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SHRINKING 1-DEMENSIONAL CELL-LIKE DECOMPOSITIONS OF 3-MANIFOLDS

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We introduce a new shrinking criterion called the *resolution disjoint disks property* and develop a new version of the disjoint disks property for generalized 3-manifolds, called the *light map separation property*. We apply these properties to the study of cell-like upper semicontinuous decompositions of 3-manifolds.

We shall be working in the category of locally compact Hausdorff spaces and continuous maps throughout this paper. Manifolds will be assumed to have no boundary unless otherwise specified. A space X is *cell-like* if there exist an n-manifold N and an embedding $f: X \to N$ such that $f(X)$ is *cellular* in N, i.e., $f(X) = \bigcap_{i=1}^{\infty} B_i^n$, where $\{B_i^n\}_{i\geq 1}$ is a properly nested decreasing sequence of n-cells in N. A map defined on an ANR X is *cell-like* if its point-inverses are cell-like sets. A closed map is *proper* if its point-inverses are compact.

Let G be a decomposition of a space X into compact and connected subsets and let $\pi: X \to X/G$ be the corresponding quotient map, H_G the collection of all *nondegenerate* (i.e., $\neq$ point) elements of G, and N_G their union. A decomposition G is upper semicontinuous if the quotient map $\pi: X \to X/G$ is closed. A decomposition G of a separable metrizable space X is *0-dimensional* if $\dim \pi(N_G) \leq 0$. A compactum $K \subset \mathbb{R}^m$ has *embedding dimension* $\leq n$, $\operatorname{dem} K \leq n$, if for every closed subpolyhedron $L \subset \mathbb{R}^m$ with $\dim L \leq m - n - 1$, there exists an arbitrarily small ambient isotopy of $\mathbb{R}^m$, with support arbitrarily close to $K \cap L$, which moves L off K (see [5] and [15]).

A space X is a *generalized n-manifold* ($n \in \mathbb{N}$) if:

(i) X is a *Euclidean neighborhood retract* (ENR), i.e., for some integer m, X embeds in $\mathbb{R}^m$ as a retract of an open subset of $\mathbb{R}^m$; and

(ii) X is a *homology n-manifold*, i.e., for every $x \in X$, $H_*(X, X - \{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^n; \mathbb{R}^n - \{0\}; \mathbb{Z})$.

In dimensions ≥ 3 X may fail to be locally Euclidean at some (or perhaps all) points. We call such exceptions *singularities* of X, and they form the *singular* set of X, $S(X) = \{x \in X | x$ *does not have a neighborhood in* X *homeomorphic to an open subset of* $\mathbb{R}^n\}$. Note that $S(X)$ is always closed, and if $S(X) \neq X$ then $M(X) = X - S(X)$ is an open n-manifold.

A *resolution* of an n-dimensional ANR X is a pair (M, f) consisting of a topological n-manifold M and a proper, cell-like map $f: M \to X$. Consequently, if X has a resolution then X is a generalized n-manifold [9]. A resolution (M, f) of X is

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called *conservative* if f is *one-to-one over* $M(X)$, i.e., the restriction $f|f^{-1}(M(X)): f^{-1}(M(X)) \to M(X)$ is one-to-one. The *nondegeneracy set* of a map $f: X \to Y$ is the set $N_f = N(f) = \{x \in X | f^{-1}(f(x)) \neq x\}$.

One of the central problems of modern geometric topology of 3-manifolds is how to *detect* topological manifolds among more general topological spaces [1], [3], [12]. Usually, the space X we are dealing with is already known to have the following properties:

(a) X is a 3-dimensional separable metrizable space;

(b) X is an ENR, so it *locally* satisfies the same *homotopy* properties as $\mathbb{R}^3$;

(c) X is a $\mathbb{Z}$-homology 3-manifold, so it *locally* satisfies the same *homology* properties as $\mathbb{R}^3$; and

(d) X is *known* to be a 3-manifold over some open subset $U \subset X$.

Let $Z = X - U$. If $\dim Z \leq 0$ then (modulo the Poincaré conjecture) we can decide whether or not X is a *singular* space by the following test, due to R. C. Lacher and D. Repovš [13]: first, blow up X to get a *conservative* (i.e., one-to-one over U) *cell-like resolution* $f: M \to X$ with M a topological 3-manifold [16], and then verify if the associated cell-like, closed 0-dimensional, upper semicontinuous decomposition $G(f) = \{f^{-1}(x) | x \in X\}$ of M is *shrinkable* by checking whether or not the quotient space $M/G = X$ satisfies the following simple general position property possessed by every topological 3-manifold, called the *map separation property* (MSP) [10]: for every finite collection of maps $f_1, \ldots, f_n: B^2 \to X$ such that for every i, f_i is an embedding on a neighborhood of ∂B^2, and for every $i \neq j$, $f_i(B^2) \cap f_j(\partial B^2) = \varnothing$, and for every open set $V \subset X$ such that $\bigcup_{i=1}^n f_i(B^2) \subset V$, there exist mappings $F_1, \ldots, F_n: B^2 \to V$ with the following properties:

(i) for every i, $F_i|\partial B^2 = f_i|\partial B^2$;

(ii) if $i \neq j$ then $F_i(B^2) \cap F_j(B^2) = \varnothing$.

In this paper we study another general position property of topological 3-manifolds, called the light map separation property (LMSP): A metric space (X, ρ) is said to have the *light map separation property* if for every $\varepsilon > 0$ and every map $f: B \to X$ of a finite collection of disks $B = \coprod_{i=1}^n B_i^2$ into X such that:

(i) $N_f \subset \operatorname{int} B$, where $N_f = \{y \in B | f^{-1}(f(y)) \neq y\}$,

(ii) $\dim N_f \leq 0$, and

(iii) $\dim Z_f \leq 0$, where $Z_f = \{x \in X | x \in f(B_i^2) \cap f(B_j^2)$ for some $i \neq j\}$,

there exists a map $F: B \to X$ such that

(1) $\rho(F, f) < \varepsilon$;

(2) $F|\partial B = f|\partial B$; and

(3) for every $i \neq j$, $F(B_i^2) \cap F(B_j^2) = \varnothing$.

Proposition. *Every topological* 3-*manifold has the light map separation property.*

We prove the following recognition theorem for topological 3-manifolds:

Recognition Theorem. *A space X is a topological* 3-*manifold if and only if X has the following properties:*

(i) *X has the light map separation property;*

(ii) *X admits a* 0-*dimensional resolution $\pi: M \to X$ (so, in particular, X is a generalized* 3-*manifold).*

The key result we need in the proof of the recognition theorem is the following shrinking criterion:

A cell-like resolution $\pi: M \to X$ of a generalized 3-manifold X is said to have the *resolution disjoint disks property* (RDDP) if for every positive integer k, every

collection of pairwise disjoint tame embeddings $f_i: B^2 \to M$, $1 \leq i \leq k$, and every $\varepsilon > 0$, there exist maps $g_i: B^2 \to X$ satisfying
(i) $\rho(g_i, \pi f_i) < \varepsilon$, and
(ii) for every $i \neq j$, $g_i(B^2) \cap g_j(B^2) = \varnothing$.

Shrinking Criterion. *Let G be a cell-like upper semicontinuous decomposition of a topological 3-manifold M, such that* $\dim N_G \leq 1$. *Then G is shrinkable if and only if the quotient map $\pi: M \to M/G$ has the resolution disjoint disks property.*

Corollary. *Let X be a generalized 3-manifold with a resolution $\pi: M \to X$ such that* $\dim N_\pi \leq 1$. *Then X is a topological 3-manifold if and only if π has the resolution disjoint disks property.*

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