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Group Gradings on Finite Dimensional Lie Algebras*

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Abstract. We study gradings by non-commutative groups on finite dimensional Lie algebras over an algebraically closed field of characteristic zero. It is shown that if L is graded by a non-abelian finite group G, then the solvable radical R of L is G-graded and there exists a Levi subalgebra $B = H_1 \oplus \cdots \oplus H_m$ homogeneous in G-grading with graded simple summands $H_1, \ldots, H_m$. All $\operatorname{Supp} H_i$ $(i = 1 \ldots, m)$ are commutative subsets of G.

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1 Introduction

Graded rings and graded algebras have been extensively studied during the last decades (cf. e.g. [2–11]). Group gradings were investigated both in the associative case ([5–8, 16]) and in the Lie case ([3, 11, 12, 15, 17]), or in other non-associative cases ([2, 3, 9]).

One of the important tasks is the description of all possible gradings on different algebras. For example, one of the well-known and actively used results in Lie theory is the description of $\mathbb{Z}$-gradings on finite dimensional complex Lie algebras (cf. [12]).

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The description of all $\mathbb{Z}_2$-gradings on matrix algebras plays an exceptional role in the theory of algebras with polynomial identities (cf. [10]).

All abelian gradings on matrix algebras were described in [6]. It was found that the so-called fine gradings on full matrix algebras play an exceptional role in the theory of orthogonal Cartan decompositions of simple complex Lie algebras (cf. [14]). For the non-commutative case, all group gradings on matrices were described in [5]. In particular, in the general case, fine gradings are closely connected with some problems of the theory of projective representations of finite groups.

All finite dimenional graded simple associative algebras were described in [7]. This description is very intensively applied in the PI-theory (cf. e.g. [1], [13]). All finite dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded simple Lie algebras were described in [2] and this result was applied for the classification of simple color Lie superalgebras. So the description of all possible gradings on algebras plays an important role both in the structure theory of finite dimensional and infinite dimensional graded algebras and in its applications.

In this paper we study general properties of commutative and non-commutative gradings on finite dimensional Lie algebras over an algebraically closed field of characteristic zero. First, we show that the description of non-commutative gradings on semisimple algebras can be reduced to abelian gradings and to the classification of graded simple algebras (cf. Propositions 2.3 and 3.1). Then we characterize finite dimensional graded simple algebras and show that all of them can split into four classical series A, B, C, D and five exceptional series, according to the classification of simple Lie algebras (cf. Proposition 3.1 and Remark 3.2). Finally, we show that in the case of finite groups, any graded Lie algebra is a split extension of a homogeneous Levi subalgebra and a solvable radical (cf. Theorem 3.4).

2 General Properties of Non-commutative Gradings

Let G be a group. Given a Lie algebra L over a field F, we say that L is G-graded if it can be decomposed into a direct sum of subspaces $L = \bigoplus_{g\in G} L_g$ such that $[L_g, L_h] \subseteq L_{gh}$ for any $g, h \in G$. The subspaces L_g are called homogeneous components, and an element $x \in L$ is called homogeneous if $x \in L_g$ for some $g \in G$. In this case, we write $\deg x = g$. A subspace (resp., subalgebra, ideal) V is said to be a graded subspace (resp., subalgebra, ideal) if $V = \bigoplus_{g\in G}(V \cap L_g)$. In other words, if $x = x_{g_1} + \cdots + x_{g_n}$, where $\deg x_{g_i} = g_i$ and $g_1, \dots, g_n \in G$ are pairwise distinct, then $x \in V$ if and only if all $x_{g_1}, \dots, x_{g_n}$ belong to V.

The support of the grading is defined as $\operatorname{Supp} L = \{g \in G \,|\, L_g \neq 0\}$. Actually, we can suppose that $\operatorname{Supp} L$ generates G. An algebra L is called graded simple if $[L, L] \neq 0$ and L does not contain any non-trivial graded ideals.

We usually use the notation $[x_1, \dots, x_n]$ for the left-normed product in a Lie algebra. That is, $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$ for all $n \geq 3$. Similarly, we write $[W_1, \dots, W_n] = [[W_1, \dots, W_{n-1}], W_n]$ for any subspaces $W_1, \dots, W_n \subseteq L$.

It is well-known that in the case of a simple Lie algebra $L = \bigoplus_{g\in G} L_g$, the group G must be commutative (cf. e.g. [15]). First, we generalize this property to the case of graded simple Lie algebras. In the following two lemmas and Proposition 2.3, G is an arbitrary group and L is a (not necessarily finite dimensional) Lie algebra over an arbitrary field F.

Lemma 2.1. *Let $L = \bigoplus_{g\in G} L_g$ be a G-graded Lie algebra and let $[L_{g_1}, \ldots, L_{g_m}] \neq 0$ for some $g_1, \ldots, g_m \in G$. Then $g_1, \ldots, g_m$ commute in G.*

Proof. First, note that an inequality $[L_g, L_h] \neq 0$ implies $gh = hg$ since $[L_g, L_h] = [L_h, L_g] \subseteq L_{gh} \cap L_{hg}$. Hence, our statement is obvious for $m = 2$. Apply induction on m. Suppose $[x_1, \ldots, x_m] \neq 0$ for some $x_1 \in L_{g_1}, \ldots, x_m \in L_{g_m}$ and $m \geq 3$. Since $[x_1, \ldots, x_m] = [u, x_{m-1}] + [v, [x_{m-1}, x_m]]$, where $u = [x_1, \ldots, x_{m-2}, x_m]$ and $v = [x_1, \ldots, x_{m-2}]$ (or $v = x_1$ in case $m = 3$), either $[u, x_{m-1}] \neq 0$ or $[v, [x_{m-1}, x_m]] \neq 0$.

In the first case, $g_1, \ldots, g_{m-2}, g_m$ commute by the inductive hypothesis and g_{m-1} commutes with the product $g_1 \cdots g_{m-2} g_m$. Since $g_1, \ldots, g_{m-1}$ also commute, by induction we have $g_m g_{m-1} = g_{m-1} g_m$, that is, g_m commutes with all $g_1, \ldots, g_{m-1}$.

In the second case, $g_1, \ldots, g_{m-1}$ commute and the product $g_{m-1} g_m$ commutes with all g_i $(1 \leq i \leq m-2)$. Since $g_i g_{m-1} = g_{m-1} g_i$ for $1 \leq i \leq m-2$, we obtain $g_i g_m = g_m g_i$ for all $i = 1, \ldots, m-2$. Clearly, $g_m g_{m-1} = g_{m-1} g_m$ and we have thus completed the proof. □

Lemma 2.2. *Let $L = \bigoplus_{g\in G} L_g$ be a G-graded Lie algebra and let $gh \neq hg$ for some $g, h \in \operatorname{Supp} L$. Then $I = [\operatorname{Id}(L_g), \operatorname{Id}(L_h)] = 0$, where $\operatorname{Id}(L_g)$ is the ideal of L generated by L_g.*

Proof. Any element of I is a linear combination of some $[x, y_1, \ldots, y_k, [z, t_1, \ldots, t_m]]$, where $x \in L_g$, $z \in L_h$ and $y_1, \ldots, y_k, t_1, \ldots, t_m$ are homogeneous elements from L. Now the statement easily follows by Lemma 2.1. □

As an immediate consequence, we get the following:

Proposition 2.3. *Let $L = \bigoplus_{g\in G} L_g$ be a G-graded simple Lie algebra. Then $\operatorname{Supp} L$ generates an abelian subgroup of G.*

In particular, if L is simple in the non-graded sense, then $\operatorname{Supp} L$ is a commutative subset of G. Note that in general, this property does not hold for semisimple algebras. For example, let $L = B_1 \oplus B_2$ be the direct sum of two simple algebras isomorphic to $sl_2(F)$. Given a group G of order 2, $G = \{e, g\}$, we can define a G-grading on $H = sl_2(F)$ by setting

$$L_e = \operatorname{Span}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}, \quad L_g = \operatorname{Span}\left\{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$$

It is now sufficient to take any group G with $gh \neq hg$ $(g, h \in G)$, $g^2 = h^2 = e$, and define a G-grading on B_i using H_i for $i = 1, 2$. Later we shall show that all non-commutative gradings on semisimple Lie algebras are of similar type.

3 Structure of Finite Dimensional Graded Algebras

Let G be a finite abelian group and F an algebraically closed field of characteristic zero. Recall a well-known duality between gradings and automorphism actions on L (cf. e.g. [6]). Let $\widehat{G}$ be the dual group for G, that is the group of all irreducible characters on G. Since G is finite abelian, the group $\widehat{G}$ is isomorphic to G. If

$L = \bigoplus_{g\in G} L_g$ is a G-graded algebra, then any $\chi \in \widehat{G}$ acts on L by the automorphism $\chi * x_g = \chi(g)x_g$, where $x_g \in L_g$ is a homogeneous element of degree g.

Clearly, all subspaces L_g are stable under the $\widehat{G}$-action. Moreover, a subspace $V \subseteq L$ is graded if and only if V is $\widehat{G}$-stable. Conversely, if one defines the $\widehat{G}$-action on L by automorphisms, then L can be decomposed into a direct sum $L = \bigoplus_{g\in G} L_g$, where $L_g = \{v \in L \mid \chi * v = \chi(g)v \text{ for all } \chi \in \widehat{G}\}$, and this is a G-grading on L.

If G is an infinite cyclic group generated by t, then we can also define an action of the infinite cyclic group generated by χ on $L = \bigoplus_{g\in G} L_g$ by setting $\chi^n * v = \lambda^{nk}v$ as soon as $v \in L_{t^k}$, where $\lambda \in F^*$ is a fixed element of infinite order. As before, a subspace V of L is graded if and only if $\chi * V = V$. If L is finite dimensional and $\operatorname{Supp} L$ generates G, then G is a finitely generated abelian group and we can again identify the G-grading of L with the G-action on L.

Using this duality, we get the following result:

Proposition 3.1. *Let $L = \bigoplus_{g\in G}$ be a finite dimensional G-graded Lie algebra over an algebraically closed field of characteristic zero.*

(i) *If L is graded simple, then G is abelian and $L = B_1 \oplus \cdots \oplus B_n$ is semisimple with isomorphic simple components $B_1, \dots, B_n$.*

(ii) *If L is semisimple, then $L = A_1 \oplus \cdots \oplus A_m$ is a direct sum of graded simple components and* $\operatorname{Supp} A_i$ *is a commutative subset of G for any $1 \le i \le m$.*

Proof. First let L be graded simple. Then G is abelian by Proposition 2.3. Suppose that L is not semisimple. Due to the duality between G-gradings and G-actions, its solvable radical R is a graded ideal since R is stable under the action of any automorphism of L. Hence, $R = L$ and $L^2 = [R, R] \neq L$ is also a graded ideal, and $L^2 \neq 0$ by the definition of graded simplicity, a contradiction. Hence, L is semisimple. Consider the decomposition $L = B_1 \oplus \cdots \oplus B_n$ into a direct sum of simple ideals.

Let $f : L \to L$ be an automorphism of L. Clearly, $f(B_1) = B_j$ for some $1 \le j \le n$. Due to the duality between G-gradings and G-actions, the orbit of B_1 under the G-action contains all summands $B_1, \dots, B_n$ since L is graded simple. In particular, all $B_1, \dots, B_n$ are isomorphic, and we have proved assertion (i) of the proposition.

Now let L be a semisimple G-graded algebra. As before, $L = B_1 \oplus \cdots \oplus B_n$ is a direct sum of minimal ideals. Consider a minimal graded ideal A_1 of L. If $A_1 = L$, then there is nothing to prove. If $A_1 \neq L$, then A_1 is a sum of some B_i, say $A_1 = B_1 \oplus \cdots \oplus B_t$. We shall prove that $B_{t+1} \oplus \cdots \oplus B_n$ is also a graded ideal of L.

First note that the centralizer of any homogeneous element in L is a graded subalgebra of L. Indeed, let $[a_h, b_{g_1} + \cdots + b_{g_m}] = 0$ for some $a_h \in L_h$, $b_{g_1} \in L_{g_1}, \dots, b_{g_m} \in L_{g_m}$ with pairwise distinct $g_1, \dots, g_m \in G$. If $[a_h, b_{g_i}] = 0$ for all $i = 1, \dots, m$, then we are done. Otherwise, fix all $g_{i_1}, \dots, g_{i_k}$ such that $c_1 = [a_h, b_{g_{i_1}}] \neq 0, \dots, c_k = [a_h, b_{g_{i_k}}] \neq 0$. Then h commutes with $g_{i_1}, \dots, g_{i_k}$ by Lemma 2.1, $c_1 \in L_{hg_{i_1}}, \dots, c_k \in L_{hg_{i_k}}$ and all $hg_{i_1}, \dots, hg_{i_k}$ are distinct. On the other hand, $c_1 + \cdots + c_k = 0$, a contradiction. Hence, the equality $[a_h, b_{g_1} + \cdots + b_{g_m}] = 0$ implies that all $b_{g_1}, \dots, b_{g_m}$ commute with a_h.

Now we observe that the intersection of two graded subalgebras is also a graded subalgebra. In particular, the centralizer of A_1 is a graded subspace in L. However, the centralizer of A_1 is $B_{t+1} \oplus \cdots \oplus B_n$ and we complete the proof of the proposition by induction on the dimension of L. □

Remark 3.2. It follows by the previous proposition that any finite dimensional graded simple algebra is associated with one of the finite dimensional simple Lie algebras. In particular, we can say that the graded simple algebra is an algebra of the type A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, G_2, F_4, E_6, E_7, or E_8.

Now we clarify the structure of a finite dimensional Lie algebra graded by a finite group.

Proposition 3.3. *Let G be a finite group and let $L = \bigoplus_{g\in G} L_g$ be a finite dimensional G-graded Lie algebra over an algebraically closed field of characteristic zero. Then the radical $R = \operatorname{Rad} L$ of L is a graded ideal and there exists a split extension $L = B + R$, where B is a maximal semisimple subalgebra of L homogeneous in G-grading.*

Proof. If G is an abelian group, as mentioned in the proof of Proposition 3.1, the solvable radical of L is graded.

Now let G be a non-commutative group. Consider a decomposition $L = B + R$, where B is a maximal semisimple subalgebra of L, $B = B_1 \oplus \cdots \oplus B_n$ and $B_1, \dots, B_n$ are minimal ideals of B. By Proposition 2.3, the algebra L cannot be graded simple. Consider a maximal proper graded ideal P of L.

First suppose $B \subseteq P$. Then $\dim L/P = 1$. Denote $R_0 = \operatorname{Rad} P$. Then $R_0 \subset R$ and $\dim R/R_0 = 1$. Moreover, $R^2 \subseteq R_0$ and $[B, R_0] \subseteq R_0$. In particular, R_0 is an ideal of L. Since $\dim P < \dim L$, we may suppose by induction that R_0 is a graded ideal of P and L. If $R_0 \neq 0$, then also by induction, R/R_0 is a graded ideal of L/R_0, hence R is a graded ideal of L.

In case $R_0 = 0$, we have $\dim R = 1$ and hence it is a trivial B-module. Since $[R, R] = 0$, we conclude that R is the center of L. By Lemma 2.1, the center of a graded algebra is homogeneous and so R is a graded ideal in this case.

Let $B \not\subseteq P$. Then L/P is graded simple and by Proposition 3.1, it is semisimple, i.e., $R \subseteq P$. Applying induction on the dimension, we again conclude that R is a homogeneous subspace of P and hence of L.

Now we prove the existence of a Levi subalgebra homogeneous in G-grading. In case G is abelian, we apply the duality between G-gradings and G-actions by automorphisms on L. By [18] there exists a maximal semisimple subalgebra B stable under the $\widehat{G}$-action. So B is a graded Levi subalgebra of L.

In the general case, we consider the graded factor-algebra L/R. By Proposition 3.1, we have $L/R = \bar{A}_1 \oplus \cdots \oplus \bar{A}_m$, where each $\bar{A}_i$ is graded simple and $S_i = \operatorname{Supp} \bar{A}_i$ is a commutative subset of G. Denote by A_i the full preimage of $\bar{A}_i$ over R. Then A_i is graded and $\operatorname{Supp} A_i \supset S_i$. Take a subalgebra C_i of A_i generated by all homogeneous $x \in A_i$ with $\deg x \in S_i$. Since S_i is commutative, by the previous remark, C_i contains a homogeneous semisimple subalgebra B_i isomorphic to $\bar{A}_i$. Note that $\dim(B_1 + \cdots + B_m) = \dim L/R$, hence $B = B_1 \oplus \cdots \oplus B_m$ is a homogeneous Levi subalgebra and we have completed the proof. □

Combining Propositions 2.3, 3.1 and 3.3, we immediately obtain the following:

Theorem 3.4. *Let $L = \bigoplus_{g\in G} L_g$ be a finite dimensional Lie algebra over an algebraically closed field of characteristic zero graded by a finite group G. Then its solvable radical is homogeneous in G-grading and there exists a Levi subalgebra B homogeneous in G-grading. Moreover, B is a direct sum $B = H_1 \oplus \cdots \oplus H_m$, where each H_i is a graded simple subalgebra with commutative support $\operatorname{Supp} H_i$, and in the non-graded sense, H_i is a direct sum of isomorphic simple components.*

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