔAOC and the graphG_f intersects a side OD of the triangle ΔBOD. But the triangle ΔAOC is congruent to the triangle ΔBOD and $AO = OB = R$. Therefore either $OC \leq (\sqrt{2}/2)R$ or $OD \leq (\sqrt{2}/s)R$. ∎

REMARKS.

a) The middlepoint O of the segment $[A, B]$ may be replaced by any point X of this segment: the inequality

$$\operatorname{dist}(X, G_f) \leq (\sqrt{2}/2)R$$

remains true.

b) The fact analoguous to the problem 1 is true for the functions satisfying the inequality $|f(x) - f(y)| \leq |x - y|$ i.e. for the Lipshitz with constant 1 functions.

c) For graphs of Lipshitz with constants k functions the following inequality may be obtained for any point X from segment $[A, B]$, where $A \in G_f$, $B \in G_f$ and $AB = 2R$

$$\operatorname{dist}(X, G_f) \leq \sin(\operatorname{arctg} k) \cdot R$$

Solution of the problem 2. Let $x(A) < x(D) \leq x(B) < x(C)$. Through the point D draw the line l parallel to the side $[A, C]$ (see fig. 2).

Then the graph G_f intersects this line l over the segment $[x(A), x(B)]$ and the graph G_f intersects this line l over the segment $[x(B), x(C)]$. Let E be a „minimal" point of intersection $G_f \cap l$ over the segment $[x(A), x(B)]$.

a) If $x(E) \leq x(D)$ (see fig. 2) then the point D lies in some segment $[E, F]$ with $E \in G_f$ and $F \in G_f$ which may be moved in the side $[A, C]$ with the help of some parallel transfer.

b) If $x(E) > x(D)$ (see fig. 3) then we may draw the line m through the point D to the side $[A, B]$. In this case we obtain that the point D lies in the some segment $[E', F]$ with $E' \in G_f$ and $F \in G_f$ which may be moved in the side $[A, B]$ with the help of some parallel transfer. ■

2.

Now we pass to the case of many variables.

The symbol $[A_1, A_2, ..., A_m]$ denotes the set of all convex combinations of the points $A_1, A_2, ..., A_m$; shortly — the simplex with vertexes $A_1, A_2, .., A_m$.

The symbol $R[A_1, A_2, ..., A_m]$ denotes the minimum of the set of radii of all balls containings the simplex $[A_1, A_2, ..., A_m]$. We denote by $\mathrm{Gr}(\mathbf{R}^{n+1})$ the class of all subsets of Euclidean space $\mathbf{R}^{n+1}$ which are graphs of some continuous functions on n variables with a convex domain of definition at some coordinate system which depends of that very subset. If we consider only an orthonormal coordinate system and graphs of only Lipshitz with constant k functions we obtain the class $G\,\mathrm{Lip}_k(\mathbf{R}^{n+1})$:

$$G\,\mathrm{Lip}_k(\mathbf{R}^{n+1}) \subset \mathrm{Gr}(\mathbf{R}^{n+1}) .$$

Theorem 1. *For every $n \in \mathbf{N}$ and $k \geq 0$ there exists a constant $\alpha = \alpha(n, k)$, $0 \leq \alpha < 1$, such that for any P from the class $G\,\mathrm{Lip}_k(\mathbf{R}^{n+1})$ for any $m \in \mathbf{N}$ and for any points $A_1, ..., A_m$ form P the inequality*

$$\mathrm{dist}(X, P) \leq \alpha \cdot R[A_1, ..., A_m]$$

holds for any points X lying in the simplex $[A_1, ..., A_m]$.

Theorem 2. [2] *Let P be an element of the class* $\mathrm{Gr}(\mathbf{R}^{n+1})$ *and $A_1, ..., A_{n+2}$ be points lying on the P. Then for any point D from the simplex $[A_1, ..., A_{n+2}]$ there exist points $B_1, ..., B_{n+1}$ from the P such that:*

- $D \in [B_1, ..., B_{n+1}]$;

- the simplex $[B_1, ..., B_{n+1}]$ may be moved in some face of the simplex $[A_1, ..., A_{n+1}]$ with the help of some parallel transfer.

Remark. If a set P from the class $\mathrm{Gr}(\mathbf{R}^{n+1})$ be a graph of some linear function then the statement of the above theorem 2 coincides with the classical Caratheodory theorem, which on the Euclidean plane (for example) confirms that any polygon may be divided in to a union of some triangles: at this case the vector of parallel transfer is equal to zero.

3.

The final part of this lecture deals with theorems about an existence of selections of manyvalued maps.

If for any element x of a set X there is some nonempty subset $F(x)$ of a set Y then there exists a singlevalued map $f : X \to Y$ such that

$$f(x) \in F(x)$$

for any $x \in X$. This statement is one of the possible statements of the axiom of choice. The siglevalued map f in this situation is called a *selection of the manyvalued map* F.

If we pass from the category of sets and maps to the more complicated category then the question about the exisetence of selections becomes more complicated, too.

We consider the category of topological spaces and continuous maps. The question here may be formulated in the following form. What topological conditions for spaces X, Y and for a manyvalued map $F : X \to Y$ are sufficient for the existence of a singlevalued continuous selection $f : X \to Y$ of manyvalued map F? One of the most important and widely used answers are given by the following E. Michael theorem [1].

Theorem M. *Let X be a paracompact, Y be a Banach space, $F : X \to Y$ be a manyvalued lower semicontinuous map with closed convex values $F(x)$ for any $x \in X$. Then F admits a singlevalued continuous selection.*

We should be reminded that manyvalued map $F : X \to Y$ is called a *lower semicontinuous* iff for any open set $W \subset Y$ the set

$$F^{-1}(W) = \{x \in X : F(x) \cap W \neq \emptyset\}$$

is an open set in space X.

There exist two standard and important examples of use of this Michael theorem.

Example 1. Let $g : A \to Y$ be a singlevalued continuous map from a closed subset A of a paracompact X to a Banach space Y. Then g admits a continuous extension $f : X \to Y$ to the whole paracompact $X : f|_A \equiv g$.

For the proof it suffices to consider the following manyvalued map from X to Y.

$$F(x) = \begin{cases} \{g(x)\}, & x \in A \\ Y, & x \notin A \end{cases}$$

It's easy to see that F is a lower semicontinuous. Therefore the application of theorem M gives a selection $f : X \to Y$ which is an extension of map g. ■

Example 2. Let $L : Y \to X$ be a linear continuous operator from a Banach space Y onto a Banach space X. Then L admits a continuous section, i.e. a map $s : X \to Y$ such that

$$L(s(x)) = x$$

for any $x \in X$.

For the proof it suffices to consider the manyvalued map $L^{-1} : X \to Y$. From the Banach Open Mapping theorem we have that L^{-1} is a lower semicontinuous and hence L^{-1} admits a selection which is a section for the L. ∎

The above theorems 1 and 2 show that the condition of the convexity of values $F(x)$ in the Michael theorem may be replaced by the the essentially weaker conditions. Briefly, convexity may be replaced by „Lipshitzivity". For example, on the plane we may assume that $F(x)$ is a graph of some monotone continuous function or $F(x)$ is a sinusoid, etc.

Theorem 3. *For every $n \in \mathbb{N}$ and $k \geq 0$, for every paracompact X and for every lower semicontinuous map $F : X \to \mathbb{R}^n$ with closed values belonging to the class $G\,\mathrm{Lip}_k(\mathbb{R}^n)$, $x \in X$, there exists a singlevalued continuous selection.*

Finally, I state two open questions. First of them deals with attempts to pass from the graphs of functions to the graphs of mappings.

Question 1. *Let A, B, C, D be points on a graph G_f of a continuous mapping $f : \mathbb{R} \to \mathbb{R}^2$. Let X be a point in the tetrahedron $[A, B, C, D]$. Then the point X lies in a some triangle $\triangle$ whose vertexes belong to G_f and which may be moved in some face of the tetrahedron $[A, B, C, D]$ with help of some parallel transfer.*

The second question deals with a more detailed estimate of teh constants $\alpha(n, k)$ (see the theorem 1, above) when $n \to \infty$.

Question 2. *Is theorem 1 true for the graphs of the Lipshitz functions with the convex domains of definition lying in the Hilbert space?*

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P. V. Semenov
Moscow Pedagogical University
Moscow, Russia

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