REGULAR NEIGHBOURHOODS OF HOMOTOPICALLY PL EMBEDDED COMPACTA IN 3-MANIFOLDS (1)

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Summary. We investigate when are regular neighbourhoods of homotopically PL embedded compacta in a given 3-manifold homeomorphic.

AMS (MOS) Subject Classification (1980):
Primary 57Q40, 57N10.
Secondary 57M05, 57M25.

Key words and phrases:
regular neighbourhood, PL embedding, compact polyhedron.

In this paper we shall investigate the following problem concerning regular neighbourhoods of compacta in 3-manifolds: consider any two homotopic PL embeddings $f_1, f_2 : K \to \text{int } M$ of a compact polyhedron K into the interior of a 3-manifold with boundary M. Let $N_i \subset \text{int } M$ be a regular neighbourhood of $f_i(K)$ in M, i = 1, 2. In general, N_1 and N_2 need not be homeomorphic as the following example shows:

(1) This paper is in its final form and no version of it will be submitted for publication elsewhere.
(2) Supported in part by the Research Council of Slovenia.

Example 1. Let $M = S^3$, $K = S^1 \vee S^2 \vee S^1$, and let $f_i : K \to M$, $i = 1, 2$, be the obvious different embeddings: in $f_1(K)$ both circles are attached to the same side of $f_1(S^2) = S^2 \subset S^3$ while in $f_2(K)$ one circle is on each of the two sides of $f_2(S^2) = f_1(S^2)$. Although f_1 is clearly homotopic to f_2, N_1 is different from N_2 because $\partial N_1 = S^2 \amalg T_2$ while $\partial N_2 = T_1 \amalg T_1$ where T_g denotes the closed orientable surface of genus $g \geq 0$ and $\amalg$ stands for the disjoint union.

However, it is nevertheless true in this example that N_1 and N_2 have the same number of boundary components and the same Euler characteristics. Our first theorem below shows that this is always the case:

Theorem 1. Let K be a compact polyhedron and let $f_1, f_2 : K \to \text{int } M$ be PL embeddings of K into a 3-manifold with boundary M. Suppose that $(f_1)_* = (f_2)_* : H_*(K; \mathbb{Z}_2) \to H_*(M; \mathbb{Z}_2)$ and let $N_i \subset \text{int } M$ be a regular neighbourhood of $f_i(K)$ in M, $i = 1, 2$. Then for every n, $\beta_n(\partial N_1; \mathbb{Z}_2) = \beta_n(\partial N_2; \mathbb{Z}_2)$, where β_n is the n-th Betti number (mod 2).

Proof. We shall suppress the $\mathbb{Z}_2$ coefficients from the notation throughout the proof. The argument is based on two assertions which we procede to state and prove below.

Assertion 1. The boundaries of N_1 and N_2 have the same Euler characteristic (mod 2), $\chi(\partial N_1; \mathbb{Z}_2) = \chi(\partial N_2; \mathbb{Z}_2)$.

Proof. Let DN_i be the double of N_i (obtained by taking two copies of N_i and identifying them along the boundary). Then $\chi(DN_i) = 0$ since N_i is odd dimensional. On the other hand, $\chi(DN_i) = 2\,\chi(N_i) - \chi(\partial N_i)$. Therefore $\chi(\partial N_1) = 2\,\chi(N_1) = 2\,\chi(N_2) = \chi(\partial N_2)$.

Assertion 2. N_1 and N_2 have the same number of boundary components.

Proof. We shall show that $\beta_2(\partial N_1) = \beta_2(\partial N_2)$. To this end we first note that

$$(*) \qquad \beta_2(\partial N_1) = \dim \ker \rho + 1$$

where $\rho = \rho_1 + \rho_2$ is from the Mayer-Vietoris homology sequence of the triple $(M, N_1, M - \text{int } N_1)$:

$$0 \to H_3(M) \to H_2(\partial N_1) \to H_2(N_1) \oplus H_2(M - \text{int } N_1) \xrightarrow{\rho} H_2(M) \to \ldots$$

and the homomorphisms $\rho_1 : H_2(N_1) \to H_2(M)$ and $\rho_2 : H_2(M - \text{int } N_1) \to H_2(M)$ are induced by inclusions [6; (IV.6)]. We wish to find another representation for $\ker \rho$ which will depend (up to homology) only on K. We shall then conclude that $\beta_2(\partial N_2) = \dim \ker \rho + 1$ as well, and the assertion will follow.

Let $\cdot : H_2(M) \times H^1(M) \to \mathbb{Z}_2$ be the (mod 2) intersection number and let $\gamma : H_1(f_1(K)) \to H_1(M)$ be the inclusion-induced homomorphism. Define

$$Z = \{\xi \in H_2(M) \mid \xi \cdot \gamma(\zeta) = 0, \text{ for every } \zeta \in H_1(f_1(K))\}$$

and consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 \to H_3(M) & \xrightarrow{\lambda'} & H_3(M, M - f_1(K)) & \xrightarrow{\delta} & H_2(M - f_1(K)) & \xrightarrow{\mu} & H_2(M) & \xrightarrow{\lambda} & H_2(M, M - f_1(K)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \phi \\
0 \to \check{H}^0(M) & \longrightarrow & \check{H}^0(f_1(K)) & \longrightarrow & \check{H}^0(M, f_1(K)) & \longrightarrow & \check{H}^1(M) & \longrightarrow & \check{H}^1(f_1(K))
\end{array}$$

where the horizontal line is the homology (resp. Čech cohomology) sequence of the pair $(M, M - f_1(K))$ (resp. $(M, f_1(K))$) and the vertical arrows are the duality isomorphisms [6; (VI.2.17)].

Since λ' is the one-to-one it follows that it is an isomorphism, hence $\ker \delta = \text{im}\, \lambda' = H_3(M, M - f_1(K))$ so $\ker \mu = \text{im}\, \delta = 0$. Therefore μ is an inclusion of $H_2(M-f_1(K))$ into $H_2(M)$ and so $H_2(M-f_1(K) \cong \text{im}\, \mu = \ker \lambda = \ker \phi\lambda$. Note that $Z = \ker \phi\lambda$ hence $H_2(M-f_1(K)) \cong Z$ via the natural identification μ, and observe that $M - \text{int}\, N_1 \approx M - f_1(K)$ so $\text{im}\, \mu = \text{im}\, \rho_2$, ρ_2 can be considered as a homomorphism onto Z, $\rho_2 : H_2(M - \text{int}\, N_1) \to Z$, and it is injective.

Next, we want to compute the dimension of $\ker \rho$. To this end let $W = \rho_1^{-1}(Z)$ and define a homomorphism $\omega : \ker \rho \to W$

by $\omega(u_1+u_2) = u_1$, for all $u_1+u_2 \in \ker \rho \subset H_2(N_1) \oplus H_2(M\text{-int}\, N_1)$. We shall verify that ω is a bijective correspondence.Indeed, if for some $u_1+u_2 \in \ker \rho$ we have that $\omega(u_1 + u_2) = 0$ it follows that $u_1 = 0$ thus $0 = \rho_1(u_1) = \rho_2(u_2)$. Since ρ_2 was shown earlier to be one-to-one, u_2 must be 0, too. Thus ω is injective. To check the surjectivity, choose any $u_1 \in W \subset H_2(N_1)$. It follows that $\rho_1(u_1) \in Z$ hence $\rho_1(u_1) \in \text{im}\, \rho_2$ so we can find $u_2 \in H_2(M - \text{int}\, N_1)$ such that $\rho_2(u_2) = \rho_1(u_1)$ hence $\rho(u_1+u_2) = \rho_1(u_1) + \rho_2(u_2) = 2\rho_1(u_1) = 0$ (recall we are over $\mathbb{Z}_2$). Consequently, $u_1 + u_2 \in \ker \rho$.

It follows that the vector spaces (over $\mathbb{Z}_2$) $\ker \rho$ and W have the same dimension, hence by (*) above,

$$\beta_2(\partial N_1) = \dim W + 1.$$

Note that W depends only on the homology class of embedding of K into M hence, in particular, the same argument would yield that

$$\beta_2(\partial N_2) = \dim W + 1.$$

The assertion now follows.

We now complete the proof of the theorem. Let $B(i,k) = \beta_i(\partial N_k)$, $0 \leq i \leq 2$, $k = 1, 2$. By Assertion 2, $B(2, 1) = B(2, 2)$. By duality, $B(0,1) = B(0, 2)$. By Assertion 1, $B(0,1) - B(1,1) + B(2,1) = \chi(\partial N_1) = \chi(\partial N_2) = B(0,2) - B(1,2) + B(2, 2)$. Consequently, $B(1,1) = B(1,2)$, too.

We now return to the original question and we prove that under certain additional conditions on K, the regular neighbourhoods N_1 and N_2 are homemorphic.

Theorem 2. Let K be a compact and connected polyhedron and let $f_1, f_2 : K \to \text{int } M$ be homotopic PL embeddings of K into a 3-manifold with boundary M. Suppose, in addition that either

(i) $\dim K \leq 1$; or

(ii) K is a surface with nonempty boundary; or

(iii) $K = S^2$; or

(iv) $K = \mathbb{R}P^2$.

Then any two regular neighbourhoods of $f_1(K)$ and $f_2(K)$ in M are PL isomorphic.

Proof. Throughout the proof let $N_i \subset \text{int } M$ be a regular neighbourhood of $f_i(K)$ in M, $i = 1, 2$.

(i) Assume dim K = 1. Then each N_i is a 3-cell with n_i (possibly nonorientable) solid 1-handles, $n_i \in \mathbb{N}$, $i = 1, 2$ [2; (II.4)]. Since f_1 and f_2 are homotopic in M it follows that $\Pi_*(N_1) = \Pi_*(N_2)$, so in particular, $n_1 = n_2$. Furthermore, N_1 and N_2 are either both orientable or both nonorientable. For, given an orientation reversing loop $\alpha \subset N_i$ we can homotope α into $f_i(K)$ and then (in M) into $f_{3-i}(K)$ thus into N_{3-i}, $i = 1, 2$. It now follows by [2; (II.2)] that $N_1 \cong N_2$.

(ii) If K is a surface with nonempty boundary then there is a finite bouquet $T \subset K$ of simple closed curves such that K collapses onto T. Therefore we can apply the preceding argument by [5; (III.29)].

(iii) If $K = S^2$ then $f_i(K)$ is two-sided in M. For, N_i is a product I-bundle ($I = [0, 1]$) because K is simply connected, hence no loop on $f_i(K)$ can reverse the orientation in M. Therefore $N_i \cong f_i(K) \times I$ and the assertion follows.

(iv) Assume first that M is orientable. Then $f_i(K)$ must be one-sided in M. Therefore N_i is a twisted I-bundle over $\mathbb{R}P^2$. Since it is known that (up to a PL isomorphism) there is just one such, we may conclude that $N_1 \cong N_2$.

Finally, suppose that M is nonorientable. If both embeddings $f_i(K)$ are one-sided in M, the preceding argument applies. If both are two-sided then $N_i \cong f_i(K) \times I$, so the assertion follows. Assume now that, say, $f_1(K)$ were one-sided and $f_2(K)$ were two-sided. Consider the orientable 3-manifold double PL covering $p : \tilde{M} \to M$ of M. Then $f_1(K)$ lifts in $\tilde{M}$ to two disjoint (PL isomorphic) copies while $p^{-1}(f_2(K))$ is connected (and double covers $f_2(K)$). Since f_1 and f_2 are homotopic in M, the number of connected components of the lifts $p^{-1}(f_i(K))$ should agree. This contradiction shows that the last case cannot occur.

We shall conclude by discussing two conjectures. The first one suggests that Theorem 2 ought to be true for the more general "genus zero" case, i.e. when $g(\partial N_i) = 0$, at least for orientable 3-manifolds:

Conjecture 1. Let K be a compact polyhedron such that $H_2(K; \mathbb{Z}_2) = 0$, and let $f_1, f_2 : K \to \text{int } M$ be homotopic PL embeddings of K into a orientable 3-manifold with boundary M. Let $N_i \subset \text{int } M$ be a regular neighbourhood of $f_i(K)$ in M and suppose that $g(\partial N_1) = 0$. Then N_1 and N_2 are homeomorphic.

Note that $H_2(K; \mathbb{Z}_2) = 0$ implies that $f_i(K)$ doesn't separate its connected neighbourhoods: consider the reduced homology sequence of the pair $(M, M\text{-}f_i(K))$ over $\mathbb{Z}_2$:

$$\ldots \to H_1(M, M - f_i(K)) \xrightarrow{\partial} \tilde{H}_0(M - f_i(K)) \to \tilde{H}_0(M) \to 0$$

Since we may assume that M is connected, ∂ is onto. By duality, $H_1(M, M - f_i(K)) \cong H^2(f_i(K)) \cong H^2(K) \cong H_2(K) \cong 0$, so $\tilde{H}_0(M - f_i(K)) \cong 0$, too.Thus ∂N_i is necessarily connected hence a 2-sphere. Therefore we could have replaced the condition "$H_2(K; \mathbb{Z}_2) \cong 0$ and $g(\partial N_1) = 0$" by "$\partial N_1 = S^2$".

The importance od Conjecture 1 is illustrated by the following result:

Theorem 3. Modulo Conjecture 1, the following two statements are equivalent:

(i) (Poincaré Conjecture) Every homotopy 3-cell is homeomorphic to the standard 3-cell.

(ii) Every homotopy 3-cell possesses a spine which PL embeds in $\mathbb{R}^3$.

Proof. Clearly, (i) implies (ii) independently of Conjecture 1, so we only need to verify the other implication. Let F be a homotopy 3-cell and choose a tame 3-cell $C \subset \text{int } F$. By hypothesis F has a spine $K \subset \text{int } F$ which PL embeds in int C via some $f : K \to \text{int } C$. Let $N \subset \text{int } C$ be a regular neighbourhood of the polyhedron f(K) in int C (hence also in int F). Since F is contractible, f is homotopic to the inclusion $K \subset F$. Also, since F collapses onto X it follows by [5; (III.30)] that F is a regular neighbourhood of K (in F). It now follows, modulo Conjecture 1, that the regular neighbourhoods F and N (of K and f(K) respectively) are homeomorphic hence N is also a homotopy 3-cell. Since $N \subset \text{int } C \approx \mathbb{R}^3$, N is a genuine 3-cell [1]. Therefore, F is homeomorphic to the standard 3-cell.

The second conjecture suggests that the "genus zero" case may just be the only case when regular neighbourhoods are always the same:

Conjecture 2. Let $N_1 \subset S^3$ be the knot space of the square knot $\Sigma_1 \subset S^3$ and let $N_2 \subset S^3$ be the knot space of the granny knot $\Sigma_2 \subset S^3$, i.e. $N_i = S^3 - \text{int } T(\Sigma_i)$ where $T(\Sigma_i)$ is a tubular neighbourhood of the knot Σ_i in S^3, i = 1, 2. Then there exists a compact connected polyhedron K and PL embeddings $f_i : K \to \text{int } N_i$ such that $f_i(K)$ is a spine of N_i, i = 1, 2. Hence N_1 and N_2 have the same spine (although $N_1 \not\cong N_2$, because their signatures are different [4; (VIII.E.15)]).

We remark that in higher dimensions one can easily construct examples of (M, K, f_1, f_2), where $N_1 \not\cong N_2$, e.g. there is a PL embedding of the dunce hat into S^4 whose regular neighbourhood has a nonsimply connected boundary [3].

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