BOLYAI SOCIETY
MATHEMATICAL STUDIES, 4

Topology with Applications,
Szekszárd (Hungary), 1993, pp. 419–425.

On Continuous Approximations

D. REPOVŠ[1] and P. V. SEMENOV

1. Introduction

Sufficient conditions for the existence of singlevalued continuous selections of multivalued maps usually assume convexity condition (in the infinite-dimensional case) or high connectivity conditions (in the finite-dimensional case) on the point inverses of the mappings or on the values of the multivalued mappings. Well-known theorems of E. Michael [1], [2], [3] illustrate the point. In [4] we investigated the problem of existence of such selections for multivalued maps without the convexity condition for arbitrary (not finite-dimensional) paracompacta (see also [5]). In particular, one of the results proved in [4] states that selections exist for any lower semicontinuous mapping F from a paracompactum P into the Euclidean plane if the values $F(p)$, $p \in P$, are graphs of polynomials

$$g(x) = a_n x^n + \ldots + a_1 x + a_0, \qquad C^{-1} \leq |a_i| \leq C.$$

Here, the domains of definitions of the polynomials are arbitrary convex subsets of the x-axis, the systems of (orthonormal) coordinates depend on the element $p \in P$ and the constant C doesn't depend on p. But is the condition

$$C^{-1} \leq |a_i| \leq C$$

[1] Supported in part by the Go West Fellowship No. 792 from the Commission of the European Communities and by a research grant from the Ministry for Science and Technology of the Republic of Slovenia No. P1-0214-101-94.

Permanent address: Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P.O.Box 64, Ljubljana 61111, Slovenia, *e-mail address*: dusan.repovs@uni-lj.si.

of uniform boundedness of the coefficients of the polynomials and of their reciprocals in fact necessary for a positive solution of the selection problem? The well-known example of E. Michael [1], which used the parts of the graph of the sinusoid $y = \sin(1/x)$, shows that for graphs of *arbitrary* continuous functions, the selection problem has (in general) a negative answer.

If we approximate every continuous function in E. Michael's example by some polynomials (Weierstrass' theorem) and if we can choose these approximations, then we will obtain the example which shows that the condition $C^{-1} \leq |a_i| \leq C$ is essential in our result from [4].

So, we reduce our problem to the following very natural question. Is it possible to choose in Weierstrass' theorem (or in the Stone-Weierstrass' theorem the polynomial v (or element v of a given algebra of functions V) so that:

(i) v is an ε-approximation of a given continuous function f; and

(ii) v continuously depends on f and ε; $v = v(f, \varepsilon)$?

The positive answer to this question can be obtained in abstract terms.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed space and $V \subset X$ a convex subset. Then the following assertions are equivalent:*

(a) *V is a dense subset of X;*

(b) *For every $\varepsilon > 0$ there exists a continuous map $v_\varepsilon : X \to V$ such that $\|x - v_\varepsilon(x)\| < \varepsilon$, for any $x \in X$;*

(c) *For every lower semicontinuous multivalued map $F : P \to X$ of a paracompactum P into the space X with convex (possibly nonclosed) values $F(p)$, $p \in P$, and for every non-decreasing function $\gamma : (0, +\infty) \to (0, +\infty)$ there exists a continuous mapping $f : P \times (0, +\infty) \to V$ such that $\operatorname{dist}(f(p, \varepsilon); F(p)) < \gamma(\varepsilon)$ for any $(p, \varepsilon) \in P \times (0, +\infty)$;*

(d) *For every continuous mapping $g : P \to X$ from a paracompactum P into the space X, for every continuous function $\varepsilon : P \to (0, +\infty)$ and for every non-decreasing function $\gamma : (0, +\infty) \to (0, +\infty)$ there exists a continuous mapping $h : P \to V$ such that $\|h(p) - g(p)\| < \gamma(\varepsilon(p))$, for any $p \in P$.*

Recall that a multivalued mapping $F : X \to Y$ is said to be *lower semicontinuous* if for any open subset G of the space Y the set

$$F^{-1}(G) = \{x \in X \mid F(x) \cap G \neq \emptyset\}$$

is an open subset of the space X. Next, a singlevalued mapping $f : X \to Y$ is said to be a *selection* of a multivalued mapping $F : X \to Y$ if $f(x) \in$

$F(x)$, for any $x \in X$. For the classical facts about the existence of continuous selections of lower semicontinuous mappings, see [1, 2, 3].

Remarks. (1) As it was pointed out by the referee, assertion (d) of Theorem 1 can be strengthened by replacing the singlevalued mapping g with a multivalued mapping, as in assertion (c). However, for our application of Theorem 1 in Chapter 3, we shall need assertion (d) in the form in which it is stated above.

(2) Clearly, Theorem 1 makes sense only for $\dim X = \infty$.

2. Proof of the Theorem

(a) $\Rightarrow$ (b).

For a fixed $\varepsilon > 0$ we consider the covering ω of the whole space X by the open balls $B(v, \varepsilon)$ with radius $\varepsilon > 0$, centered at the points $v \in V$, where V is a convex dense subset of the normed space $(X, \|\cdot\|)$. Let $\{e_\alpha\}$, $\alpha \in A$, be a locally finite continuous partition of unity, inscribed into the covering ω. For any index $\alpha \in A$, we pick an element $v_\alpha \in V$ such that $\operatorname{supp} e_\alpha \subset B(v_\alpha, \varepsilon)$, where $\operatorname{supp} e_\alpha$ is the support of the continuous function $e_\alpha : X \to [0, 1]$. Next, we define the mapping $v_\varepsilon : X \to V$ by the equality

$$v_\varepsilon(x) = \sum e_\alpha(x) v_\alpha, \qquad a \in A.$$

In a sufficiently small neighbourhood of a point $x \in X$ the mapping v_ε is the sum of a finite number of continuous mappings. Hence, v_ε is a continuous mapping from X into V.

Now, for a given point $x \in X$, let $e_{\alpha(1)}, \ldots, e_{\alpha(n)}$ be all elements from $\{e_\alpha\}$, $\alpha \in A$, such that $e_{\alpha(i)}(x) > 0$. Then

$$x \in \operatorname{supp} e_{\alpha(i)} \subset B\big(v_{\alpha(i)}, \varepsilon\big)$$

i.e. $\|x - v_{\alpha(i)}\| < \varepsilon$. Hence

$$\|x - v_\varepsilon(x)\| = \Big\|\sum e_{\alpha(i)}(x)\big(x - v_{\alpha(i)}\big)\Big\| \le \sum e_{\alpha(i)}(x)\|x - v_{\alpha(i)}\| < \varepsilon.$$

(b) $\Rightarrow$ (c).

Let Y be the completion of X, $Y = \overline{X}$, and let $\Phi(p) = \mathrm{cl}\{F(p)\}$, $p \in P$. Then Φ is a lower-semicontinuous mapping from the paracompactum P into the Banach space Y with convex, closed values. By the classical E. Michael's selection theorem [1], there exists a singlevalued continuous mapping $\varphi : P \to Y$ such that $\varphi(p) \in \Phi(p)$, for any $p \in P$. Clearly, (b) implies the density of V in X and hence, the density of V in Y. So, we can apply statement (b) to the pair (Y, V). Pick a monotone decreasing sequence $\varepsilon_1, \varepsilon_2, \dots$ of the positive numbers ε_n, which tends to zero and denote by v_n a continuous mapping $v_n : Y \to V$ such that $\|y - v_n(y)\| < \gamma(\varepsilon_n)$ for any $y \in Y$ and $n \in \mathbb{N}$.

For $\varepsilon \geq \varepsilon_1$ let

$$f(p, \varepsilon) = v_2(\varphi(p)), \qquad p \in P$$

For every $\varepsilon \in [\varepsilon_{n+1}, \varepsilon_n]$, defined as follows:

$$\varepsilon = (1 - t)\varepsilon_{n+1} + t\varepsilon_n, \qquad t \in [0, 1]$$

let

$$f(p, \varepsilon) = (1 - t)v_{n+2}(\varphi(p)) + tv_{n+1}(\varphi(p)), \qquad p \in P.$$

Then the continuity of the mapping $f : P \times (0, +\infty) \to V$ is a corollary of the continuity of the mappings $\varphi, v_1, v_2, \dots$ and of the construction of the mapping f. Finally, for $y = \varphi(p)$ and for

$$\varepsilon = (1 - t)\varepsilon_{n+1} + t\varepsilon_n, \qquad t \in [0, 1]$$

it follows that

$$\|f(p, \varepsilon) - y\| = \|(1 - t)(v_{n+2}(y) - y) + t(v_{n+1}(y) - y)\| < \\ < (1 - t)\gamma(\varepsilon_{n+2}) + t\gamma(\varepsilon_{n+1}) \leq \gamma(\varepsilon_{n+1}) \leq \gamma(\varepsilon),$$

whereas for $\varepsilon \geq \varepsilon_1$, we have that

$$\|f(p, \varepsilon) - y\| = \|v_2(y) - y\| < \gamma(\varepsilon$$

(d) $\Rightarrow$ (a).

For a given $\varepsilon_0 > 0$ let in (d): $P = X$; $g(p) = g(x) = x$; $\varepsilon(p) = \varepsilon(x) \equiv \varepsilon_0$; and $\gamma(t) = t$, $t > 0$. Then for any $p = x \in X$, the element $v = h(p) = h(x) \in V$, where h is a (continuous) mapping from (d), is an ε_0-approximation of x. This completes the proof of Theorem 1. ∎

The proof of the implication (a) $\Rightarrow$ (b) practically coincides with the first step of induction in the proof of the classical convex-valued selection theorem [1] in which the existence is established of a continuous ε-selection f_ε of a given convex-valued lower semicontinuous map F, where ε-selection means that $\operatorname{dist}\left(f_\varepsilon(x), F(x)\right) < \varepsilon$, for all $x \in X$. However, in [1], the proof started with the covering of the space X by open balls $\left\{ B(x,\varepsilon) \mid x \in X \right\}$. For our purpose however, it is sufficient to consider only the open balls $B(x,\varepsilon)$ centered at the elements $x \in V$ of the convex dense subset V of the space X. Notice also that in our situation Michael's theorem doesn't apply directly because X isn't a complete space.

3. An Example

We return now to graph-valued maps.

Example. *There exists a lower semicontinuous (in fact, continuous) mapping F from the segment $[0,1]$ to the Euclidean plane such that:*

(i) *all values of the mapping F are graphs of some polynomials; and*

(ii) *the mapping F doesn't admit any singlevalued continuous selection.*

Proof. Consider the paracompactum $P = (0,1]$ and for any $p \in P$ let $\ell_p : [p/2, p] \to [0,1]$ be a linear function from the segment $[p/2,p]$ onto the segment $[0,1]$ with $\ell_p(p) = 1$. For any $x \in [0,1]$, let

$$[g(p)](x) = \sin\left(1/\ell_p^{-1}(x)\right).$$

Then $g(p)$ is an element of the Banach space $C[0,1]$ of all continuous functions on $[0,1]$ with the usual sup-norm and $g : P \to C[0,1]$ is a singlevalued continuous mapping. By the statement (d) of Theorem 1, there exists a continuous mapping $h : P \to C[0,1]$ such that $h(p)$ is a polynomial, for any $p \in P$ and $\left\| h(p) - g(p) \right\| < p$, $p \in P$.

Now, define the mapping $F : [0,1] \to \mathbb{R}^2$ as follows:

$$F(0) = \{(0,t) \mid -1 \le t \le 1\} \quad \text{and}$$
$$F(p) = \{t, [h(p)](\ell_p(t)) \mid p/2 \le t \le p\}, \quad p \in P.$$

It is clear that the values of the mapping F are graphs of some polynomials over some segments.

Suppose to the contrary that F admits a continuous selection f, i.e. $f(p) = (f_1(p), f_2(p)) \in F(p)$. From the continuity of f_1 and from the condition $p/2 \le f_1(p) \le p$ we obtain that $\mathrm{Im}(f_1) \supset (0,1/2]$. For an arbitrary $a \in [-1,1]$ let: $y_n \in (0,1/2]$; $y_n \to 0$; and $\sin(1/y_n) = a$. Then for some $t_n > 0$, $f_1(t_n) = y_n$, $n \in \mathbb{N}$.

Hence $t_n \to 0$ and $f_2(t_n) = [h(t_n)](\ell_{t_n}(y_n))$, for all $n \in \mathbb{N}$; $\| [h(t_n)](\ell_{t_n}(y_n)) - [g(t_n)](\ell_{t_n}(y_n)) \| < t_n$; and $[g(t_n)](\ell_{t_n}(y_n)) = \sin(1/\ell_{t_n}^{-1}(\ell_{t_n}(y_n))) = a$.

If we pass in the equality $f(t_n) = (y_n, f_2(t_n))$ to the limit, when $n \to \infty$ then we obtain that $f_2(0) = a$, i.e. f cannot be singlevalued. Contradiction. ■

To obtain such a contradiction it suffices to consider only 1-approximations of continuous functions from Michael's example by polynomials. But, 1-approximations aren't sufficient for the continuity of the above constructed mapping F. We omit the direct verification of the fact that the continuity of the multivalued map F is a corollary of the inequality $\| h(p) - g(p) \| < p$, $p \in P$.

Acknowledgements. This research was initiated during the visit by the first author to the Steklov Mathematical Institute in 1993, on the basis of the long term joint research program of the Slovenian Academy of Arts and Sciences and the Russian Academy of Sciences (1991–1995). The authors wish to acknowledge the referee for a very close reading of the manuscript and for providing a detailed list of corrections and remarks, including on how to improve the exposition.

REFERENCES

[1] E. Michael, Continuous selections I, II, *Ann. of Math.*, **63** (1956), 361–382; **64** (1956), 375–390.

[2] E. Michael, Selected selection theorems, *Amer. Math. Monthly*, **63** (1956), 233–238.

[3] E. Michael, Convex structure and continuous selections, *Can. J. Math.*, **11** (1959), 556–575.

[4] D. Repovš and P. V. Semenov, *On function of non-convexity for graphs of continuous functions* (to appear).

[5] P. V. Semenov, Selections of graphvalued maps, *Serdica. Math.*, **18** (1992), 3–9.

Dušan Repovš
University of Trieste
1, Piazzale Europa
34100 Trieste
Italy

e-mail: dusan@ictp.trieste.it

Pavel V. Semenov
Moscow State Pedagogical University
M. Pyrogovskaya 1
119882 Moscow
Russia

e-mail: semenov.matan@mpgu.msk.su