Anisotropic Singular Neumann Equations with Unbalanced Growth

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Abstract

We consider a nonlinear parametric Neumann problem driven by the anisotropic (p, q)-Laplacian and a reaction which exhibits the combined effects of a singular term and of a parametric superlinear perturbation. We are looking for positive solutions. Using a combination of topological and variational tools together with suitable truncation and comparison techniques, we prove a bifurcation-type result describing the set of positive solutions as the positive parameter λ varies. We also show the existence of minimal positive solutions u_λ^* and determine the monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$.

Keywords Modular function · Truncation · Comparison principle · Minimal solution · Anisotropic regularity

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1 Introduction

This paper was motivated by several recent contributions to the qualitative analysis of non-linear problems with unbalanced growth. We mainly refer to the pioneering contributions of Marcellini [22–24] who studied lower semicontinuity and regularity properties of minimizers of certain quasiconvex integrals. Problems of this type arise in nonlinear elasticity and are connected with the deformation of an elastic body, cf. Ball [4, 5].

We are concerned with the qualitative analysis of a class of anisotropic singular problems with Neumann boundary condition and driven by a differential operator with unbalanced growth. The features of this paper are the following:

(i)   the problem studied in the present work is associated to a *double phase energy* with *variable exponents* (variational integral with anisotropic unbalanced growth);
(ii) the reaction is both *singular* and *anisotropic*;
(iii) we assume a *Neumann boundary condition*.

To the best of our knowledge, this is the first paper dealing with the combined effects generated by the above features.

1.1 Unbalanced Problems and Their Historical Traces

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with a smooth boundary. If $u : \Omega \to \mathbb{R}^N$ is the displacement and Du is the $N \times N$ matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$I(u) = \int_\Omega f(z, Du(z))dz, \tag{1}$$

where the energy function $f = f(z, \xi) : \Omega \times \mathbb{R}^{N \times N} \to \mathbb{R}$ is quasiconvex with respect to ξ, see Morrey [26]. One of the simplest examples considered by Ball is given by functions f of the type

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $N \times N$ matrix ξ, and g, h are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geq c_1 |\xi|^p; \quad \lim_{t \to +\infty} h(t) = +\infty,$$

where c_1 is a positive constant and $1 < p < N$. The condition $p \leq N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the integral (1) that are discontinuous at one point where a cavity forms; in fact, every u with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, and thus it is a continuous function if $p > N$. In accordance with these problems arising in nonlinear elasticity, Marcellini [22, 23] considered continuous functions $f = f(x, u)$ with *unbalanced growth* that satisfy

$$c_1 |u|^p \leq |f(z, u)| \leq c_2 (1 + |u|^q) \quad \text{for all } (x, u) \in \Omega \times \mathbb{R},$$

where c_1, c_2 are positive constants and $1 \leq p \leq q$. Regularity and existence of solutions of elliptic equations with p, q–growth conditions were studied in [23].

The study of non-autonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point has been continued in a series of remarkable papers by Mingione et al. [6, 7, 12]. These contributions are in relationship with the work of Zhikov [43, 44], which describe the behavior of phenomena arising in nonlinear elasticity. In fact, Zhikov intended to provide models for strongly

anisotropic materials in the context of homogenisation. In particular, he considered the following model functional

$$\mathcal{P}_{p,q}(u) := \int_\Omega (|Du|^p + a(z)|Du|^q)dz, \quad 0 \le a(x) \le L, \ 1 < p < q, \tag{2}$$

where the modulating coefficient $a(x)$ dictates the geometry of the composite made of two differential materials, with hardening exponents p and q, respectively.

In the present paper we are concerned with a problem whose energy is of the type defined in Eq. 2 but such that the exponents p and q are variable (they depend on the point).

1.2 Statement of the Problem

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric singular anisotropic (p, q)-equation:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)u(z)^{p(z)-1} = u(z)^{-\eta(z)} + \lambda f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u(z) > 0 \text{ for all } z \in \Omega, \ \lambda > 0. \end{cases} \tag{P_λ}$$

In this problem, we make the following hypotheses for the exponents $p(\cdot)$, $q(\cdot)$, $\eta(\cdot)$:

$$p, \ q, \ \eta \in C^1(\overline{\Omega}), \ q_- \le q_+ < p_- \le p_+, \ 0 < \eta(z) < 1 \text{ for all } z \in \overline{\Omega},$$

where for every $r \in C(\overline{\Omega})$ we define

$$r_- := \min_{\overline{\Omega}} r, \ r_+ := \max_{\overline{\Omega}} r.$$

Also for $r \in C(\overline{\Omega})$ with $1 < r(z) < \infty$ for all $z \in \overline{\Omega}$, we denote by $\Delta_{r(z)}$ the $r(z)$-Laplace differential operator defined by

$$\Delta_{r(z)}u = \text{div}\,(|Du|^{r(z)-2}Du) \text{ for all } u \in W^{1,r(z)}(\Omega).$$

The potential function $\xi \in L^\infty(\Omega)$ satisfies $\xi(z) > 0$ for a.a. $z \in \Omega$. In the reaction we have two terms. One is the singular term $x \mapsto x^{-\eta(z)}$ with $0 < \eta(z) < 1$ for all $z \in \overline{\Omega}$ and the other is a parametric perturbation $\lambda f(z, x)$ with $\lambda > 0$ being the parameter. The function $f(z, x)$ is a Carathéodory function, that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z, x)$ is continuous. We assume that for a.a. $z \in \Omega$, the function $f(z, \cdot)$ exhibits a $(p_+ - 1)$-superlinear growth near $+\infty$ with $p_+ = \max_{\overline{\Omega}} p$, but without satisfying the so-called Ambrosetti-Rabinowitz condition (the AR-condition for short), which is common in the literature when dealing with superlinear problems. Instead, we use a less restrictive condition which incorporates in our framework superlinear nonlinearities with slower growth near $+\infty$. The precise hypotheses on $f(z, x)$ can be found in Section 2 (see hypotheses H_1).

We are looking for positive solutions and our aim is to determine how the set of positive solutions changes as the parameter $\lambda > 0$ varies. In this direction we prove a bifurcation-type result describing the changes in the set of positive solutions of (P_λ) as the positive parameter λ increases. We also show that if $\lambda > 0$ is admissible (that is, problem (P_λ) admits positive solutions), then there is a minimal positive solution u_λ^* (that is, a smallest solution) and we examine the monotonicity and continuity of the map $\lambda \mapsto u_\lambda^*$.

Analogous studies for p-Laplacian equations with constant exponent, were conducted by Giacomoni et al. [18] and Papageorgiou and Winkert [28]. More general equations driven by nonhomogeneous differential operators, were considered recently by Papageorgiou et

al. [33–36], Papageorgiou and Scapellato [27], Papageorgiou et al. [31], Papageorgiou and Zhang [30], and Ragusa and Tachikawa [38]. We should also mention the very recent works of De Filippis and Mingione [13] and Marcellini [25], on the regularity of solutions of double phase problems. This is a very interesting area with several issues remaining open and requiring further investigation. Finally, we mention the work of Bahrouni et al. [3] on a class of double phase problems with convection. Singular anisotropic equations driven by the $p(z)$-Laplacian, were studied by Byun and Ko [9], Zhang and Rădulescu [42], and Saudi and Ghanmi [40]. To the best of our knowledge, there are no works on singular anisotropic (p, q)-equations.

Boundary value problems driven by a combination of differential operators (such as (p, q)-equations) arise in many mathematical models of physical processes. We mention the historically first such work of Cahn and Hilliard [10], which deals with the process of separation of binary alloys and the more recent works of Benci et al. [8] on quantum physics, of Cherfils and Ilyasov [11] on reaction diffusion systems and of Bahrouni et al. [1, 2] on transonic flow problems. Boundary value problems involving differential operators with variable exponents, are studied in the book of Rădulescu and Repovš [39], while a comprehensive discussion of semilinear singular problems and a rich relevant bibliography can be found in the book of Ghergu and Rădulescu [17].

2 Mathematical Background and Hypotheses

Although as we already mentioned in the previous section, we require that our exponents $p(\cdot)$, $q(\cdot)$, $\eta(\cdot)$ are smooth (in order to exploit the existing anisotropic regularity theory), the introduction of the variable exponent spaces does not require such regularity restrictions.

We introduce the following spaces

$$M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\}$$
$$L_1^\infty(\Omega) = \{p \in L^\infty(\Omega) : 1 \le \operatorname*{essinf}_\Omega p\}.$$

As usual, we identify in $M(\Omega)$ two functions which differ only on a set of measure zero. If $p \in L_1^\infty(\Omega)$, then we set

$$p_- = \operatorname*{essinf}_\Omega p \text{ and } p_+ = \operatorname*{esssup}_\Omega p.$$

Given $p \in L_1^\infty(\Omega)$, the variable exponent Lebesgue space $L^{p(z)}(\Omega)$ is defined by

$$L^{p(z)}(\Omega) = \left\{u \in M(\Omega) : \int_\Omega |u|^{p(z)} dz < \infty\right\}.$$

We equip this space with the so-called "Luxemburg norm" defined by

$$\|u\|_{p(z)} = \inf \left\{\lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda}\right)^{p(z)} dz \le 1\right\}.$$

Variable exponent Lebesgue spaces are similar to the classical Lebesgue spaces. More precisely, they are separable Banach spaces, they are reflexive if and only if $1 < p_- \le p_+ < \infty$ (in fact, they are uniformly convex). Moreover, simple functions and continuous functions of compact support are dense in $L^{p(z)}(\Omega)$.

Suppose that $p, q \in L_1^\infty(\Omega)$. Then we have the following property:

$$\text{``}L^{p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \text{ continuously}$$

if and only if

$$q(z) \leq p(z) \text{ for a.a. } z \in \Omega".$$

Let $p, p' \in L_1^\infty(\Omega)$ such that $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ for a.a. $z \in \Omega$. Then $L^{p(z)}(\Omega)^* = L^{p'(z)}(\Omega)$ and the following Hölder-type inequality is true

$$\int_\Omega |uv|dz \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(z)} \|v\|_{p'(z)}$$

for all $u \in L^{p(z)}(\Omega)$ and all $v \in L^{p'(z)}(\Omega)$.

Using the variable exponent Lebesgue spaces, we can define in the usual way variable exponent Sobolev spaces. So, if $p \in L_1^\infty(\Omega)$, then we set

$$W^{1,p(z)}(\Omega) = \{u \in L^{p(z)}(\Omega) : |Du| \in L^{p(z)}(\Omega)\}.$$

This space is furnished with the following norm

$$\|u\| = \|u\|_{p(z)} + \||Du|\|_{p(z)}.$$

Evidently, an equivalent norm is given by

$$|u| = \inf\left\{\lambda > 0 : \int_\Omega \left[\left(\frac{|Du|}{\lambda}\right)^{p(z)} + \left(\frac{|u|}{\lambda}\right)^{p(z)}\right]dz \leq 1\right\}.$$

The anisotropic Sobolev space $W^{1,p(z)}(\Omega)$ is a separable Banach space and if $1 < p_- \leq p_+ < \infty$, then $W^{1,p(z)}(\Omega)$ is reflexive (in fact, uniformly convex). Note that $W^{1,p(z)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega)$ continuously. Also, $W_0^{1,p(z)}(\Omega)$ is the closure of the set of $W^{1,p(z)}(\Omega)$-functions with compact support, that is, of the set

$$\left\{u \in W^{1,p(z)}(\Omega) : u = u\chi_K \text{ with } K \subseteq \Omega \text{ compact}\right\}.$$

If $p \in C^1(\overline{\Omega})$, then $W_0^{1,p(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|}$.

We set

$$p^*(z) = \begin{cases} \frac{Np(z)}{N-p(z)}, & \text{if } p(z) < N \\ +\infty, & \text{if } N \leq p(z). \end{cases}$$

Suppose that $p, q \in L_1^\infty(\Omega) \cap C(\overline{\Omega})$ with $p_+ < N$ and $q(z) \leq p^*(z)$ (resp., $q(z) < p^*(z)$) for all $z \in \overline{\Omega}$, then we have that $W^{1,p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega)$ continuously (resp., compactly). A comprehensive presentation of variable exponent Lebesgue and Sobolov spaces can be found in the book of Diening et al. [14].

Let $r \in L_1^\infty(\Omega)$ and consider the Lebesgue space $L^{r(z)}(\Omega)$. The modular function for this space is given by

$$\rho_{r(z)}(u) = \int_\Omega |u|^{r(z)}dz.$$

This function is basic in the study of $L^{r(z)}(\Omega)$ and is closely related to the norm $\|\cdot\|_{p(z)}$ introduced above. More specifically, we have the following result.

Proposition 2.1 (a)   *For $u \in L^{r(z)}(\Omega)$, $u \neq 0$, we have*

$$\|u\|_{r(z)} \leq \lambda \Leftrightarrow \rho_{r(z)}\left(\frac{u}{\lambda}\right) \leq 1;$$

(b) $\|u\|_{r(z)} < 1$ *(resp. $= 1$, > 1)* $\Leftrightarrow \rho_{r(z)}(u) < 1$ *(resp. $= 1$, > 1);*

(c) $\|u\|_{r(z)} < 1 \Rightarrow \|u\|_{r(z)}^{r_+} \leq \rho_{r(z)}(u) \leq \|u\|_{r(z)}^{r_-}$,
    $\|u\|_{r(z)} > 1 \Rightarrow \|u\|_{r(z)}^{r_-} \leq \rho_{r(z)}(u) \leq \|u\|_{r(z)}^{r_+}$;

(d) $\|u_n\|_{r(z)} \to 0 \Leftrightarrow \rho_{r(z)}(u_n) \to 0$;

(e) $\|u_n\|_{r(z)} \to +\infty \Leftrightarrow \rho_{r(z)}(u_n) \to +\infty$.

We consider the map $A_{r(z)} : W^{1,r(z)}(\Omega) \to W^{1,r(z)}(\Omega)^*$ defined by

$$\langle A_{r(z)}(u), h \rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,r(z)}(\Omega).$$

This map has the following properties (see Gasinski and Papageorgiou [16, Proposition 2.5]).

Proposition 2.2 *The map* $A_{r(z)} : W^{1,p(z)}(\Omega) \to W^{1,p(z)}(\Omega)^*$ *is bounded (that is, maps bounded sets to bounded sets), continuous, monotone, hence also maximal monotone and of type* $(S)_+$, *that is,*

"$u_n \xrightarrow{w} u$ *in* $W^{1,r(z)}(\Omega)$, $\limsup_{n\to\infty} \langle A_{r(z)}(u_n), u_n - u \rangle \leq 0 \Rightarrow u_n \to u$ *in* $W^{1,r(z)}(\Omega)$."

In addition to the variable exponent spaces, we will use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has nonempty interior given by

$$\text{int } C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will also use another open cone in $C^1(\overline{\Omega})$ given by

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega \cap u^{-1}(0)} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Combining the proofs of Proposition 2.5 of [37] and of Proposition 6 in [36] we have the following strong comparison principle.

Proposition 2.3 *If* $p, q, \eta \in C^1(\overline{\Omega})$, $1 < q_- \leq q_+ < p_- \leq p_+$, $0 < \eta(z) < 1$ *for all* $z \in \overline{\Omega}$, $\hat{\xi}, h, g \in L^\infty(\Omega)$, $\hat{\xi}(z) \geq 0$ *for a.a.* $z \in \Omega$, $0 < \mu \leq g(z) - h(z)$ *for a.a.* $z \in \Omega$ *and* $u, v \in C^1(\overline{\Omega})$ *satisfy* $0 \leq u \leq v$ *and*

$$-\Delta_{p(z)}u - \Delta_{q(z)}u + \hat{\xi}(z)u^{p(z)-1} - u^{-\eta(z)} = h(z) \text{ in } \Omega,$$
$$-\Delta_{p(z)}v - \Delta_{q(z)}v + \hat{\xi}(z)v^{p(z)-1} - v^{-\eta(z)} = g(z) \text{ in } \Omega,$$

then $v - u \in D_+$.

If $u, v \in W^{1,p(z)}(\Omega)$ ($p \in L_1^\infty(\Omega)$) with $u \leq v$, then we define

$$[u, v] = \left\{h \in W^{1,p(z)}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\right\}$$

and

$$[u) = \left\{h \in W^{1,p(z)}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega\right\}.$$

If X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then we denote by K_φ the critical set of φ, that is, the set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Also, we say that $\varphi \in C^1(X, \mathbb{R})$ satisfies the "C-condition", if the following property holds:

"Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

and $(1+\|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence".

This is a compactness-type condition on the functional $\varphi(\cdot)$. It compensates for the fact that X is not locally compact, being in general infinite dimensional. The C-condition plays a crucial role in the minimax theory of the critical values of the functional $\varphi(\cdot)$.

Now we are ready to introduce the hypotheses on the data of (P_λ).

H_0: $p, q, \eta \in C^1(\overline{\Omega})$, $1 < q_- \leq q_+ < p_- \leq p_+$, $0 < \eta(z) < 1$ for all $z \in \overline{\Omega}$, $\xi \in L^\infty(\Omega)$, $\xi(z) > 0$ for a.a. $z \in \Omega$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, $f(z, x)$ is measurable in $z \in \Omega$ and continuous in $x \in \mathbb{R}$) such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \leq f(z, x) \leq a(z)[1 + x^{r(z)-1}]$ for a.a. $z \in \Omega$, all $x \geq 0$ with $a \in L^\infty(\Omega)$ and $r \in C(\overline{\Omega})$ with $p_+ < r_- \leq r_+ < p^*(z)$ for all $z \in \overline{\Omega}$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x \to +\infty} \dfrac{F(z, x)}{x^{p_+}} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) if $\xi_\lambda(z, x) = \left(1 - \dfrac{p_+}{1-\eta(z)}\right)x^{1-\eta(z)} + \lambda\left[f(z, x)x - p_+ F(z, x)\right]$, then there exists $\hat{\vartheta}_\lambda \in L^1(\Omega)$ such that

$$\xi_\lambda(z, x) \leq \xi_\lambda(z, y) + \hat{\vartheta}_\lambda(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq y;$$

(iv) for every $s > 0$, we can find $\mu_s > 0$ such that

$$0 < \hat{\mu}_s \leq f(z, x) \text{ for a.a. } z \in \Omega, \text{ all } s \leq x$$

and $0 < \hat{c} \leq \liminf_{x \to 0^+} \dfrac{f(z, x)}{x^{q_+ - 1}}$ uniformly for a.a. $z \in \Omega$;

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z, x) + \hat{\xi}_\rho x^{p(z)-1}$$

is nondecreasing on $[0, \rho]$.

Remarks Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, \infty)$, without any loss of generality, we may assume that

$$f(z, x) = 0 \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0. \tag{3}$$

On account of hypotheses $H_1(ii), (iii)$ we see that for a.a. $z \in \Omega$, $f(z, \cdot)$ is $(p_+ - 1)$-superlinear. However the superlinearity property of $f(z, \cdot)$ is not expressed in terms of AR-condition. We recall that in the present anisotropic setting the AR-condition says that there exist $\vartheta > p_+$ and $M > 0$ such that

$$0 < \vartheta F(z, x) \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } x \geq M, \tag{4}$$

$$0 < \operatorname*{essinf}_\Omega F(\cdot, M). \tag{5}$$

In fact this is a unilateral version of the AR-condition due to Eq. 3. Integrating Eq. 4 and using Eq. 5, we obtain the weaker condition

$$C_0 x^\vartheta \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M, \text{ some } C_0 > 0,$$
$$\Rightarrow C_0 x^{\vartheta - 1} \leq f(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M.$$

So, the AR-condition implies that $f(z, \cdot)$ eventually has $(\vartheta - 1)$-polynomial growth. Here we replace the AR-condition by the quasimonotonicity hypothesis $H_1(iii)$. This hypothesis is a slightly more general version of a condition used by Li and Yang [20]. Note that there exists $M > 0$ such that for a.a. $z \in \Omega$

$$x \mapsto \frac{x^{-\eta(z)} + \lambda f(z, x)}{x^{p_+ - 1}} \text{ is nondecreasing on } x \geq M$$

or $x \mapsto d_\lambda(z, x)$ is nondecreasing on $x \geq M$.

Examples Consider the following two functions

$$f_1(z, x) = (x^+)^{r(z) - 1} \text{ and } f_2(z, x) = (x^+)^{p(z) - 1} \ln(1 + x^+).$$

Both functions satisfy hypothesis H_1, but only f_1 satisfies the AR-condition.

By $L^{p(z)}(\Omega, \xi)$ we will denote the weighted $L^{p(z)}$-space with weight $\xi(\cdot)$. Therefore

$$L^{p(z)}(\Omega, \xi) = \left\{ u \in M(\Omega) : \int_\Omega \xi(z) |u|^{p(z)} dz < \infty \right\}.$$

This space is furnished with the norm

$$\|u\|_{L^{p(z)}(\Omega, \xi)} = \inf \left\{ \lambda > 0 : \int_\Omega \xi(z) \left| \frac{u}{\lambda} \right|^{p(z)} dz \leq 1 \right\}.$$

Note that since by hypothesis $\xi(z) > 0$ for a.a. $z \in \Omega$ (see hypothesis H_0), the function

$$\rho_{p(z), \xi}(u) = \int_\Omega \xi(z) |u|^{p(z)} dz,$$

is a modular function (see Diening et al. [14, Definition 2.1.1, p. 20]).

On $W^{1, p(z)}(\Omega)$ we consider the norm $\| \cdot \|$ defined earlier and a new norm given by

$$|u| = \|Du\|_{p(z)} + \|u\|_{L^{p(z)}(\Omega, \xi)}.$$

Proposition 2.4 *If hypothesis H_0 holds, then $\| \cdot \|$ and $| \cdot |$ are equivalent norms on $W^{1, p(z)}(\Omega)$.*

Proof From the definitions of the two norms, we have

$$|u| \leq C_1 \|u\| \text{ for some } C_1 > 0, \text{ all } u \in W^{1, p(z)}(\Omega).$$

Claim. There exists $C_2 > 0$ such that $\|u\|_{p(z)} \leq C_2 |u|$ for all $u \in W^{1, p(z)}(\Omega)$.

We argue indirectly. So, suppose that the claim is not true. Then we can find $\{u_n\}_{n \geq 1} \subseteq W^{1, p(z)}(\Omega)$ such that

$$\|u_n\|_{p(z)} > n|u_n| \text{ for all } n \in \mathbb{N}.$$

Normalizing in $L^{p(z)}(\Omega)$, we see that we have

$$|u_n| < \frac{1}{n} \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow |u_n| \to 0,$$
$$\Rightarrow \|Du_n\|_{p(z)} \to 0 \text{ and } \|u_n\|_{L^{p(z)}(\Omega, \xi)} \to 0 \text{ as } n \to \infty. \tag{6}$$

Evidently $\{u_n\}_{n \geq 1} \subseteq W^{1, p(z)}(\Omega)$ is bounded. So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1, p(z)}(\Omega) \text{ and } u_n \to u \text{ in } L^{p(z)}(\Omega),$$
$$\Rightarrow \|u\|_{p(z)} = 1. \tag{7}$$

From Eqs. 6 and 7, we have $u = \tilde{c} \in \mathbb{R} \setminus \{0\}$ and

$$\int_{\Omega} \xi(z)|u_n|^{p(z)}dz \to \int_{\Omega} \xi(z)|\tilde{c}|^{p(z)}dz.$$

On account of Eq. 6, we have

- If $0 < |\tilde{c}| \leq 1$, then $0 < |\tilde{c}|^{p_+} \int_{\Omega} \xi(z)dz \leq 0$, a contradiction.
- If $1 < |\tilde{c}|$, then $0 \leq |\tilde{c}|^{p_-} \int_{\Omega} \xi(z)dz \leq 0$, a contradiction.

This proves the Claim.

Using the Claim and the definitions of the two norms, we conclude that $\|\cdot\|$ and $|\cdot|$ are equivalent. $\qquad\square$

In what follows, we denote by $\gamma_{p(z)} : W^{1,p(z)}(\Omega) \to \mathbb{R}$ the C^1-functional defined by

$$\gamma_{p(z)}(u) = \int_{\Omega} \frac{1}{p(z)}|Du|^{p(z)}dz + \int_{\Omega} \frac{\xi(z)}{p(z)}|u|^{p(z)}dz \text{ for every } u \in W^{1,p(z)}(\Omega).$$

For every $\lambda > 0$ the energy functional $\varphi_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ for problem (P_λ), is given by

$$\varphi_\lambda(u) = \gamma_{p(z)}(u) + \int_{\Omega} \frac{1}{q(z)}|Du|^{q(z)}dz - \int_{\Omega} \frac{1}{1-\eta(z)}(u^+)^{1-\eta(z)}dz - \int_{\Omega} F(z, u^+)dz$$

for all $u \in W^{1,p(z)}(\Omega)$.

On account of the third term, this functional is not C^1 and so the minimax theorems from the critical point theory are not directly applicable to this functional. For this reason we use truncation techniques in order to bypass the singularity and have C^1-functionals on which the critical point theory applies. For this reason in the next section, we deal with a purely singular problem.

Finally, we mention that, as usual, by a solution of (P_λ), we understand a function $u \in W^{1,p(z)}(\Omega)$ such that

$$u \geq 0, u \not\equiv 0, u^{-\eta(\cdot)}h \in L^1(\Omega) \text{ for all } h \in W^{1,p(z)}(\Omega)$$

and

$$\langle A_{p(z)}(u), h \rangle + \langle A_{q(z)}(u), h \rangle + \int_{\Omega} \xi(z)u^{p(z)-1}hdz$$

$$= \int_{\Omega} \left[u^{-\eta(z)} + \lambda f(z, u) \right] hdz \text{ for all } h \in W^{1,p(z)}(\Omega).$$

3 A Purely Singular Problem

In this section we deal with the following purely singular problem

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)u(z)^{p(z)-1} = u(z)^{-\eta(z)} \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u > 0. \end{cases} \tag{8}$$

To solve Eq. 8, we first consider a perturbation of Eq. 8 which removes the singularity. So, we consider the following approximation of problem Eq. 8:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)u(z)^{p(z)-1} = [u(z) + \varepsilon]^{-\eta(z)} \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u > 0. \end{cases} \tag{8_ε}$$

We solve this problem using a topological approach (fixed point theory). So, given $g \in L^{p(z)}(\Omega)$, $g \geq 0$ and $\varepsilon \in (0, 1]$, we consider the following problem:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)u(z)^{p(z)-1} = [g(z) + \varepsilon]^{-\eta(z)} \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u > 0. \end{cases} \tag{9}$$

For this problem we have the following result.

Proposition 3.1 *If hypotheses H_0 hold, problem* Eq. 9 *admits a unique solution $\tilde{u}_\varepsilon \in$ int C_+.*

Proof Let $K_{p(z)} : L^{p(z)}(\Omega) \to L^{p'(z)}(\Omega)$ be the map defined by

$$K_{p(z)}(u) = |u|^{p(z)-2}u \text{ for all } u \in L^{p(z)}(\Omega).$$

Evidently this map is bounded, continuous, strictly monotone. Then we consider the operator $V : W^{1,p(z)}(\Omega) \to W^{1,p(z)}(\Omega)^*$ defined by

$$V(u) = A_{p(z)}(u) + A_{q(z)}(u) + \xi(z)K_{p(z)}(u) \text{ for all } u \in W^{1,p(z)}(\Omega).$$

This operator is bounded, continuous, strictly monotone (thus, maximal monotone too). Also, if $u \in W^{1,p(z)}(\Omega)$, we have

$$\begin{aligned} \langle V(u), u \rangle &= \rho_{p(z)}(Du) + \rho_{q(z)}(Du) + \int_\Omega \xi(z)|u|^{p(z)}dz \\ &\geq \rho_{p(z)}(Du) + \int_\Omega \xi(z)|u|^{p(z)}dz \\ &\geq \|Du\|_{p(z)} + \|u\|_{L^{p(z)}(\Omega,\xi)} - 1 \\ &\quad \left(\text{see Corollary 2.1.15 of [14, p. 25] and recall that } \rho_{p(z),\xi}(\cdot) \text{ is modular}\right) \\ &\geq C_2\|u\| \text{ for some } C_2 > 0 \text{ (see Proposition 2.4),} \\ &\Rightarrow V(\cdot) \text{ is coercive.} \end{aligned}$$

We know that a maximal monotone coercive operator is surjective (see Papageorgiou et al. [32, Corollary 2.8.7, p. 135]). Since $[g(\cdot) + \varepsilon]^{-\eta(\cdot)} \in L^\infty(\Omega)$, we can find $\tilde{u}_\varepsilon \in W^{1,p(z)}(\Omega)$ such that

$$V(\tilde{u}_\varepsilon) = [g(\cdot) + \varepsilon]^{-\eta(\cdot)}.$$

Moreover, the strict monotonicity of $V(\cdot)$ implies that this solution is unique. Proposition 3.1 of Gasinski and Papageorgiou [16] implies that $\tilde{u}_\varepsilon \in L^\infty(\Omega)$. Then by Theorem 1.3 of Fan [15] (see also Lieberman [21]), we have that $\tilde{u}_\varepsilon \in C_+ \setminus \{0\}$. Finally the anisotropic maximum principle of Zhang [41] implies that $\tilde{u}_\varepsilon \in$ int C_+. □

Using Proposition 3.1 we can define the solution map $L_\varepsilon : L^{p(z)}(\Omega) \to L^{p(z)}(\Omega)$ for problem Eq. 9 by

$$L_\varepsilon(g) = \tilde{u}_\varepsilon. \tag{10}$$

Clearly, a fixed point of this map will be a solution for problem (8_ε).

Proposition 3.2 *If hypotheses H_0 hold, then problem (8_ε) admits a unique solution $\overline{u}_\varepsilon \in$ int C_+.*

Proof From Proposition 3.1, we have

$$\langle A_{p(z)}(\tilde{u}_\varepsilon), h\rangle + \langle A_{q(z)}(\tilde{u}_\varepsilon), h\rangle + \int_\Omega \xi(z)\tilde{u}_\varepsilon^{p(z)-1}hdz = \int_\Omega [g(z)+\varepsilon]^{-\eta(z)}hdz \quad (11)$$

for all $h \in W^{1,p(z)}(\Omega)$.

In Eq. 11 we choose $h = \tilde{u}_\varepsilon = L_\varepsilon(g) \in W^{1,p(z)}(\Omega)$ (see Eq. 10) and we obtain

$$\rho_{p(z)}(D\tilde{u}_\varepsilon) + \rho_{q(z)}(D\tilde{u}_\varepsilon) + \int_\Omega \xi(z)\tilde{u}_\varepsilon^{p(z)}dz = \int_\Omega \frac{\tilde{u}_\varepsilon}{[g(z)+\varepsilon]^{\eta(z)}}dz,$$

$$\Rightarrow \begin{cases} \|L_\varepsilon(g)\|^{p_+} \le C_3\|L_\varepsilon(g)\| \text{ if } \|L_\varepsilon(g)\| \le 1 \\ \|L_\varepsilon(g)\|^{p_-} \le C_3\|L_\varepsilon(g)\| \text{ if } \|L_\varepsilon(g)\| > 1 \text{ for some } C_3 = C_3(\varepsilon) > 0, \text{ all } g \in L^\infty(\Omega) \end{cases}$$

(see Proposition 2.4). $\quad (12)$

Next, we show that $L_\varepsilon(\cdot)$ is continuous. To this end let $g_n \to g$ in $L^{p(z)}(\Omega)$. From Eq. 12 we have that

$$\{L_\varepsilon(g_n) = \tilde{u}_{\varepsilon_n} = \tilde{u}_n\}_{n\ge 1} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\tilde{u}_n \xrightarrow{w} \tilde{u} \text{ in } W^{1,p(z)}(\Omega) \text{ and } \tilde{u}_n \to \tilde{u} \text{ in } L^{p(z)}(\Omega). \quad (13)$$

We have

$$\langle A_{p(z)}(\tilde{u}_n), h\rangle + \langle A_{q(z)}(\tilde{u}_n), h\rangle + \int_\Omega \xi(z)(\tilde{u}_n)^{p(z)-1}hdz = \int_\Omega \frac{h}{[g_n+\varepsilon]^{\eta(z)}}dz \quad (14)$$

for all $h \in W^{1,p(z)}(\Omega)$, all $n \in \mathbb{N}$.

In Eq. 14 we choose $h = \tilde{u}_n - \tilde{u} \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use Eq. 13. Then we have

$$\lim_{n\to\infty}\left[\langle A_{p(z)}(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle + \langle A_{q(z)}(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle\right] = 0,$$

$$\Rightarrow \limsup_{n\to\infty}\left[\langle A_{p(z)}(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle + \langle A_{q(z)}(\tilde{u}), \tilde{u}_n - \tilde{u}\rangle\right] \le 0$$

$$\text{(since } A_{q(z)}(\cdot) \text{ is monotone)},$$

$$\Rightarrow \limsup_{n\to\infty}\langle A_{p(z)}(\tilde{u}_n), \tilde{u}_n - \tilde{u}\rangle \le 0 \text{ (see Eq. 13)},$$

$$\Rightarrow \tilde{u}_n \to \tilde{u} \text{ in } W^{1,p(z)}(\Omega) \text{ (see Proposition 2.2).} \quad (15)$$

If in Eq. 14 we pass to the limit as $n \to \infty$ and use Eq. 15, we obtain that

$$\langle A_{p(z)}(\tilde{u}), h\rangle + \langle A_{q(z)}(\tilde{u}), h\rangle + \int_\Omega \xi(z)\tilde{u}^{p(z)-1}hdz = \int_\Omega \frac{h}{[g+\varepsilon]^{\eta(z)}}dz$$

$$\text{for all } h \in W^{1,p(z)}(\Omega),$$

$$\Rightarrow \tilde{u} = L_\varepsilon(g),$$

$$\Rightarrow L_\varepsilon(\cdot) \text{ is continuous.}$$

The continuity of $L_\varepsilon(\cdot)$ together with Eq. 12 and the compact embedding of $W^{1,p(z)}(\Omega)$ into $L^{p(z)}(\Omega)$, permit the use of Schauder-Tychonov fixed point theorem (see, for example, Papageorgiou and Winkert [29, Theorem 6.8.5, p. 581]) and we find $\overline{u}_\varepsilon \in$ int C_+ such that

$$L_\varepsilon(\overline{u}_\varepsilon) = \overline{u}_\varepsilon,$$

$$\Rightarrow \overline{u}_\varepsilon \in \text{int } C_+ \text{ is a positive solution of Eq. 8.}$$

Next we show the uniqueness of this solution. Suppose that $\hat{u}_\varepsilon \in W^{1,p(z)}(\Omega)$ is another positive solution of Eq. 8. Again we have $\hat{u}_\varepsilon \in \text{int}\, C_+$. Also, we have

$$0 \leq \langle A_{p(z)}(\overline{u}_\varepsilon) - A_{p(z)}(\hat{u}_\varepsilon), (\overline{u}_\varepsilon - \hat{u}_\varepsilon)^+ \rangle + \langle A_{q(z)}(\overline{u}_\varepsilon) - A_{q(z)}(\hat{u}_\varepsilon), (\overline{u}_\varepsilon - \hat{u}_\varepsilon)^+ \rangle$$

$$+ \int_\Omega \xi(z)(\overline{u}_\varepsilon^{p(z)-1} - \hat{u}_\varepsilon^{p(z)-1})(\overline{u}_\varepsilon - \hat{u}_\varepsilon)^+ dz =$$

$$= \int_\Omega \left[\frac{1}{[\overline{u}_\varepsilon + \varepsilon]^{\eta(z)}} - \frac{1}{[\hat{u}_\varepsilon + \varepsilon]^{\eta(z)}} \right] (\overline{u}_\varepsilon - \hat{u}_\varepsilon)^+ dz \leq 0$$

$$\Rightarrow \overline{u}_\varepsilon \leq \hat{u}_\varepsilon.$$

Interchanging the roles of $\overline{u}_\varepsilon$ and $\hat{u}_\varepsilon$ in the above argument, we also have that $\hat{u}_\varepsilon \leq \overline{u}_\varepsilon$, therefore $\overline{u}_\varepsilon = \hat{u}_\varepsilon$. This proves the uniqueness of the positive solution $\overline{u}_\varepsilon \in \text{int}\, C_+$ of problem (8_ε). □

Evidently, to produce a positive solution of Eq. 8, we will let $\varepsilon \to 0^+$.
To this end, the following monotonicity property of the map $\varepsilon \mapsto \overline{u}_\varepsilon$ will be useful.

Proposition 3.3 *If hypotheses H_0 hold, then the map $\varepsilon \mapsto \overline{u}_\varepsilon$ from $(0, 1]$ into C_+ is nonincreasing, that is,*

$$0 < \varepsilon' < \varepsilon \leq 1 \Rightarrow \overline{u}_\varepsilon \leq \overline{u}_{\varepsilon'}.$$

Proof Let $0 < \varepsilon' < \varepsilon \leq 1$ and consider $\overline{u}_{\varepsilon'}, \overline{u}_\varepsilon \in \text{int}\, C_+$ the corresponding unique positive solutions of problems $(8_{\varepsilon'})$ and Eq. 8_ε respectively, established in Proposition 3.2.
We have

$$-\Delta_{p(z)}\overline{u}_{\varepsilon'} - \Delta_{q(z)}\overline{u}_{\varepsilon'} + \xi(z)\overline{u}_{\varepsilon'}^{p(z)-1}$$

$$= [\overline{u}_{\varepsilon'} + \varepsilon']^{-\eta(z)}$$

$$\geq [\overline{u}_{\varepsilon'} + \varepsilon]^{-\eta(z)} \text{ in } \Omega \text{ (since } 0 < \varepsilon' < \varepsilon). \tag{16}$$

We introduce the Carathéodory function $e_\varepsilon(z, x)$ defined by

$$e_\varepsilon(z, x) = \begin{cases} \dfrac{1}{[x^+ + \varepsilon]^{\eta(z)}}, & \text{if } x \leq \overline{u}_{\varepsilon'}(z) \\ \dfrac{1}{[\overline{u}_{\varepsilon'}(z) + \varepsilon]^{\eta(z)}}, & \text{if } \overline{u}_{\varepsilon'}(z) < x. \end{cases} \tag{17}$$

We set $E_\varepsilon(z, x) = \int_0^x e_\varepsilon(z, s)ds$ and consider the C^1-functional $\tau_\varepsilon : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\tau_\varepsilon(u) = \gamma_{p(z)}(u) + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz - \int_\Omega E_\varepsilon(z, u)dz \text{ for all } u \in W^{1,p(z)}(\Omega).$$

We have

$$\tau_\varepsilon(u) \geq \rho_{p(z)}(Du) + \rho_{p(z),\xi}(u) - C_4 \text{ for some } C_4 > 0 \text{ (see Eq. 17)}$$

$$\geq |u| - C_4 - 1,$$

$$\Rightarrow \tau_\varepsilon(\cdot) \text{ is coercive (see Proposition 2.4).}$$

Also, exploiting the compact embedding of $W^{1,p(z)}(\Omega)$ into $L^{p(z)}(\Omega)$, we have that

$$\tau_\varepsilon(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

Invoking the Weierstrass-Tonelli theorem, we can find $\hat{u}_\varepsilon \in W^{1,p(z)}(\Omega)$ such that

$$\tau_\varepsilon(\hat{u}_\varepsilon) = \min\left\{\tau_\varepsilon(u) : u \in W^{1,p(z)}(\Omega)\right\},$$

$$\Rightarrow \tau_\varepsilon'(\hat{u}_\varepsilon) = 0,$$

$$\Rightarrow \langle A_{p(z)}(\hat{u}_\varepsilon), h \rangle + \langle A_{q(z)}(\hat{u}_\varepsilon), h \rangle + \int_\Omega \xi(z)|\hat{u}_\varepsilon|^{p(z)-2}\hat{u}_\varepsilon h\,dz = \int_\Omega e_\varepsilon(z, \hat{u}_\varepsilon)h\,dz \quad (18)$$

for all $h \in W^{1,p(z)}(\Omega)$.

In Eq. 18 we choose $h = -\hat{u}_\varepsilon^- \in W^{1,p(z)}(\Omega)$ and obtain

$$\rho_{p(z)}(D\hat{u}_\varepsilon^-) + \rho_{q(z)}(D\hat{u}_\varepsilon^-) + \rho_{p(z),\xi}(\hat{u}_\varepsilon^-) = 0 \text{ (see Eq. 17)},$$

$$\Rightarrow \rho_{p(z)}(Du_\varepsilon^-) + \rho_{p(z),\xi}(\hat{u}_\varepsilon^-) \le 0$$

$$\Rightarrow \hat{u}_\varepsilon \ge 0, \ \hat{u}_\varepsilon \ne 0 \text{ (see Eq. 18)}.$$

Next, in Eq. 18 we choose $h = [\hat{u}_\varepsilon - \overline{u}_{\varepsilon'}]^+ \in W^{1,p(z)}(\Omega)$. We obtain

$$\langle A_{p(z)}(\hat{u}_\varepsilon), (\hat{u}_\varepsilon - \overline{u}_{\varepsilon'})^+ \rangle + \langle A_{q(z)}(\hat{u}_\varepsilon), (\hat{u}_\varepsilon - \overline{u}_{\varepsilon'})^+ \rangle + \int_\Omega \xi(z)\hat{u}_\varepsilon^{p(z)-1}(\hat{u}_\varepsilon - \overline{u}_{\varepsilon'})^+\,dz$$

$$= \int_\Omega \frac{[\hat{u}_\varepsilon - \overline{u}_{\varepsilon'}]^+}{[\overline{u}_{\varepsilon'} + \varepsilon]^{\eta(z)}}\,dz \text{ (see Eq. 17)}$$

$$\le \langle A_{p(z)}(\overline{u}_{\varepsilon'}), (\hat{u}_\varepsilon - \overline{u}_{\varepsilon'})^+ \rangle + \langle A_{q(z)}(\overline{u}_{\varepsilon'}), (\hat{u}_\varepsilon - \overline{u}_{\varepsilon'})^+ \rangle + \int_\Omega \xi(z)\overline{u}_{\varepsilon'}^{p(z)-1}(\hat{u}_\varepsilon - \overline{u}_{\varepsilon'})^+\,dz,$$

$$\Rightarrow \hat{u}_\varepsilon \le \overline{u}_{\varepsilon'}.$$

So, we have proved that

$$\hat{u}_\varepsilon \in [0, \overline{u}_{\varepsilon'}], \ \hat{u}_\varepsilon \ne 0,$$

$$\Rightarrow \hat{u}_\varepsilon = \overline{u}_\varepsilon \text{ (see Eqs. 18, 17 and Proposition 3.2)},$$

$$\Rightarrow \overline{u}_\varepsilon \le \overline{u}_{\varepsilon'}.$$

The proof is now complete. $\qquad\qquad\square$

Now we will pass to the limit as $\varepsilon \to 0^+$ and produce a solution for problem (8).

Proposition 3.4 *If hypotheses H_0 hold, then problem* Eq. 8 *admits a unique solution* $\overline{u} \in int\,C_+$.

Proof Let $\{\varepsilon_n\}_{n\ge 1} \subseteq (0, 1]$ be such that $\varepsilon_n \to 0^+$ and let $\overline{u}_n = \overline{u}_{\varepsilon_n} \in int\,C_+$ be as in Proposition 3.2. We have

$$\langle A_{p(z)}(\overline{u}_n), h \rangle + \langle A_{q(z)}(\overline{u}_n), h \rangle + \int_\Omega \xi(z)\overline{u}_n^{p(z)-1}h\,dz = \int_\Omega \frac{h}{[\overline{u}_n + \varepsilon_n]^{\eta(z)}}\,dz \quad (19)$$

for all $h \in W^{1,p(z)}(\Omega)$, all $n \in \mathbb{N}$.

We choose $h = \overline{u}_n \in W^{1,p(z)}(\Omega)$. We obtain

$$\rho_{p(z)}(D\overline{u}_n) + \rho_{q(z)}(D\overline{u}_n) + \int_\Omega \xi(z)\overline{u}_n^{p(z)}\,dz \le \int_\Omega \frac{\overline{u}_n}{\overline{u}_1^{\eta(z)}}$$

(since $\overline{u}_1 \le \overline{u}_n$ for all $n \in \mathbb{N}$, see Proposition 3.3)

$$\Rightarrow \{\overline{u}_n\}_{n\ge 1} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded}$$

(see Proposition 2.4 and recall that $p_- > 1$).

Proposition 3.1 of Gasinski and Papageorgiou [16], implies that we can find $C_5 > 0$ such that

$$\|\overline{u}_n\|_\infty \le C_5 \text{ for all } n \in \mathbb{N}.$$

Then the anisotropic regularity theory of Fan [15, Theorem 1.3] implies that we can find $\alpha \in (0, 1)$ and $C_6 > 0$ such that

$$\overline{u}_n \in C^{1,\alpha}(\overline{\Omega}), \ \|\overline{u}_n\|_{C^{1,\alpha}(\overline{\Omega})} \le C_6 \text{ for all } n \in \mathbb{N}.$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and the monotonicity of the sequence $\{\overline{u}_n\}_{n \ge 1}$ (see Proposition 3.3), imply that

$$\overline{u}_n \to \overline{u} \text{ in } C^1(\overline{\Omega}). \tag{20}$$

Since $\overline{u}_1 \le \overline{u}_n$ for all $n \in \mathbb{N}$, we have $\overline{u} \ne 0$ and so $\overline{u} \in \text{int } C_+$. Moreover, passing to the limit as $n \to \infty$ in Eq. 19 and using Eq. 20, we conclude that $\overline{u} \in \text{int } C_+$ is a positive solution of problem Eq. 8.

Finally, we show the uniqueness of this positive solution. So, suppose that $\tilde{u} \in W^{1,p(z)}(\Omega)$ is another positive solution of Eq. 8. As in the proof of Proposition 3.2, using the fact that the map $x \mapsto x^{-\eta(z)}$ is strictly decreasing on $(0, +\infty)$, we obtain

$$\tilde{u} = \overline{u},$$
$$\Rightarrow \overline{u} \in \text{int } C_+ \text{ is the unique positive solution of Eq. 8.}$$

The proof is now complete. $\square$

In the next section we will use $\overline{u} \in \text{int } C_+$ and truncation techniques to bypass the singularity and show that problem (P_λ) has positive solutions for certain values of the parameter $\lambda > 0$.

4 Positive Solutions

We introduce the following two sets

$$\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\},$$
$$S_\lambda = \text{set of positive solutions of problem } (P_\lambda).$$

We start by showing the nonemptiness of $\mathcal{L}$ (=the set of admissible parameter values).

Proposition 4.1 *If hypotheses H_0, H_1 hold, then $\mathcal{L} \ne \emptyset$.*

Proof Let $\overline{u} \in \text{int } C_+$ be the unique positive solution of problem Eq. 8 produced in Proposition 3.4.

We consider the following auxiliary problem:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)u(z)^{p(z)-1} = \overline{u}(z)^{-\eta(z)} + 1 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u > 0. \end{cases} \tag{21}$$

From Proposition 3.1, we know that this problem has a unique positive solution $\hat{u} \in \text{int} \, C_+$. We have

$$\langle A_{p(z)}(\overline{u}), (\overline{u} - \hat{u})^+ \rangle + \langle A_{q(z)}(\overline{u}), (\overline{u} - \hat{u})^+ \rangle + \int_\Omega \xi(z) \overline{u}^{p(z)-1} (\overline{u} - \hat{u})^+ dz$$

$$= \int_\Omega \overline{u}^{-\eta(z)} (\overline{u} - \hat{u})^+ dz \text{ (see Eq. 8)}$$

$$\leq \int_\Omega \left[\overline{u}^{-\eta(z)} + 1 \right] (\overline{u} - \hat{u})^+ dz$$

$$= \langle A_{p(z)}(\hat{u}), (\overline{u} - \hat{u})^+ \rangle + \langle A_{q(z)}(\hat{u}), (\overline{u} - \hat{u})^+ \rangle + \int_\Omega \xi(z) \hat{u}^{p(z)-1} (\overline{u} - \hat{u})^+ dz$$

(since $\hat{u} \in \text{int} \, C_+$ solves Eq. 21),

$$\Rightarrow \overline{u} \leq \hat{u}. \tag{22}$$

Since $\hat{u} \in \text{int} \, C_+$, on account of hypothesis $H_1(i)$, we have

$$0 \leq f(\cdot, \hat{u}(\cdot)) \in L^\infty(\Omega).$$

So, we can find $\lambda_0 > 0$ such that

$$0 \leq \lambda f(z, \hat{u}(z)) \leq 1 \text{ for a.a. } z \in \Omega, \text{ all } \lambda \in (0, \lambda_0]. \tag{23}$$

We introduce the Carathéodory function $\beta_\lambda(z, x)$ defined by

$$\beta_\lambda(z, x) = \begin{cases} \overline{u}(z)^{-\eta(z)} + \lambda f(z, \overline{u}(z)), & \text{if } x < \overline{u}(z) \\ x^{-\eta(z)} + \lambda f(z, x), & \text{if } \overline{u}(z) \leq x \leq \hat{u}(z) \text{ (see Eq. 22)} \\ \hat{u}(z)^{-\eta(z)} + \lambda f(z, \hat{u}(z)), & \text{if } \hat{u}(z) < x. \end{cases} \tag{24}$$

We set $B_\lambda(z, x) = \int_0^x \beta_\lambda(z, s) ds$ and consider the C^1-functional $\tilde{\Psi}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\Psi}_\lambda(u) = \gamma_{p(z)}(u) + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz - \int_\Omega B_\lambda(z, u) dz \text{ for all } u \in W^{1,p(z)}(\Omega), \text{ all } \lambda \in (0, \lambda_0].$$

From Eq. 22 it is clear that $\tilde{\Psi}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. Hence, we can find $\tilde{u} \in W^{1,p(z)}(\Omega)$ such that

$$\tilde{\Psi}_\lambda(\tilde{u}) = \min \left\{ \tilde{\Psi}_\lambda(u) : u \in W^{1,p(z)}(\Omega) \right\},$$

$$\Rightarrow \tilde{\Psi}'_\lambda(\tilde{u}) = 0,$$

$$\Rightarrow \langle A_{p(z)}(\tilde{u}), h \rangle + \langle A_{q(z)}(\tilde{u}), h \rangle + \int_\Omega \xi(z) |\tilde{u}|^{p(z)-2} \tilde{u} h dz = \int_\Omega \beta_\lambda(z, \tilde{u}) h dz \tag{25}$$

for all $h \in W^{1,p(z)}(\Omega)$.

In Eq. 25 first we choose $h = (\overline{u} - \tilde{u})^+ \in W^{1,p(z)}(\Omega)$. We have

$$
\langle A_{p(z)}(\tilde{u}), (\overline{u} - \tilde{u})^+ \rangle + \langle A_{q(z)}(\tilde{u}), (\overline{u} - \tilde{u})^+ \rangle + \int_\Omega \xi(z)|\tilde{u}|^{p(z)-2}\tilde{u}(\overline{u} - \tilde{u})^+ dz
$$

$$
= \int_\Omega \left[\overline{u}^{-\eta(z)} + \lambda f(z, \overline{u}) \right] (\overline{u} - \tilde{u})^+ dz \text{ (see Eq. 24)}
$$

$$
\geq \int_\Omega \overline{u}^{-\eta(z)}(\overline{u} - \tilde{u})^+ dz \text{ (see hypothesis } H_1(i))
$$

$$
= \langle A_{p(z)}(\overline{u}), (\overline{u} - \tilde{u})^+ \rangle + \langle A_{q(z)}, (\overline{u} - \tilde{u})^+ \rangle + \int_\Omega \xi(z)\overline{u}^{p(z)-1}(\overline{u} - \tilde{u})^+ dz,
$$

$$
\Rightarrow \overline{u} \leq \tilde{u}.
$$

Next, in Eq. 25 we choose $h = (\tilde{u} - \hat{u})^+ \in W^{1,p(z)}(\Omega)$. We have

$$
\langle A_{p(z)}(\tilde{u}), (\tilde{u} - \hat{u})^+ \rangle + \langle A_{q(z)}, (\tilde{u} - \hat{u})^+ \rangle + \int_\Omega \xi(z)\tilde{u}^{p(z)-1}(\tilde{u} - \hat{u})^+ dz
$$

$$
= \int_\Omega \left[\hat{u}^{-\eta(z)} + \lambda f(z, \hat{u}) \right] (\tilde{u} - \hat{u})^+ dz \text{(see Eq. 24)}
$$

$$
\leq \int_\Omega \left[\hat{u}^{-\eta(z)} + 1 \right] (\tilde{u} - \hat{u})^+ dz \text{ (since } \lambda \in (0, \lambda_0], \text{ see Eq. 23)}
$$

$$
\leq \int_\Omega \left[\overline{u}^{-\eta(z)} + 1 \right] (\tilde{u} - \hat{u})^+ dz \text{ (see Eq. 22)}
$$

$$
= \langle A_{p(z)}(\hat{u}), (\tilde{u} - \hat{u})^+ \rangle + \langle A_{q(z)}(\hat{u}), (\tilde{u} - \hat{u})^+ \rangle + \int_\Omega \xi(z)\tilde{u}^{p(z)-1}(\tilde{u} - \hat{u})^+ dz,
$$

$$
\Rightarrow \tilde{u} \leq \hat{u}.
$$

So, we have proved that

$$
\tilde{u} \in [\overline{u}, \hat{u}]. \tag{26}
$$

From Eqs. 26, 24 and 25, we infer that

$$
\tilde{u} \in S_\lambda,
$$
$$
\Rightarrow (0, \lambda_0] \subseteq \mathcal{L} \neq \emptyset,
$$

which concludes the proof.   □

Proposition 4.2 *If hypotheses H_0, H_1 hold and $\lambda \in \mathcal{L}$, then $\overline{u} \leq u$ for all $u \in S_\lambda$.*

Proof Let $u \in S_\lambda$ and consider the following function

$$
\hat{\mu}_+(z, x) = \begin{cases} x^{-\eta(z)}, & \text{if } 0 \leq x \leq u(z) \\ u(z)^{-\eta(z)}, & \text{if } u(z) < x. \end{cases} \tag{27}
$$

Evidently, this function is Carathéodory on $\Omega \times (\mathbb{R} \setminus \{0\})$ and is singular at $x = 0$. We consider the purely singular problem

$$
\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)u(z)^{p(z)-1} = \hat{\mu}_+(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ u > 0. \end{cases} \tag{28}
$$

Reasoning as for problem (8), we show that problem (28) admits a positive solution $u^* \in \text{int } C_+$ (see also Papageorgiou et al. [36, Proposition 10]). We have

$$\langle A_{p(z)}(u^*), (u^* - u)^+ \rangle + \langle A_{q(z)}(u^*), (u^* - u)^+ \rangle + \int_\Omega \xi(z)(u^*)^{p(z)-1}(u^* - u)^+ dz$$

$$= \int_\Omega u^{-\eta(z)}(u^* - u)^+ dz \text{ (see Eq. 27)}$$

(recall that, by definition, $u^{-\eta(z)}h \in L^1(\Omega)$ for all $h \in W^{1,p(z)}(\Omega)$)

$$\leq \int_\Omega \left[u^{-\eta(z)} + \lambda f(z, u) \right] (u^* - u)^+ dz$$

$$= \langle A_{p(z)}(u), (u^* - u)^+ \rangle + \langle A_{q(z)}(u), (u^* - u)^+ \rangle$$

$$+ \int_\Omega \xi(z)u^{p(z)-1}(u^* - u)^+ dz \text{ (since } u \in S_\lambda),$$

$$\Rightarrow u^* \leq u.$$

So, we have

$$u^* \in [0, u], \ u^* \neq 0. \tag{29}$$

From Eqs. 29, 27 and Proposition 3.4, it follows that

$$u^* = \overline{u}$$
$$\Rightarrow \overline{u} \leq u \text{ for all } u \in S_\lambda.$$

The proof is now complete. $\qquad\square$

According to the previous proposition, if $\lambda \in \mathcal{L}$ and $u \in S_\lambda$, then

$$0 \leq u^{-\eta(z)} \leq \overline{u}^{-\eta(z)} \text{ with } \overline{u}(\cdot)^{-\eta(\cdot)} \in L^\infty(\Omega).$$

Then the anisotropic regularity theory (see Fan [15]) and the anisotropic maximum principle (see Zhang [41]), imply the following result concerning the solution set S_λ.

Proposition 4.3 *If hypotheses H_0, H_1 hold and $\lambda \in \mathcal{L}$, then $S_\lambda \subseteq \text{int } C_+$.*

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 4.4 *If hypotheses H_0, H_1 hold, then $\lambda^* < \infty$.*

Proof On account of hypotheses $H_1(ii)$, (iii), (iv), we can find $\lambda_0 > 0$ big such that

$$\lambda_0 f(z, x) \geq \xi(z)x^{p(z)-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{30}$$

Let $\lambda > \lambda_0$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \text{int } C_+$ (see Proposition 4.3). Let $m_\lambda = \min_{\overline{\Omega}} u_\lambda > 0$ (since $u_\lambda \in \text{int } C_+$). Let $\delta \in (0, 1]$, $\rho =$

$\max\{\|u_\lambda\|_\infty, m_\lambda + 1\}$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(v)$. We set $m_\lambda^\delta = m_\lambda + \delta$

$$-\Delta_{p(z)}m_\lambda^\delta - \Delta_{q(z)}m_\lambda^\delta + [\xi(z) + \hat{\xi}_\rho](m_\lambda^\delta)^{p(z)-1} - (m_\lambda^\delta)^{-\eta(z)}$$

$$\leq [\xi(z) + \hat{\xi}_\rho]m_\lambda^{p(z)-1} + \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+$$

$$\leq \lambda_0 f(z, m_\lambda) + \hat{\xi}_\rho m_\lambda^{p(z)-1} + \chi(\delta) \text{ (see Eq. 30)}$$

$$= \lambda f(z, m_\lambda) + \hat{\xi}_\rho m_\lambda^{p(z)-1} - [\lambda - \lambda_0]f(z, m_\lambda) + \chi(\delta)$$

$$\leq \lambda f(z, m_\lambda) + \hat{\xi}_\rho m_\lambda^{p(z)-1} - [\lambda - \lambda_0]\hat{\mu}_\lambda + \chi(\delta)$$

$$\text{with } \hat{\mu}_\lambda > 0 \text{ (see hypothesis } H_1(iv))$$

$$\leq \lambda f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} \text{ for } \delta > 0 \text{ small (see hypothesis } H_1(v))$$

$$= -\Delta_{p(z)}u_\lambda - \Delta_{q(z)}u_\lambda + [\xi(z) + \hat{\xi}_\rho]u_\lambda^{p(z)-1} - u_\lambda^{-\eta(z)}(\text{ since } u_\lambda \in S_\lambda),$$

$$\Rightarrow u_\lambda - m_\lambda^\delta \in D_+ \text{ for } \delta > 0 \text{ small (see Proposition 2.3),}$$

a contradiction to the definition of $m_\lambda > 0$.

So, we have $\lambda^* \leq \lambda_0 < \infty$.   □

Next, we show that $\mathcal{L}$ is, in fact, an interval.

Proposition 4.5 *If hypotheses H_0, H_1 hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \subseteq \text{int } C_+$ (see Proposition 4.3). Then we have

$$-\Delta_{p(z)}u_\lambda - \Delta_{q(z)}u_\lambda + \xi(z)u_\lambda^{p(z)-1}$$

$$= u_\lambda^{-\eta(z)} + \lambda f(z, u_\lambda)$$

$$\geq u_\lambda^{-\eta(z)} + \mu f(z, u_\lambda) \text{ in } \Omega \tag{31}$$

$$\text{(recall that } 0 < \mu < \lambda \text{ and see hypothesis } H_1(i)).$$

Also we have

$$-\Delta_{p(z)}\overline{u} - \Delta_{q(z)}\overline{u} + \xi(z)\overline{u}^{p(z)-1}$$

$$= \overline{u}^{-\eta(z)}$$

$$\leq \overline{u}^{-\eta(z)} + \mu f(z, \overline{u}) \text{ in } \Omega \text{ (see hypothesis } H_1(i)). \tag{32}$$

From Proposition 4.2 we know that $\overline{u} \leq u_\lambda$. So, we can define the following truncation of the reaction of problem (P_μ)

$$\hat{\tau}_\mu(z, x) = \begin{cases} \overline{u}^{-\eta(z)} + \mu f(z, \overline{u}(z)), & \text{if } x < \overline{u}(z) \\ x^{-\eta(z)} + \mu f(z, x), & \text{if } \overline{u}(z) \leq x \leq u_\lambda(z) \\ u_\lambda(z)^{-\eta(z)} + \mu f(z, u_\lambda(z)), & \text{if } u_\lambda(z) < x. \end{cases} \tag{33}$$

This is a Carathéodory function. We set $\hat{T}_\mu(z, x) = \int_0^x \hat{\tau}_\mu(z, s)ds$ and consider the C^1-functional $\hat{w}_\mu : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{w}_\mu(u) = \gamma_{p(z)}(u) + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz - \int_\Omega \hat{T}_\mu(z, u)dz \text{ for all } u \in W^{1,p(z)}(\Omega).$$

As before (see the proof of Proposition 4.1), using the direct method of the calculus of variations and Eqs. 31, 32, we can find $u_\mu \in W^{1,p(z)}(\Omega)$ such that

$$u_\mu \in K_{w_\lambda} \subseteq [\overline{u}, u_\lambda] \cap \text{int } C_+,$$
$$\Rightarrow u_\mu \in S_\mu \text{ (see Eq. 33)},$$
$$\Rightarrow \mu \in \mathcal{L}.$$

The proof is now complete. $\qquad\square$

Remark 4.6 As a byproduct of this proof we have that if $0 < \mu < \lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda \subseteq$ int C_+ then we can find $u_\mu \in S_\mu \subseteq$ int C_+ such that

$$u_\lambda - u_\mu \in C_+ \setminus \{0\}.$$

In fact we can improve this result using the strong comparison principle (see Proposition 2.3).

Proposition 4.7 *If hypotheses H_0, H_1 hold, $0 < \mu < \lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda \subseteq$ int C_+, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq$ int C_+ such that $u_\lambda - u_\mu \in D_+$.*

Proof From Proposition 4.5 and its proof, we already know that $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq$ int C_+ such that

$$u_\mu \leq u_\lambda, \ u_\mu \neq u_\lambda. \tag{34}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$-\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + [\xi(z) + \hat{\xi}_\rho] u_\mu^{p(z)-1} - u_\mu^{-\eta(z)}$$
$$= \mu f(z, u_\mu) + \hat{\xi}_\rho u_\mu^{p(z)-1}$$
$$= \lambda f(z, u_\mu) + \hat{\xi}_\rho u_\mu^{p(z)-1} - (\lambda - \mu) f(z, u_\mu)$$
$$\leq \lambda f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} \text{ (see Eq. 34 and hypothesis } H_1(v))$$
$$= -\Delta_{p(z)} u_\lambda - \Delta_{q(z)} u_\lambda + [\xi(z) + \hat{\xi}_\rho] u_\lambda^{p(z)-1} \text{ in } \Omega. \tag{35}$$

On account of hypothesis $H_1(iv)$ and since $u_\mu \in$ int C_+, we have

$$0 < \hat{\mu}_\mu \leq (\lambda - \mu) f(z, u_\mu(z)) \text{ for a.a. } z \in \Omega.$$

Hence from Eq. 35 and Proposition 2.3, it follows that

$$u_\lambda - u_\mu \in D_+.$$

The proof is now complete. $\qquad\square$

Proposition 4.8 *If hypotheses H_0, H_1 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits at least two positive solutions*

$$u_0, \hat{u} \in \text{int } C_+, \ u_0 \leq \hat{u}, \ u_0 \neq \hat{u}.$$

Proof Let $0 < \lambda < \vartheta < \lambda^*$. We know that $\vartheta \in \mathcal{L}$ (see Proposition 4.5). Moreover, according to Proposition 4.7, we can find $u_\vartheta \in S_\vartheta \subseteq$ int C_+ and $u_0 \in S_\lambda \subseteq$ int C_+ such that

$$u_\vartheta - u_0 \in \text{int } C_+. \tag{36}$$

We introduce the Carathéodory function $d_\lambda(z, x)$ defined by

$$d_\lambda(z, x) = \begin{cases} u_0(z)^{-\eta(z)} + \lambda f(z, u_0(z)), & \text{if } x \leq u_0(z) \\ x^{-\eta(z)} + \lambda f(z, x), & \text{if } u_0(z) < x. \end{cases} \tag{37}$$

Also we consider the following truncation of $d_\lambda(z, \cdot)$:

$$\hat{d}_\lambda(z, x) = \begin{cases} d_\lambda(z, x), & \text{if } x \le u_\vartheta(z) \\ d_\lambda(z, u_\vartheta(z)), & \text{if } u_\vartheta(z) < x. \end{cases} \tag{38}$$

This is also a Carathéodory function. We set

$$D_\lambda(z, x) = \int_0^x d_\lambda(z, s)ds \text{ and } \hat{D}_\lambda(z, x) = \int_0^x \hat{d}_\lambda(z, s)ds.$$

Then we consider the C^1-functionals $v_\lambda, \hat{v}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$v_\lambda(u) = \gamma_{p(z)}(u) + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)} - \int_\Omega D_\lambda(z, u)dz,$$

$$\hat{v}_\lambda(u) = \gamma_{p(z)}(u) + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz - \int_\Omega \hat{D}_\lambda(z, u)dz \text{ for all } u \in W^{1,p(z)}(\Omega).$$

From Eqs. 37, 38 and the anisotropic regularity theory, we have

$$K_{v_\lambda} \subseteq [u_0) \cap \text{int } C_+, \tag{39}$$

$$K_{\hat{v}_\lambda} \subseteq [u_0, u_\vartheta] \cap \text{int } C_+. \tag{40}$$

On account of Eqs. 37 and 39, we see that we may assume that

$$K_{v_\lambda} \cap [u_0, u_\vartheta] = \{u_0\}. \tag{41}$$

Otherwise we already have a second positive smooth solution bigger that u_0 (see Eqs. 37, 39). Also from Eqs. 37 and 38 we see that

$$v_\lambda\Big|_{[0,u_\vartheta]} = \hat{v}_\lambda\Big|_{[0,u_\vartheta]}, \ v'_\lambda\Big|_{[0,u_\vartheta]} = \hat{v}'_\lambda\Big|_{[0,u_\vartheta]}. \tag{42}$$

The functional $\hat{v}_\lambda(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W^{1,p(z)}(\Omega)$ such that

$$\hat{v}_\lambda(\tilde{u}_0) = \min\left\{\hat{v}_\lambda(u) : u \in W^{1,p(z)}(\Omega)\right\},$$

$$\Rightarrow \tilde{u}_0 \in K_{\hat{v}_\lambda} \subseteq [u_0, u_\vartheta] \cap \text{int } C_+ \text{ (see Eq. 40)},$$

$$\Rightarrow \tilde{u}_0 = u_0 \text{ (see Eqs. 41, 42)},$$

$$\Rightarrow u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } v_\lambda(\cdot) \text{ (see Eqs. 36, 42)},$$

$$\Rightarrow u_0 \text{ is a local } W^{1,p(z)}(\Omega)\text{-minimizer of } v_\lambda(\cdot) \tag{43}$$

(see Gasinski and Papageorgiou [16, Proposition 3.3]).

From Eqs. 39 and 37 it is clear that we may assume that

$$K_{v_\lambda} \text{ is finite.} \tag{44}$$

Indeed, otherwise we already have a whole sequence of positive smooth solutions all bigger than u_0 and so we are done.

From Eqs. 43, 44 and Theorem 5.7.6 of Papageorgiou et al. [32, p. 449], we know that we can find $\rho \in (0, 1)$ small such that

$$v_\lambda(u_0) < \inf\{v_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda. \tag{45}$$

Also hypothesis $H_1(ii)$ and Eq. 37 imply that if $u \in \text{int } C_+$, then

$$v_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{46}$$

Claim: $v_\lambda(\cdot)$ satisfies the C-condition.

We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p(z)}(\Omega)$ such that

$$|v_\lambda(u_n)| \leq C_7 \text{ for some } C_7 > 0, \text{ all } n \in \mathbb{N}, \tag{47}$$

$$(1 + \|u_n\|)v'_\lambda(u_n) \to 0 \text{ in } W^{1,p(z)}(\Omega)^* \text{ as } n \to \infty. \tag{48}$$

From Eq. 48 we have

$$\left| \langle A_{p(z)}(u_n), h \rangle + \langle A_{q(z)}(u_n), h \rangle + \int_\Omega \xi(z)|u_n|^{p(z)-2}u_n h\,dz - \int_\Omega d_\lambda(z, u_n)h\,dz \right| \leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{49}$$

for all $h \in W^{1,p(z)}(\Omega)$, with $\varepsilon_n \to 0^+$.

In Eq. 49, we choose $h = -u_n^- \in W^{1,p(z)}(\Omega)$ and obtain

$$\rho_{p(z)}(Du_n^-) + \rho_{q(z)}(Du_n^-) + \int_\Omega \xi(z)(u_n^-)^{p(z)}dz \leq C_8$$

for some $C_8 > 0$, all $n \in \mathbb{N}$ (see Eq. 37),

$$\Rightarrow \{u_n^-\}_{n\geq 1} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded.} \tag{50}$$

Next, we choose in Eq. 49 $h = u_n^+ \in W^{1,p(z)}(\Omega)$ and we obtain

$$-\rho_{p(z)}(Du_n^+) - \rho(Du_n^+) - \int_\Omega \xi(z)(u_n^+)^{p(z)}dz + \int_\Omega d_\lambda(z, u_n^+)u_n^+dz \leq \varepsilon_n \tag{51}$$

for all $n \in \mathbb{N}$.

From Eqs. 47 and 50, we have

$$\rho_{p(z)}(Du_n^+) + \rho_{q(z)}(Du_n^+) + \int_\Omega \xi(z)(u_n^+)^{p(z)}dz - \int_\Omega p_+ D_\lambda(z, u_n^+)dz \leq C_9 \tag{52}$$

for some $C_9 > 0$, all $n \in \mathbb{N}$ (recall $q_- < p_-$).

We add Eqs. 51 and 52. Then

$$\int_\Omega I_\lambda(z, u_n^+)dz \leq C_{10} \tag{53}$$

for some $C_{10} > 0$, all $n \in \mathbb{N}$ (see Eq. 37).

Suppose that $\{u_n^+\}_{n\geq 1} \subseteq W^{1,p(z)}(\Omega)$ is not bounded. So, we may assume that

$$\|u_n^+\| \to +\infty \text{ as } n \to \infty. \tag{54}$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p(z)}(\Omega) \text{ and } y_n \to y \text{ in } L^{p(z)}(\Omega), \ y \geq 0. \tag{55}$$

Initially we assume that $y \not\equiv 0$. Let $\hat{\Omega} = \{z \in \Omega : y(z) > 0\}$. From Eq. 55 we see that $|\hat{\Omega}|_N > 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$) and

$$u_n^+(z) \to +\infty \text{ for a.a. } z \in \hat{\Omega}.$$

Hypothesis $H_1(ii)$ implies

$$\frac{F(z, u_n^+(z))}{\|u_n^+\|^{p_+}} = \frac{F(z, u_n^+(z))}{u_n^+(z)^{p_+}} y_n(z)^{p_+} \to +\infty \text{ for a.a. } z \in \hat{\Omega},$$

$$\Rightarrow \int_{\hat{\Omega}} \frac{F(z, u_n^+)}{\|u_n^+\|^{p_+}}dz \to +\infty \text{ (by Fatou's lemma)},$$

$$\Rightarrow \int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^{p_+}}dz \to +\infty \text{ (since } F \geq 0, \text{ see hypothesis } H_1(i)). \tag{56}$$

From Eqs. 47 and 50, we have

$$-\gamma_{p(z)}(u_n^+) - \int_\Omega \frac{1}{q(z)} |Du_n^+|^{q(z)} dz - \int_\Omega \frac{\xi(z)}{p(z)} (u_n^+)^{p(z)} dz + \int_\Omega D_\lambda(z, u_n^+) dz \le C_{11}$$

for some $C_{11} > 0$, all $n \in \mathbb{N}$,

$$\Rightarrow \int_\Omega \frac{\lambda F(z, u_n^+)}{\|u_n^+\|^{p_+}} dz \le C_{12} \tag{57}$$

for some $C_{12} > 0$, all $n \in \mathbb{N}$ (see Eqs. 54 and 37).

Comparing Eqs. 56 and 57, we have a contradiction.

Now we assume that $y \equiv 0$. We consider the C^1-functional $\tilde{v}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{v}_\lambda(u) = \frac{1}{p_+} \left[\int_\Omega |Du|^{p(z)} dz + \int_\Omega \xi(z) |u|^{p(z)} dz \right] - \int_\Omega D_\lambda(z, u) dz$$

for all $u \in W^{1,p(z)}(\Omega)$.

Evidently we have

$$\tilde{v}_\lambda \le v_\lambda. \tag{58}$$

Let $k_n(t) = \tilde{v}_\lambda(tu_n^+)$ for all $t \in [0, 1]$, all $n \in \mathbb{N}$. We can find $t_n \in [0, 1]$ such that

$$k_n(t_n) = \max_{0 \le t \le 1} k_n(t). \tag{59}$$

For $\hat{\eta} \ge 1$ let $w_n = (2\hat{\eta})^{\frac{1}{p_-}} y_n \ n \in \mathbb{N}$. Then

$$w \to 0 \text{ in } L^{p(z)}(\Omega) \text{ (see Eq. 55 and recall that } y \equiv 0),$$

$$\Rightarrow \int_\Omega D_\lambda(z, w_n) dz \to 0 \text{ as } n \to \infty. \tag{60}$$

On account of Eq. 54, we see that we can find $n_0 \in \mathbb{N}$ such that

$$(2\hat{\eta})^{\frac{1}{p_-}} \frac{1}{\|u_n^+\|} \le 1 \text{ for all } n \ge n_0. \tag{61}$$

From Eqs. 59 and 61, we have

$$k_n(t_n) \ge k_n \left(\frac{(2\hat{\eta})^{1/p_-}}{\|u_n^+\|} \right),$$

$$\Rightarrow \tilde{v}_\lambda(t_n u_n^+) \ge \tilde{v}_\lambda \left((2\hat{\eta})^{1/p_-} y_n \right) = \tilde{v}_\lambda(w_n),$$

$$\Rightarrow \tilde{v}_\lambda(t_n u_n^+) \ge \frac{2\hat{\eta}}{p_+} \left[\rho_{p(z)}(Dy_n) + \rho_{p(z),\xi}(y_n) \right] - \int_\Omega D_\lambda(z, w_n) dz$$

$$\ge \frac{2\hat{\eta}}{p_+} C_{13} \|y_n\| - \int_\Omega D_\lambda(z, w_n) dz \text{ for some } C_{13} > 0$$

$$\ge \frac{\hat{\eta}}{p_+} C_{13} \text{ for all } n \ge n_1 \ge n_0$$

(see Eq. 60 and recall that $\|y_n\| = 1$).

But $\hat{\eta} \ge 1$ is arbitrary. So, it follows that

$$\tilde{v}_\lambda(t_n u_n^+) \to +\infty \text{ as } n \to \infty. \tag{62}$$

We have

$$0 \le t_n u_n^+ \le u_n^+ \text{ for all } n \in \mathbb{N}.$$

Hence hypothesis $H_1(iii)$ implies that

$$\int_\Omega I_\lambda(z, t_n u_n^+) dz \leq \int_\Omega I_\lambda(z, u_n^+) dz + \|\hat{\vartheta}_\lambda\|_1 \leq C_{14} \tag{63}$$

for some $C_{14} > 0$, all $n \in \mathbb{N}$ (see Eq. 53).

We know that

$$\tilde{v}_\lambda(0) = 0 \text{ and } \tilde{v}_\lambda(u_n) \leq C_7 \text{ for all } n \in \mathbb{N} \text{ (see Eqs. 47 and 58)}.$$

Then from Eq. 62 it follows that

$$t_n \in (0, 1) \text{ for all } n \geq n_2.$$

So, from Eq. 59 we have

$$0 = t_n \frac{d}{dt} \tilde{v}_\lambda(tu_n^+)\Big|_{t=t_n}$$

$$= \langle \tilde{v}'_\lambda(t_n u_n^+), t_n u_n^+ \rangle \text{ for all } n \geq n_2 \text{ (by the chain rule)},$$

$$\Rightarrow \quad \rho_{p(z)}(D(t_n u_n^+)) + \rho_{p(z),\xi}(t_n u_n^+) = \int_\Omega d_\lambda(z, t_n u_n^+)(t_n u_n^+) dz \text{ for all } n \geq n_2,$$

$$\Rightarrow \quad \rho_{p(z)}(D(t_n u_n^+)) + \rho_{p(z),\xi}(t_n u_n^+) - \int_\Omega p_+ D_\lambda(z, t_n u_n^+) dz \leq C_{15}$$

for some $C_{15} > 0$, all $n \geq n_2$ (see Eqs. 63 and 37),

$$\Rightarrow \quad p_+ \tilde{v}_\lambda(t_n u_n^+) \leq C_{15} \text{ for all } n \geq n_2. \tag{64}$$

Comparing Eqs. 62 and 64 we have a contradiction.
Therefore we infer that

$$\{u_n^+\}_{n \geq 1} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded},$$

$$\Rightarrow \{u_n\}_{n \geq 1} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded (see Eq. 50)}.$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p(z)}(\Omega) \text{ and } u_n \to u \text{ in } L^{r(z)}(\Omega) \text{ as } n \to \infty. \tag{65}$$

In Eq. 49 we choose $h = u_n - u \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use Eq. 63. Then reasoning as in the proof of Proposition 3.2, using Proposition 2.2, we obtain that

$$u_n \to u \text{ in } W^{1,p(z)}(\Omega),$$

$$\Rightarrow v_\lambda(\cdot) \text{ satisfies the } C\text{-condition}.$$

This proves the Claim.

On account of Eqs. 45, 46 and the Claim, we can apply the mountain pass theorem and find $\hat{u} \in W^{1,p(z)}(\Omega)$ such that

$$\hat{u} \in K_{v_\lambda} \subseteq [u_0) \cap \text{int } C_+ \text{ and } m_\lambda \leq v_\lambda(\hat{u}). \tag{66}$$

From Eqs. 66, 45 and 37, we obtain

$$\hat{u} \in S_\lambda \subseteq \text{int } C_+, \ u_0 \leq \hat{u}, \ u_0 \neq \hat{u},$$

which concludes the proof. □

We introduce the Carathéodory function $\hat{\beta}_\lambda(z, x)$ defined by

$$\hat{\beta}_\lambda(z, x) = \begin{cases} \overline{u}(z)^{-\eta(z)} + \lambda f(z, \overline{u}(z)), & \text{if } x \leq \overline{u}(z) \\ x^{-\eta(z)} + \lambda f(z, x), & \text{if } \overline{u}(z) < x. \end{cases} \tag{67}$$

We set $\hat{B}_\lambda(z, x) = \int_0^x \hat{\beta}(z, s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \gamma_{p(z)}(u) + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz - \int_\Omega \hat{B}_\lambda(z, u)dz$$

for all $u \in W^{1,p(z)}(\Omega)$.

Using this functional, we can establish the admissibility of the critical parameter value $\lambda^* > 0$.

Proposition 4.9 *If hypotheses H_0, H_1 hold, then $\lambda^* \in \mathcal{L}$.*

Proof Let $\{\lambda_n\}_{n\geq 1} \subseteq (0, \lambda^*) \subseteq \mathcal{L}$ and assume that $\lambda_n \to \lambda^*$. We can find $u_n \in S_\lambda \subseteq \text{int } C_+$ $n \in \mathbb{N}$ such that

$$\hat{\varphi}_{\lambda_n}(u_n) \leq \hat{\varphi}_{\lambda_n}(\overline{u}) < 0 \text{ for all } n \in \mathbb{N} \tag{68}$$

$$\text{(see the proof of Proposition 4.5 and Eq. 67).}$$

Also we have

$$\hat{\varphi}'_{\lambda_n}(u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{69}$$

Using Eqs. 68, 69 and reasoning as in the Claim in the proof of Proposition 4.8, we obtain

$$u_n \to u^* \text{ in } W^{1,p(z)}(\Omega), \overline{u} \leq u^* \text{ (see Proposition 4.2)}$$

$$\Rightarrow u^* \in S_{\lambda^*} \subseteq \text{int } C_+ \text{ and so } \lambda^* \in \mathcal{L}.$$

The proof is now complete.   □

According to the above proposition, we have

$$\mathcal{L} = (0, \lambda^*].$$

5 Minimal Positive Solutions

Recall that a set $S \subseteq W^{1,p(z)}(\Omega)$ is said to be downward directed, if for all $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leq u_1, u \leq u_2$.

As in the proof of Proposition 18 of Papageorgiou et al. [36], we prove that for every $\lambda \in \mathcal{L}$, the solution set $S_\lambda \subseteq \text{int } C_+$ is downward directed. We will show that S_λ has a minimal element.

Proposition 5.1 *If hypotheses H_0, H_1 hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in S_\lambda \subseteq \text{int } C_+$ (that is, $u_\lambda^* \leq u$ for all $u \in S_\lambda$).*

Proof On account of Lemma 3.10 of Hu and Papageorgiou [19, p. 176], we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda$ decreasing (since S_λ is downward directed) such that

$$\inf_{n\geq 1} u_n = \inf S_\lambda.$$

We have

$$\hat{\varphi}'_\lambda(u_n) = 0 \text{ for all } n \in \mathbb{N}, \tag{70}$$

$$\overline{u} \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{71}$$

From Eqs. 70 and 71, it follows that

$$\{u_n\}_{n \geq 1} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded.}$$

Then as in the proof of Proposition 4.5 and using the fact that $\{u_n\}_{n \geq 1}$ is decreasing, we obtain

$$u_n \to u^*_\lambda \text{ in } W^{1,p(z)}(\Omega),$$

$$\Rightarrow u^*_\lambda \in S_\lambda \subseteq \text{int } C_+, u^*_\lambda = \inf S_\lambda,$$

which concludes the proof. $\square$

Next we examine the map $\lambda \mapsto u^*_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C^1(\overline{\Omega})$.

Proposition 5.2 *If hypotheses H_0, H_1 hold, then*

(a) *the map $\lambda \mapsto u^*_\lambda$ from $\mathcal{L} = (0, \lambda^*]$ into $C^1(\overline{\Omega})$ is strictly increasing in the sense that*

$$0 < \mu < \lambda \leq \lambda^* \Rightarrow u^*_\lambda - u^*_\mu \in D_+;$$

(b) $\lambda \mapsto u^*_\lambda$ *is left continuous from $\mathcal{L}$ into $C^1(\overline{\Omega})$.*

Proof (a) Let $0 < \mu < \lambda \leq \lambda^*$. On account of Proposition 4.7, we can find $u_\mu \in S_\mu \subseteq$ int C_+ such that

$$u^*_\lambda - u_\mu \in D_+,$$

$$\Rightarrow u^*_\lambda - u^*_\mu \in D_+.$$

(b) Let $\lambda_n \to \lambda^-$, $\lambda_n \in \mathcal{L}$ for all $n \in \mathbb{N}$. From Proposition 4.2 and part (*a*) of this proposition, we have

$$\overline{u} \leq u^*_\lambda \leq u^*_{\lambda_1} \text{ for all } n \in \mathbb{N}.$$

It follows that

$$\{u^*_n = u^*_{\lambda_n}\}_{n \geq 1} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded.}$$

Then Proposition 3.1 of Gasinski and Papageorgiou [16] implies that

$$\|u^*_n\|_\infty \leq C_{16} \text{ for some } C_{16} > 0, \text{ all } n \in \mathbb{N}.$$

Using Theorem 1.3 of Fan [15] (see also Lieberman [21]), we see that we can find $\alpha \in (0, 1)$ and $C_{17} > 0$ such that

$$u^*_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|u^*_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_{17} \text{ for all } n \in \mathbb{N}. \tag{72}$$

From Eq. 72, the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and the monotonicity of the sequence $\{u^*_n\}_{n \geq 1}$, we have that

$$u^*_n \to \hat{u}^*_\lambda \text{ in } C^1(\overline{\Omega}). \tag{73}$$

If $\hat{u}^*_\lambda \neq u^*_\lambda$, then there exists $z_0 \in \overline{\Omega}$ such that

$$u^*_\lambda(z_0) < \hat{u}^*_\lambda(z_0),$$

$$\Rightarrow u^*_\lambda(z_0) < u^*_n(z_0) \text{ for all } n \geq n_0 \text{ (see Eq. 73),}$$

contradicting part (*a*). So $\hat{u}^*_\lambda = u^*_\lambda$ and we have the left continuity of the map $\lambda \mapsto u^*_\lambda$. $\square$

The next theorem summarizes our main contributions in this paper concerning problem (P_λ).

Theorem 5.3 *If hypotheses H_0, H_1 hold, then there exists $\lambda^* < \infty$ such that*

(a)   *for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in int\, C_+$, $u_0 \le \hat{u}$, $u_0 \ne \hat{u}$;*

(b)   *for $\lambda = \lambda^*$, problem (P_λ) has at least one positive solution $u^* \in int\, C_+$;*

(c)   *for all $\lambda > \lambda^*$, problem (P_λ) has no positive solutions;*

(d)   *for all $\lambda \in \mathcal{L} = (0, \lambda^*]$, problem (P_λ) has a smallest positive solution $u_\lambda^* \in int\, C_+$ and the map $\lambda \mapsto u_\lambda^*$ is strictly increasing and left-continuous from $\mathcal{L}$ into $C^1(\overline{\Omega})$.*

Declarations

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