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ON THE FAILURE OF THE URYSOHN-MENGER SUM FORMULA FOR COHOMOLOGICAL DIMENSION

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ABSTRACT. We prove that the classical Urysohn-Menger sum formula, $\dim(A \cup B) \leq \dim A + \dim B + 1$, which is also known to be true for cohomological dimension over the integers (and some other abelian groups), does not hold for cohomological dimension over an arbitrary abelian group of coefficients. In particular, we prove that there exist subsets A, $B \subset \mathbb{R}^4$ such that $4 = \dim_{\mathbb{Q}/\mathbb{Z}}(A \cup B) > \dim_{\mathbb{Q}/\mathbb{Z}} A + \dim_{\mathbb{Q}/\mathbb{Z}} B + 1 = 3$.

1. INTRODUCTION

One of the key properties of covering dimension is that the Urysohn-Menger sum formula holds: Let A, $B \subset X$ be arbitrary subsets of a separable metric space X. Then

$$\dim(A \cup B) \leq \dim A + \dim B + 1. \tag{1}$$

It was shown by Rubin [13] that this is also true for cohomological dimension with integer coefficients, $G = \mathbb{Z}$

$$\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1. \tag{2}$$

Recently, Dydak and Walsh [9] have proved the formula (2) for the case when G is either the integers modulo a prime p, $\mathbb{Z}_p$, or the integers localized at a subset l of primes, $\mathbb{Z}_l$, *provided* that $\dim_G A \geq 2$ and $\dim_G B \geq 2$.

The purpose of the present paper is to exhibit an example which shows that formula (2) does *not* generalize to cohomological dimension over *arbitrary* abelian groups G. Theorem 1.1 also solves a long-standing problem in cohomological dimension theory (cf. [11, Problem 10]).

Theorem 1.1. *There exist subsets A, $B \subset \mathbb{R}^4$ such that:*

(i) $\dim_{\mathbb{Q}/\mathbb{Z}} A = 1$;

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(ii) $\dim_{\mathbb{Q}/\mathbb{Z}} B = 1$; *and*
(iii) $\mathbb{R}^4 = A \cup B$.

Remark 1.1. Theorem 1.1 remains valid if one replaces the group $\mathbb{Q}/\mathbb{Z}$ with the group $\mathbb{Z}_{p^\infty}$, p any prime, where $\mathbb{Z}_{p^\infty}$ is the p-adic circle, $\mathbb{Z}_{p^\infty} = \mathbb{Q}/\mathbb{Z}_{(p)}$, and $\mathbb{Z}_{(p)}$ is the p-localization of the integers, $\mathbb{Z}_{(p)} = \{\frac{m}{n} | n$ is not divisible by $p\}$ (cf. [6]).

Our example from Theorem 1.1 shows that the equation

$$\dim_G(X \cup Y) - \dim_G X - \dim_G Y = k \tag{3}$$

has a solution for $G = \mathbb{Q}/\mathbb{Z}$ and $k = 2$. On the other hand, note that Dydak [8] has shown that $k \le 3$, *provided* that $\dim_G X \ge 2$ and $\dim_G Y \ge 2$.

2. Preliminaries

A subset Y of a space S is said to be *negligible* with respect to some compactum X (*X-negligible*) (cf. [6]), if mappings of X into S are approximable by mappings whose images miss Y. Compacta X and Y are said to be of the same *dimension type*, $\mathrm{DIM}\, X = \dim Y$, if $\dim_G X = \dim_G Y$, for every abelian group G. Note that this is equivalent to the requirement that $\dim(X \times Z) = \dim(Y \times Z)$, for every compactum Z (cf. [6]).

Proposition 2.1. *Let $X \subset \mathbb{R}^n$ be any compactum. Then there exists a subset $X^\sigma \subset \mathbb{R}^n$ such that*:

(i) *X^σ is σ-compact*;
(ii) $\mathrm{DIM}\, X^\sigma = \mathrm{DIM}\, X$; *and*
(iii) *for every compactum $Y \subset \mathbb{R}^n \backslash X^\sigma$*, $\dim(X \times Y) < n$.

Proof. Consider a sequence $\{\tau_k\}_{k\in\mathbb{N}}$ of triangulations of $\mathbb{R}^n$ such that $\lim_{k\to\infty} \operatorname{mesh} \tau_k = 0$, and let $L = \{l : \mathbb{R}^n \to \mathbb{R}^n | l$ is simplicial with respect to some subdivision of τ_k and has compact support$\}$. Set $X^\sigma = L(X)$, where $L(X) = \bigcup\{l(X) | l \in L\}$. Clearly, X^σ is σ-compact, so assertion (i) is verified. Next, since l is a PL map, it follows by [6] that, for every compactum $C \subset \mathbb{R}^n$, $\mathrm{DIM}\, l(C) = \mathrm{DIM}\, C$. Therefore, $\mathrm{DIM}\, X^\sigma = \mathrm{DIM}\, X$, by the Countable Sum Theorem, so (ii) also holds.

It remains to verify (iii). Let $Y \subset \mathbb{R}^n \backslash X^\sigma$ be any compactum. Then Y is X-negligible in $\mathbb{R}^n$, so by [4], $\dim(X \times Y) < n$. □

We conjecture that the inequality $\mathrm{DIM}(Y) \le \mathrm{DIM}(\mathbb{R}^n \backslash X^\delta)$ holds for every compactum Y such that $\dim(X \times Y) < n$ (cf. [6, Theorem 1.14]).

Proposition 2.2. *Let $X^\sigma \subset \mathbb{R}^n$ be a σ-compact subset of $\mathbb{R}^n$ such that, for every prime p, $\dim_{\mathbb{Z}_p} X^\sigma = 1$. Then there exists a G_δ-set $X^\delta \subset \mathbb{R}^n$ such that $X^\sigma \subset X^\delta$ and $\dim_{\mathbb{Q}/\mathbb{Z}} X^\delta = \dim_{\mathbb{Z}_p} X^\delta = 1$, for all primes p.*

For the proof of Proposition 2.2 we shall require two lemmas.

Lemma 2.1. *Suppose that X is a σ-compact metric space such that, for every prime p, $\dim_{\mathbb{Z}_p} X = 1$. Then there exists a metrizable compactification cX of X such that $\dim_{\mathbb{Z}_p} cX = 1$, for all primes p.*

Proof. This is a result of Švedov (cf. [11, Chapter VII]). □

Lemma 2.2. *Let S and T be complete metric spaces and $h \colon A \to B$ any homeomorphism between subsets $A \subset S$ and $B \subset T$. Then there exist G_δ-sets $\widetilde{A} \subset S$*

and $\widetilde{B} \subset T$ *such that* $A \subset \widetilde{A}$, $B \subset \widetilde{B}$, *and* h *extends to a homeomorphism* $\tilde{h}: \widetilde{A} \subset \widetilde{B}$.

Proof. This is a classical result of Lavrent′ev [12] (cf. [10, p. 335]). □

Proof of Proposition 2.2. By Lemma 2.1, there is a metrizable compactification cX^σ of X^σ such that $\dim_{\mathbb{Z}_p} cX^\sigma = 1$, for all primes p. Hence, by the Bockstein inequalities [1] (cf. [11, 6]), $\dim_{\mathbb{Q}/\mathbb{Z}} cX^\sigma = 1$. Consider the canonical inclusion $i: X^\sigma \hookrightarrow cX^\sigma$.

Apply Lemma 2.2 to get G_δ-sets $\widetilde{A} \subset \mathbb{R}^n$ and $\widetilde{B} \subset cX^\sigma$, such that $X^\sigma \subset \widetilde{A}$ and $i(X^\sigma) \subset \widetilde{B}$, and an extension of i to a homeomorphism $\tilde{i}: \widetilde{A} \to \widetilde{B}$. Then $X^\delta = \widetilde{A}$ is the required G_δ-set from the assertion. Indeed, since $\dim_{\mathbb{Q}/\mathbb{Z}} cX^\sigma = 1$, we have that $\dim_{\mathbb{Q}/\mathbb{Z}} \widetilde{B} = 1$, so $\dim_{\mathbb{Q}/\mathbb{Z}} X^\delta = 1$. □

3. Proof of Theorem 1.1

By [2], there exists a compactum X such that $\dim_{\mathbb{Z}_{(p)}} X = \dim X = 2$ and $\dim_{\mathbb{Z}_p} X = \dim_{\mathbb{Q}} X = 1$, for all primes p. Therefore, by [11], $\dim(X \times X) = 3$, so it follows by [5] (cf. also [14]) that X embeds in $\mathbb{R}^4$. Also, by the Bockstein inequalities [1] (cf. [11, 6]), $\dim_{\mathbb{Z}_{p^\infty}} X = 1$, for all primes p, so $\dim_{\mathbb{Q}/\mathbb{Z}} X = 1$, by the Bockstein theorem [11].

Let X^σ be as in Proposition 2.1. Let $A = X^\delta$, where X^δ is the completion of X^σ, guaranteed by Proposition 2.2, with $\dim_{\mathbb{Z}_p} X^\delta = 1$, for all primes p, and $\dim_{\mathbb{Q}/\mathbb{Z}} X^\delta = 1$. Define $B = \mathbb{R}^4 \backslash X^\delta$. Thus B is of F_σ-type. In order to complete the proof it suffices, by the Countable Sum Theorem [11], to verify the following.

Assertion. *For every compactum* $C \subset B$, $\dim_{\mathbb{Z}_{p^\infty}} C \le 1$, *for all primes* p.

Proof. Suppose that $\dim_{\mathbb{Z}_{p^\infty}} C \ge 2$ were true for some prime p. Since $\dim C \le 2$ (if $\dim C \ge 3$, then $\dim(C \times D) > \dim C$ for every non-zero-dimensional compactum D), it follows that

$$\dim_{\mathbb{Z}_{(p)}} C = \dim_{\mathbb{Z}_{(p)}} C = \dim_{\mathbb{Z}_{p^\infty}} C;$$

that is, C is p-regular (cf. [6]), and hence

$$\dim_{\mathbb{Z}_{(p)}} (C \times X) = \dim_{\mathbb{Z}_{(p)}} C + \dim_{\mathbb{Z}_{(p)}} X = 2 + 2.$$

By assertion (iii) of Proposition 2.1 it follows that $\dim(C \times X) < 4$, which yields a contradiction. This proves $\dim_{\mathbb{Z}_{p^\infty}} C \le 1$, for all primes p. □

Thus, by Bockstein's theorem [11], $\dim_{\mathbb{Q}/\mathbb{Z}} C \le 1$, and by the Countable Sum Theorem, $\dim_{\mathbb{Q}/\mathbb{Z}} B \le 1$. □

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