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Research Article

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Existence and Multiplicity of Solutions for Resonant $(p, 2)$-Equations

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Abstract: We consider Dirichlet elliptic equations driven by the sum of a p-Laplacian $(2 < p)$ and a Laplacian. The conditions on the reaction term imply that the problem is resonant at both $\pm\infty$ and at zero. We prove an existence theorem (producing one nontrivial smooth solution) and a multiplicity theorem (producing five nontrivial smooth solutions, four of constant sign and the fifth nodal; the solutions are ordered). Our approach uses variational methods and critical groups.

Keywords: Resonance, Variational Eigenvalues, Critical Groups, Nonlinear Regularity, Multiple Solutions, Nodal Solutions

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear and nonhomogeneous Dirichlet problem:

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega \ (2 < p), \qquad u|_{\partial\Omega} = 0, \tag{1.1}$$

Here, for $r \in (1, \infty)$, we denote by Δ_r the r-Laplacian defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2} Du) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

When $r = 2$, we write $\Delta_2 = \Delta$ (the standard Laplace differential operator). The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z, x)$ is continuous). We assume that for almost all $z \in \Omega$, $f(z, \cdot)$ is $(p-1)$-sublinear near $\pm\infty$, and asymptotically as $x \to \pm\infty$, the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ interacts with the variational part of the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$ (resonant problem). Equations driven by the sum of a p-Laplacian and a Laplacian (known as $(p, 2)$-equations) have recently been studied in [3, 11, 27, 28, 30, 31, 34, 35]. The aforementioned works, either do not consider resonant at $\pm\infty$ equations (see [3, 11, 34, 35]) or the resonance is with respect to the principal eigenvalue (see [27, 28, 30, 31]). For $p \neq 2$, we do not have a complete knowledge of the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$, the eigenspaces are not linear subspaces of $W_0^{1,p}(\Omega)$, and the Sobolev space $W_0^{1,p}(\Omega)$ cannot be expressed as a direct sum of the eigenspaces. All these negative facts make difficult the study of problems with resonance

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at higher parts of the spectrum. Our present paper is closer to [11, 27]. Compared to [11], we allow resonance to occur and so we improve their existence theorem. Compared to [27], the resonance is with respect to any variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, not only the principal one.

Using tools from Morse theory and variational methods based on the critical point theory, we prove existence and multiplicity theorems for resonant $(p, 2)$-equations. We mention that $(p, 2)$-equations arise in problems of mathematical physics. The Dirichlet $(p, 2)$-problem treated in this paper models some phenomena in quantum physics as first pointed out by Benci, Fortunato and Pisani [6]. We refer to the works of Benci, D'Avenia, Fortunato and Pisani [5] (in quantum physics), and Cherfils and Ilyasov [10] (in plasma physics). Related results on (p, q)-Laplacian problems are due to Marano, Mosconi and Papageorgiou [21], and Mugnai and Papageorgiou [24].

In the next section we briefly recall the main mathematical tools which will be used in the sequel.

2 Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the dual pair (X^*, X). Also, let $\varphi \in C^1(X, \mathbb{R})$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short) if the following property holds:

- Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and
$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$
admits a strongly convergent subsequence.

This compactness-type condition on the functional φ leads to a deformation theorem from which one derives the minimax theory of the critical values of φ. A basic result in this theory is the celebrated "mountain pass theorem" due to Ambrosetti and Rabinowitz [4]. Here, we state the result in a slightly more general form (see, for example, [16, p. 648]).

Theorem 2.1. *Let X be a Banach space, and assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$,*
$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$$
and
$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)), \quad \text{where } \Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$
Then $c \geqslant m_\rho$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi'(u) = 0$, $\varphi(u) = c$).

Three Banach spaces will be central in our analysis of problem (1.1). We refer to the Dirichlet Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$, and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$.

By Poincaré's inequality, the norm of $W_0^{1,p}(\Omega)$ can be defined by
$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The space $H_0^1(\Omega)$ is a Hilbert space and again the Poincaré inequality implies that we can choose as inner product
$$(u, h) = (Du, Dh)_{L^2(\Omega,\mathbb{R}^N)} \quad \text{for all } u, h \in H_0^1(\Omega).$$
The corresponding norm is
$$\|u\|_{H_0^1(\Omega)} = \|Du\|_2 \quad \text{for all } u \in H_0^1(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive cone
$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \Big\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\Big\}.$$

Here, $\frac{\partial u}{\partial n}$ is the usual normal derivative defined by $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Recall that $C_0^1(\overline{\Omega})$ is dense in both $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$.

Given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and then define $u^\pm(\cdot) = u(\cdot)^\pm$ for all $u \in W_0^{1,p}(\Omega)$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also, we denote the Lebesgue measure on $\mathbb{R}^N$ by $|\cdot|_N$, and if $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), we define the Nemytskii map corresponding to $g(\cdot,\cdot)$ by

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We will use the spectra of the operators $(-\Delta_p, W_0^{1,p}(\Omega))$ and $(-\Delta, H_0^1(\Omega))$. We start with the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, consider the following nonlinear eigenvalue problem:

$$-\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega\ (1 < p < \infty), \qquad u|_{\partial\Omega} = 0. \tag{2.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ if problem (2.1) admits a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$, known as the eigenfunction corresponding to $\hat{\lambda}$. We know that there exists the smallest eigenvalue $\hat{\lambda}_1(p) > 0$, which has the following properties:

- $\hat{\lambda}_1(p)$ is isolated in the spectrum $\hat{\sigma}(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$; in other words, there exists $\epsilon > 0$ such that $(\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon) \cap \hat{\sigma}(p) = \emptyset$.
- $\hat{\lambda}_1(p)$ is simple; that is, if $\hat{u}, \tilde{u} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to $\hat{\lambda}_1(p)$, then $\hat{u} = \xi\tilde{u}$ with $\xi \in \mathbb{R} \setminus \{0\}$.
- We have

$$\hat{\lambda}_1(p) = \inf\Big\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0\Big\}. \tag{2.2}$$

In (2.2) the infimum is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(p)$. The above properties imply that the elements of this eigenspace do not change sign. We point out that the nonlinear regularity theory (see, for example, [16, p. 737]) implies that all eigenfunctions of $(-\Delta_p, W_0^{1,p}(\Omega))$ belong to $C_0^1(\overline{\Omega})$. By $\hat{u}_1(p)$ we denote the positive L^p-normalized (that is, $\|\hat{u}_1(p)\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1(p) > 0$. As we have already mentioned, $\hat{u}_1(p) \in C_+ \setminus \{0\}$ and, in fact, the nonlinear maximum principle (see, for example, [16, p. 738]) implies that $\hat{u}_1(p) \in \operatorname{int} C_+$. An eigenfunction $\hat{u}$ which corresponds to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(p)$ is nodal (sign changing). Since $\hat{\sigma}(p)$ is closed and $\hat{\lambda}_1(p) > 0$ is isolated, the second eigenvalue $\hat{\lambda}_2(p)$ is well defined by

$$\hat{\lambda}_2(p) = \min\{\hat{\lambda} \in \hat{\sigma}(p) : \hat{\lambda} > \hat{\lambda}_1(p)\}.$$

For additional eigenvalues, we employ the Ljusternik–Schnirelmann minimax scheme, which gives the entire nondecreasing sequence of eigenvalues $\{\hat{\lambda}_k(p)\}_{k\geqslant 1}$ such that $\hat{\lambda}_k(p) \to +\infty$. These eigenvalues are known as "variational eigenvalues" and, depending on the index used in the Ljusternik–Schnirelmann scheme, we can have various such sequences of variational eigenvalues, which all coincide in the first two elements $\hat{\lambda}_1(p)$ and $\hat{\lambda}_2(p)$, defined as described above. For the other elements we do not know if their sequences coincide. Here, we use the sequence constructed by using the Fadell–Rabinowitz [14] cohomological index (see [32]). Note that we do not know if the variational eigenvalues exhaust the spectrum $\hat{\sigma}(p)$. We have full knowledge of the spectrum if $N = 1$ (ordinary differential equations) and when $p = 2$ (linear eigenvalue problem). In the latter case, we have $\hat{\sigma}(2) = \{\hat{\lambda}_k(2)\}_{k\geqslant 1}$ with $0 < \hat{\lambda}_1(2) < \hat{\lambda}_2(2) < \cdots < \hat{\lambda}_k(2) \to +\infty$ as $k \to \infty$. The corresponding eigenspaces, denoted by $E(\hat{\lambda}_k(2))$, are linear spaces, and we have the orthogonal direct sum decomposition

$$H_0^1(\overline{\Omega}) = \overline{\bigoplus_{k\geqslant 1} E(\hat{\lambda}_k(2))}.$$

For all $k \in \mathbb{N}$, each $E(\hat{\lambda}_k(2))$ is finite dimensional, $E(\hat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$, and has the so-called Unique Continuation Property (UCP for short), that is, if $u \in E(\hat{\lambda}_k(2))$ vanishes on a set of positive measure in Ω, then $u \equiv 0$. For every $k \in \mathbb{N}$, we define

$$\bar{H}_k = \bigoplus_{i=1}^{k} E(\hat{\lambda}_i(2)) \quad \text{and} \quad \hat{H}_{k+1} = \overline{\bigoplus_{i \geqslant k+1} E(\hat{\lambda}_i(2))} = \bar{H}_k^{\perp}.$$

We have

$$H_0^1(\Omega) = \bar{H}_k \oplus \hat{H}_{k+1}.$$

In this case all eigenvalues admit variational characterizations and we have

$$\hat{\lambda}_1(2) = \inf\Big\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\ u \neq 0\Big\}, \tag{2.3}$$

$$\hat{\lambda}_k(2) = \sup\Big\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \bar{H}_k,\ u \neq 0\Big\} = \inf\Big\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \hat{H}_k,\ u \neq 0\Big\}, \quad k \geqslant 2. \tag{2.4}$$

Again, the infimum in (2.3) is realized on the one-dimensional eigenspace $E(\hat{\lambda}_1(2))$, while both the supremum and the infimum in (2.4) are realized on $E(\hat{\lambda}_k(2))$.

As a consequence of the UCP, we have the following convenient inequalities.

Lemma 2.2. (a) *If $\vartheta \in L^\infty(\Omega)$ and, for $k \in \mathbb{N}$, $\vartheta(z) \geqslant \hat{\lambda}_k(2)$ for almost all $z \in \Omega$, with $\vartheta \not\equiv \hat{\lambda}_k(2)$, then there exists a constant $c_0 > 0$ such that*

$$\|Du\|_2^2 - \int_\Omega \vartheta(z)u^2\,dz \leqslant -c_0\|u\|^2 \quad \text{for all } u \in \bar{H}_k.$$

(b) *If $\vartheta \in L^\infty(\Omega)$ and, for $k \in \mathbb{N}$, $\vartheta(z) \leqslant \hat{\lambda}_k(2)$ for almost all $z \in \Omega$, with $\vartheta \not\equiv \hat{\lambda}_k(2)$, then there exists a constant $c_1 > 0$ such that*

$$\|Du\|_2^2 - \int_\Omega \vartheta(z)u^2\,dz \geqslant c_1\|u\|^2 \quad \text{for all } u \in \hat{H}_k.$$

In what follows, let $A_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ ($\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$) be the map defined by

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\,dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

By [23, p. 40], we have the following proposition.

Proposition 2.3. *The map $A_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ ($1 < p < \infty$) is bounded (that is, it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too) and of type $(S)_+$, that is, if*

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to n$ in $W_0^{1,p}(\Omega)$.

If $p = 2$, then $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

Consider a Carathéodory function $f_0 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$|f_0(z, x)| \leqslant a_0(z)(1 + |x|^{r-1}) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\mathbb{R})$ and $1 < r < p^*$, where the critical Sobolev exponent is defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N,\\ +\infty & \text{if } p \geqslant N.\end{cases}$$

Let $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next proposition is a special case of a more general result by Aizicovici, Papageorgiou and Staicu [2], see also [26, 29] for similar results in different spaces. All these results are consequences of the nonlinear regularity theory of Lieberman [20].

Proposition 2.4. *Let $u_0 \in W_0^{1,p}(\Omega)$ be a local $C_0^1(\bar{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \leqslant \rho_0.$$

Then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \|h\| \leqslant \rho_1.$$

Finally, we recall some basic definitions and facts from Morse theory (critical groups), which we will use in the sequel.

So, let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}, \quad K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}, \quad \varphi^c = \{u \in X : \varphi(u) \leqslant c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Given an isolated $u \in K_\varphi^c$, the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that φ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see, for example, [16, p. 628]) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

In the next section we prove an existence theorem under conditions of resonance both at $\pm\infty$ and at zero.

3 Existence of Nontrivial Solutions

The hypotheses on the reaction term $f(z,x)$ are the following.

Hypotheses 3.1. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the following properties:

(i) For every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leqslant a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant \rho.$$

(ii) There exists an integer $m \geqslant 1$ such that

$$\lim_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = \hat{\lambda}_m(p) \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) There exists $\tau \in (2,p)$ such that

$$0 < \beta_0 \leqslant \liminf_{x \to \pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\tau} \quad \text{uniformly for almost all } z \in \Omega,$$

where $F(z,x) = \int_0^x f(z,s)\,ds$.

(iv) There exist integer $l \geqslant 1$, with $d_l \neq m$ ($d_l = \dim \bar{H}_l$), $\delta > 0$ and $\eta \in L^\infty(\Omega)$ such that

$$\begin{aligned} \hat{\lambda}_l(2) &\leqslant \eta(z) && \text{for almost all } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_l(2), \\ \eta(z)x^2 \leqslant f(z,x)x &\leqslant \hat{\lambda}_{l+1}x^2 && \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant \delta, \end{aligned}$$

and for every $x \neq 0$ the second inequality is strict on a subset of positive Lebesgue measure.

Remark 3.2. Hypothesis 3.1 (ii) says that asymptotically as $x \to \pm\infty$, we have resonance with respect to some variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Similarly, Hypothesis 3.1 (iv) permits resonance at zero with respect to the eigenvalue $\hat{\lambda}_{l+1}(2)$ of $(-\Delta, H_0^1(\Omega))$. So, in a sense, we have a double resonance setting.

Let $\varphi\colon H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.3. *If Hypotheses 3.1* (i), (ii), (iii) *hold, then φ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \in \mathbb{N}, \tag{3.1}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*. \tag{3.2}$$

By (3.2), we have

$$\left| \langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f(z,u_n)h\,dz \right| \leqslant \frac{\epsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \epsilon_n \to 0^+. \tag{3.3}$$

In (3.3) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and obtain

$$-\|Du_n\|_p^p - \|Du_n\|_2^2 + \int_\Omega f(z,u_n)u_n\,dz \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

On the other hand, from (3.1) we have

$$\|Du_n\|_p^p + \frac{p}{2}\|Du_n\|_2^2 - \int_\Omega pF(z,u_n)\,dz \leqslant pM_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

We add (3.4) and (3.5) and obtain

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]\,dz \leqslant M_2 + \left(1 - \frac{p}{2}\right)\|Du_n\|_2^2 \quad \text{for all } n \in \mathbb{N}, \tag{3.6}$$

for some $M_2 > 0$. Hypotheses 3.1 (i), (iii) imply that we can find $\beta_1 \in (0, \beta_0)$ and $c_2 > 0$ such that

$$\beta_1|x|^\tau - c_2 \leqslant f(z,x)x - pF(z,x) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.7}$$

Returning to (3.6) and using (3.7), we have (recall that $\tau > 2$)

$$\|u_n\|_\tau^\tau \leqslant c_3(1 + \|Du_n\|_2^2) \quad \text{for all } n \in \mathbb{N}, \tag{3.8}$$

for some $c_3 > 0$.

Claim 1. $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

Arguing by contradiction, suppose that the claim is not true. By passing to a subsequence if necessary, we have

$$\|u_n\| \to \infty. \tag{3.9}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^p(\Omega). \tag{3.10}$$

From (3.3) we have

$$\left| \langle A_p(y_n), h\rangle + \frac{1}{\|u_n\|^{p-2}} \langle A(y_n), h\rangle - \int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}} h\, dz \right| \leqslant \frac{\epsilon_n \|h\|}{(1 + \|u_n\|)\|u_n\|^{p-1}} \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

Hypotheses 3.1 (i), (ii) imply that

$$|f(z,x)| \leqslant c_4(1 + |x|^{p-1}) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}, \tag{3.12}$$

for some $c_4 > 0$, and hence

$$\left\{ \frac{N_f(u_n)}{\|u_n\|^{p-1}} \right\}_{n \geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{3.13}$$

In (3.11) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.9), (3.10), (3.13) and the fact that $p > 2$. Then $\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0$, which implies (see Proposition 2.3)

$$y_n \to y \quad \text{in } W_0^{1,p}(\Omega) \implies \|y\| = 1. \tag{3.14}$$

From (3.8) we have

$$\|y_n\|_\tau^\tau \leqslant \frac{c_3}{\|u_n\|^\tau} + \frac{c_3}{\|u_n\|^{\tau-2}} \|Dy_n\|_2^2 \leqslant \frac{c_5}{\|u_n\|^{\tau-2}} \quad \text{for all } n \geqslant n_0 \geqslant 1,$$

for some $c_5 > 0$. This yields (see (3.9) and recall that $\tau > 2$)

$$y_n \to 0 \quad \text{in } L^\tau(\Omega) \text{ as } n \to \infty.$$

Thus, $y = 0$ (see (3.10)), a contradiction to (3.14). This proves the claim.

Because of Claim 1, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^p(\Omega). \tag{3.15}$$

From (3.12) we see that

$$\{N_f(u_n)\}_{n \geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{3.16}$$

So, if in (3.3) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.15) and (3.16), then

$$\lim_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0,$$

and since $A(\cdot)$ is monotone, we have

$$\limsup_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle] \leqslant 0.$$

From (3.15),

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leqslant 0,$$

which implies (see Proposition 2.3)

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

Thus, φ satisfies the C-condition. □

We can have two approaches in the proof of the existence theorem. We present both because we believe that the particular tools used in each of them are of independent interest and can be used in different circumstances.

In the first approach we compute directly the critical groups at infinity of the energy functional φ. Note that Proposition 3.3 permits this computation.

Proposition 3.4. *If Hypotheses 3.1* (i), (ii), (iii) *hold, then* $C_m(\varphi,\infty)\neq 0$.

Proof. Let $\lambda\in(\hat\lambda_m(p),\hat\lambda_{m+1}(p))\setminus\hat\sigma(p)$ and consider the C^2-functional $\psi\colon W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\psi(u)=\frac{1}{p}\|Du\|_p^p-\frac{\lambda}{p}\|u\|_p^p\quad\text{for all } u\in W_0^{1,p}(\Omega).$$

We also consider the homotopy $h(t,u)$ defined by

$$h(t,u)=(1-t)\varphi(u)+t\psi(u)\quad\text{for all }(t,u)\in[0,1]\times W_0^{1,p}(\Omega).$$

Claim 2. There exist $\eta\in\mathbb{R}$ and $\hat\delta>0$ such that

$$h(t,u)\leqslant\eta\implies(1+\|u\|)\|h'_u(t,u)\|_*\geqslant\hat\delta\quad\text{for all } t\in[0,1].$$

We argue indirectly. So, suppose that the claim is not true. Since $h(\cdot,\cdot)$ maps bounded sets to bounded sets, we can find $\{t_n\}_{n\geqslant1}\subseteq[0,1]$ and $\{u_n\}_{n\geqslant1}\subseteq W_0^{1,p}(\Omega)$ such that

$$t_n\to t,\qquad\|u_n\|\to\infty,\qquad h(t_n,u_n)\to-\infty\qquad\text{and}\qquad(1+\|u_n\|)h'_u(t_n,u_n)\to0\quad\text{in } W^{-1,p'}(\Omega).\tag{3.17}$$

From the last convergence in (3.17), we have

$$\Big|\langle A_p(u_n),h\rangle+(1-t_n)\langle A(u_n),h\rangle-(1-t_n)\int_\Omega f(z,u_n)h\,dz-t_n\int_\Omega\lambda|u_n|^{p-2}u_nh\,dz\Big|\leqslant\frac{\epsilon_n\|h\|}{1+\|u_n\|}\tag{3.18}$$

for all $h\in W_0^{1,p}(\Omega)$ with $\epsilon_n\to0^+$.

Let $y_n=\frac{u_n}{\|u_n\|}$, $n\in\mathbb{N}$. Then $\|y_n\|=1$ for all $n\in\mathbb{N}$, and so we may assume that

$$y_n\xrightarrow{w}y\quad\text{in } W_0^{1,p}(\Omega)\qquad\text{and}\qquad y_n\to y\quad\text{in } L^p(\Omega).\tag{3.19}$$

From (3.18) we have

$$\Big|\langle A_p(y_n),h\rangle+\frac{1-t_n}{\|u_n\|^{p-2}}\langle A(y_n),h\rangle-(1-t_n)\int_\Omega\frac{N_f(u_n)}{\|u_n\|^{p-1}}h\,dz-t_n\int_\Omega\lambda|y_n|^{p-2}y_nh\,dz\Big|\leqslant\frac{\epsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}}\tag{3.20}$$

for all $n\in\mathbb{N}$.

From (3.12) and (3.19), we see that

$$\Big\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\Big\}_{n\geqslant1}\subseteq L^{p'}(\Omega)\text{ is bounded.}$$

Hence, by passing to a subsequence if necessary and using Hypothesis 3.1 (ii), we obtain (see [15])

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}}\xrightarrow{w}\hat\lambda_m(p)|y|^{p-2}y\quad\text{in } L^{p'}(\Omega).\tag{3.21}$$

In (3.20) we choose $h=y_n-y\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (3.17), (3.19), (3.21) and the fact that $2<p$. Then $\lim_{n\to\infty}\langle A_p(y_n),y_n-y\rangle=0$, which implies (see Proposition 2.3)

$$y_n\to y\quad\text{in } W_0^{1,p}(\Omega)\implies\|y\|=1.\tag{3.22}$$

We return to (3.20), pass to the limit as $n\to\infty$ and use (3.21) and (3.22). We obtain

$$\langle A_p(y),h\rangle=\int_\Omega\lambda_t|y|^{p-2}yh\,dz\quad\text{for all } h\in W_0^{1,p}(\Omega),\text{ with }\lambda_t=(1-t)\hat\lambda_m(p)+t\lambda,$$

hence

$$-\Delta_py(z)=\lambda_t|y(z)|^{p-2}y(z)\quad\text{for almost all } z\in\Omega,\qquad y|_{\partial\Omega}=0.\tag{3.23}$$

If $\lambda_t\notin\hat\sigma(p)$, then from (3.23) it follows that $y=0$, a contradiction (see (3.22)).

If $\lambda_t \in \hat{\sigma}(p)$, then for $E = \{z \in \Omega : y(z) \neq 0\}$, we have $|E|_N > 0$. Hence,

$$|u_n(z)| \to +\infty \quad \text{for almost all } z \in \Omega,$$

and thus

$$\liminf_{n\to\infty} \frac{f(z, u_n(z))u_n(z) - pF(z, u_n(z))}{|u_n(z)|^\tau} \geqslant \beta_0 > 0 \quad \text{for almost all } z \in E. \tag{3.24}$$

From (3.24), Hypothesis 3.1 (iii) and Fatou's lemma, we have

$$\liminf_{n\to\infty} \frac{1}{\|u_n\|^\tau} \int_E [f(z, u_n)u_n - pF(z, u_n)]\,dz > 0. \tag{3.25}$$

Note that Hypothesis 3.1 (iii) implies that we can find $M_3 > 0$ such that

$$f(z, x)x - pF(z, x) \geqslant 0 \quad \text{for almost all } z \in \Omega \text{ and all } |x| \geqslant M_3. \tag{3.26}$$

Then, in view of (3.26) and Hypothesis 3.1 (i), we have

$$\begin{aligned}
&\frac{1}{\|u_n\|^\tau} \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\,dz \\
&\quad= \frac{1}{\|u_n\|^\tau} \int_{\Omega\cap\{|u_n|\geqslant M_3\}} [f(z, u_n)u_n - pF(z, u_n)]\,dz + \frac{1}{\|u_n\|^\tau} \int_{\Omega\cap\{|u_n|<M_3\}} [f(z, u_n)u_n - pF(z, u_n)]\,dz \\
&\quad\geqslant \frac{1}{\|u_n\|^\tau} \int_{E\cap\{|u_n|\geqslant M_3\}} [f(z, u_n)u_n - pF(z, u_n)]\,dz - \frac{c_6}{\|u_n\|^\tau} \\
&\quad\geqslant \frac{1}{\|u_n\|^\tau} \int_E [f(z, u_n)u_n - pF(z, u_n)]\,dz - \frac{c_7}{\|u_n\|^\tau} \quad \text{for all } n \in \mathbb{N},
\end{aligned}$$

for some $c_6, c_7 > 0$. Hence, by (3.25),

$$\liminf_{n\to\infty} \frac{1}{\|u_n\|^\tau} \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\,dz > 0. \tag{3.27}$$

From the third convergence in (3.17), we see that we can find $n_0 \in \mathbb{N}$ such that

$$\|Du_n\|_p^p + \frac{(1 - t_n)p}{2}\|Du_n\|_2^2 - (1 - t_n)\int_\Omega pF(z, u_n)\,dz - t_n \int_\Omega \lambda|u_n|^p\,dz \leqslant -1 \quad \text{for all } n \geqslant n_0. \tag{3.28}$$

In (3.18) we choose $h = u_n \in W_0^{1,p}(\Omega)$. Then

$$-\|Du_n\|_p^p - (1 - t_n)\|Du_n\|_2^2 + (1 - t_n)\int_\Omega f(z, u_n)u_n\,dz + t_n \int_\Omega \lambda|u_n|^p\,dz \leqslant \epsilon_m \quad \text{for all } n \in \mathbb{N}. \tag{3.29}$$

Since $\epsilon_n \to 0^+$, by choosing $n_0 \in \mathbb{N}$ even bigger if necessary, we can get

$$\epsilon_n \in (0, 1) \quad \text{for all } n \geqslant n_0. \tag{3.30}$$

By adding (3.28) and (3.29), and using (3.30), we have

$$(1 - t_n)\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\,dz \leqslant (1 - t_n)\Big(1 - \frac{p}{2}\Big)\|Du_n\|_2^2.$$

We may assume that $t_n \neq 1$ for all $n \in \mathbb{N}$. Otherwise, $t = 1$, and so $\lambda_t = \lambda \notin \sigma(p)$, hence $y = 0$, a contradiction to (3.22). Then

$$\frac{1}{\|u_n\|^\tau} \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\,dz \leqslant \Big(1 - \frac{p}{2}\Big)\frac{1}{\|u_n\|^{\tau-2}}\|Dy_n\|_2^2 \quad \text{for all } n \in \mathbb{N}.$$

Since $p > \tau > 2$, it follows from (3.17) and (3.22) that

$$\limsup_{n\to\infty} \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\, dz \leqslant 0,$$

which contradicts (3.27). This proves the claim.

In fact, the above argument with minor changes shows that for every $t \in [0, 1]$, $h(t, \cdot)$ satisfies the C-condition. So, [9, Theorem 5.1.12] (see also [19, Proposition 3.2]) implies that

$$C_k(h(0, \cdot), \infty) = C_k(h(1, \cdot), \infty) \quad \text{for all } k \in \mathbb{N}_0,$$

and therefore

$$C_k(\varphi, \infty) = C_k(\psi, \infty) \quad \text{for all } k \in \mathbb{N}_0.$$

Since $\lambda \notin \hat{\sigma}(p)$, we have $K_\psi = \{0\}$, and so $C_k(\psi, \infty) = C_k(\psi, 0)$ for all $k \in \mathbb{N}_0$. Hence,

$$C_k(\varphi, \infty) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

But by [32, Proposition 1.1], we have $C_m(\psi, 0) \neq 0$. So, $C_m(\varphi, \infty) \neq 0$. □

In the second approach, we avoid the computation of the critical groups of φ at infinity. Instead we use the following result which is essentially due to Perera [32, Lemma 4.1], here adapted to our setting.

Proposition 3.5. *If Hypotheses 3.1* (i), (ii), (iii) *hold, then there exist $r > 0$ and $\varphi_0 \in C^1(W_0^{1,p}(\Omega))$ such that*

$$\varphi_0(u) = \begin{cases} \varphi(u) & \textit{if } \|u\| \leqslant r, \\ \psi(u) & \textit{if } \|u\| \geqslant 2^{1/p} r, \end{cases}$$

$K_{\varphi_0} = K_\varphi$ and $C_m(\varphi_0, \infty) \neq 0$.

Proof. Let $\psi \in C^2(W_0^{1,p}(\Omega))$ be as in the proof of Proposition 3.4. Also let $\tau\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\tau(u) = \int_\Omega F(z, u)\, dz - \frac{\lambda}{p}\|u\|_p^p - \frac{1}{2}\|Du\|_2^2 \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently, we have

$$\varphi(u) = \psi(u) - \tau(u) \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3.31}$$

Since $\lambda \notin \hat{\sigma}(p)$, the functional ψ satisfies the C-condition, and so

$$\mu = \inf\{\|\psi'(u)\|_* : u \in W_0^{1,p}(\Omega),\ \|u\| = 1\} > 0.$$

We have

$$\psi'(u) = A_p(u) - \lambda|u|^{p-2}u,$$

hence the $(p-1)$-homogeneity of $\psi'(\cdot)$ implies that

$$\inf\{\|\psi'(u)\|_* : u \in W_0^{1,p}(\Omega),\ \|u\| = r\} = r^{p-1}\mu > 0 \quad (r > 0). \tag{3.32}$$

Since $\lambda > \hat{\lambda}_m(p)$ and $p > 2$, it follows that

$$\limsup_{\|u\|\to\infty} \frac{\tau'(u)}{\|u\|^{p-1}} \leqslant 0. \tag{3.33}$$

From (3.31) we have

$$\varphi'(u) = \psi'(u) - \tau'(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Hence, using (3.32) and (3.33),

$$\varphi'(u) > 0 \quad \text{and} \quad \varphi'(u) + \tau'(u) > 0 \quad \text{for all } \|u\| > r. \tag{3.34}$$

Let $\xi\colon \mathbb{R}_+ \to [0,1]$ be a C^1-function such that $|\xi'(t)| \leqslant 1$ for all $t \geqslant 0$ and

$$\xi(t) = \begin{cases} 0 & \text{if } t \in [0,1], \\ 1 & \text{if } t \geqslant 2. \end{cases} \tag{3.35}$$

We define

$$\varphi_0(u) = \varphi(u) + \xi\Big(\frac{\|u\|^p}{r^p}\Big)\tau(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $\varphi_0 \in C^1(W_0^{1,p}(\Omega))$ and from (3.34) and (3.35), it follows that

$$\varphi_0(u) = \begin{cases} \varphi(u) & \text{if } \|u\| \leqslant r, \\ \psi(u) & \text{if } \|u\| \geqslant 2^{1/p} r, \end{cases} \qquad K_{\varphi_0} = K_\varphi \subseteq \bar{B}_r. \tag{3.36}$$

Moreover, by (3.36), it is clear that

$$C_k(\varphi_0, \infty) = C_k(\psi, \infty) \quad \text{for all } k \in \mathbb{N}_0,$$

and, since $K_\psi = \{0\}$ and $\lambda \notin \hat{\sigma}(p)$, we have

$$C_k(\varphi_0, \infty) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}.$$

Thus, $C_m(\varphi_0, \infty) \neq 0$, see [32, Proposition 1.1]. □

Next, we turn our attention to the critical groups of φ at the origin. To compute them we only need a subcritical growth on $f(z,\cdot)$ and the behavior of $f(z,\cdot)$ near zero. So, we introduce the following weaker set of hypotheses on $f(z,x)$.

Hypotheses 3.6. $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the following properties:
(i) $|f(z,x)| \leqslant a(z)(1 + |x|^{r-1})$ for almost all $z \in \Omega$ and all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $p \leqslant r < p^*$.
(ii) There exist $l \in \mathbb{N}$, $\delta > 0$ and $\eta \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_l(2) \leqslant \eta(z) \quad \text{for almost all } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_l(2),$$
$$\eta(z)x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_{l+1}(2)x^2 \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant \delta,$$

and for every $x \neq 0$, the second inequality is strict on a set of positive Lebesgue measure.

Proposition 3.7. *If Hypotheses 3.6 hold and the functional φ satisfies the C-condition, then $C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ with $d_l = \dim \bar{H}_l$.*

Proof. We consider the C^2-functional $\hat{\psi}\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

We set $\psi = \hat{\psi}|_{W_0^{1,p}(\Omega)}$ (recall that $p > 2$).

Claim 3. $C_k(\psi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.

To prove this claim, let $\vartheta \in (\hat{\lambda}_l(2), \hat{\lambda}_{l+1}(2))$ and consider the C^2-functional $\tau\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\tau(u) = \frac{1}{2}\|Du\|_2^2 - \frac{\vartheta}{2}\|u\|_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

We also consider the homotopy $h(t,u)$ defined by

$$h(t,u) = (1-t)\hat{\psi}(u) + t\tau(u) \quad \text{for all } (t,u) \in [0,1] \times H_0^1(\Omega).$$

First consider $t \in (0,1]$. Let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$, where $\delta > 0$ is as in Hypothesis 3.6 (ii). Let $\langle \cdot,\cdot \rangle_0$ denote the duality brackets for the pair $(H^{-1}(\Omega), H_0^1(\Omega))$. Then we have

$$\langle h_u'(t,u), v\rangle = (1-t)\langle \hat{\psi}'(u), v\rangle_0 + t\langle \tau'(u), v\rangle_0 \quad \text{for all } v \in H_0^1(\Omega). \tag{3.37}$$

Recall that

$$\bar{H}_l = \bigoplus_{k=1}^{l} E(\hat{\lambda}_k(2)), \quad \hat{H}_{l+1} = \bar{H}_l^{\perp} = \bigoplus_{k\geqslant l+1} E(\hat{\lambda}_k(2))$$

and consider the orthogonal direct sum decomposition

$$H_0^1(\Omega) = \bar{H}_l \oplus \hat{H}_{l+1}.$$

So, every $u \in H_0^1(\Omega)$ admits a unique sum decomposition

$$u = \bar{u} + \hat{u}, \quad \text{with } \bar{u} \in \bar{H}_l,\ \hat{u} \in \hat{H}_{l+1}.$$

In (3.37) we choose $v = \hat{u} - \bar{u}$. Exploiting the orthogonality of the component spaces, we have

$$\langle \hat{\psi}(u), \hat{u} - \bar{u} \rangle_0 = \|D\hat{u}\|_2^2 - \|D\bar{u}\|_2^2 - \int_\Omega f(z, u)(\hat{u} - \bar{u})\, dz. \tag{3.38}$$

Hypothesis 3.6 (ii) implies that

$$\eta(z) \leqslant \frac{f(z, x)}{x} \leqslant \hat{\lambda}_{l+1}(2) \quad \text{for almost all } z \in \Omega \text{ and all } 0 < |x| \leqslant \delta, \tag{3.39}$$

and the second inequality is, for every $x \neq 0$, strict on a set of positive Lebesgue measure. Set $y = \hat{u} - \bar{u}$. Then, using (3.39), we have

$$\begin{aligned} f(z, u)(\hat{u} - \bar{u}) = f(z, u)y = \frac{f(z, u)}{u} uy &\leqslant \begin{cases} \hat{\lambda}_{l+1}(2)(\hat{u}^2 - \bar{u}^2) & \text{if } uy \geqslant 0, \\ \eta(z)(\hat{u}^2 - \bar{u}^2) & \text{if } uy < 0 \end{cases} \\ &\leqslant \hat{\lambda}_{l+1}(2)\hat{u}^2 - \eta(z)\bar{u}^2 \quad \text{for almost all } z \in \Omega. \end{aligned} \tag{3.40}$$

Returning to (3.38) and using (3.40), we obtain (see Hypothesis 3.6 (ii) and (2.4))

$$\langle \hat{\psi}'(u), \hat{u} - \bar{u} \rangle_0 \geqslant \|D\hat{u}\|_2^2 - \hat{\lambda}_{l+1}(2)\|\hat{u}\|_2^2 - [\|D\bar{u}\|_2^2 - \hat{\lambda}_l(2)\|\bar{u}\|_2^2] \geqslant 0. \tag{3.41}$$

Also, using Lemma 2.2, for some $c_9 > 0$, we have

$$\langle \tau'(u), \hat{u} - \bar{u} \rangle_0 = \|D\hat{u}\|_2^2 - \vartheta\|\hat{u}\|_2^2 - [\|D\bar{u}\|_2^2 - \vartheta\|\bar{u}\|_2^2] \geqslant c_9\|u\|^2. \tag{3.42}$$

So, if we use (3.41) and (3.42) in (3.37), then

$$\langle h_u'(t, u), \hat{u} - \bar{u} \rangle \geqslant tc_9\|u\|^2 > 0 \quad \text{for all } t \in (0, 1].$$

Standard regularity theory implies that

$$K_{h(t,\cdot)} \subseteq C_0^1(\overline{\Omega}) \quad \text{for all } t \in [0, 1].$$

Therefore, we infer that for all $t \in (0, 1]$, $u = 0$ is isolated in $K_{h(t,\cdot)}$.

We have $h(0, \cdot) = \hat{\psi}(\cdot)$. Next, we show that $0 \in K_{\hat{\psi}}$ s isolated. Arguing by contradiction, suppose that we could find $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(\Omega)$ such that

$$u_n \to 0 \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad \hat{\psi}'(u_n) = 0 \quad \text{for all } n \in \mathbb{N}_0. \tag{3.43}$$

From the equation in (3.43), we have

$$-\Delta u_n(z) = f(z, u_n(z)) \quad \text{for almost all } z \in \Omega, \qquad u_n|_{\partial\Omega} = 0, \qquad n \in \mathbb{N}. \tag{3.44}$$

From (3.44) and standard regularity theory (see, for example, [16, pp. 737–738]), we can find $\alpha \in (0, 1)$ and $c_{10} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_{10} \quad \text{for all } n \in \mathbb{N}. \tag{3.45}$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (3.45) and (3.43), we obtain

$$u_n \to 0 \quad \text{in } C_0^1(\overline{\Omega}).$$

Therefore, we can find $n_0 \in \mathbb{N}$ such that

$$|u_n(z)| \leqslant \delta \quad \text{for all } n \geqslant n_0 \text{ and all } z \in \overline{\Omega},$$

hence (see Hypothesis 3.6 (ii))

$$\eta(z)u_n(z)^2 \leqslant f(z, u_n(z))u_n(z) \leqslant \hat{\lambda}_{l+1}(2)u_n(z)^2 \quad \text{for almost all } z \in \Omega \text{ and all } n \geqslant n_0.$$

Then from (3.45) and the previous argument, we have

$$f(z, u_n(z))(\hat{u}_n - \bar{u}_n)(z) \leqslant \hat{\lambda}_{l+1}(2)\hat{u}_n(z)^2 - \eta(z)\bar{u}_n(z)^2 \quad \text{for almost all } z \in \Omega \text{ and all } n \geqslant n_0. \tag{3.46}$$

From (3.44) we have

$$\langle A(u_n), v\rangle = \int_\Omega f(z, u_n)v \, dz \quad \text{for all } v \in H_0^1(\Omega).$$

Choosing $v = \hat{u}_n - \bar{u}_n \in H_0^1(\Omega)$ and using the orthogonality of the component spaces and (3.46), we obtain

$$\int_\Omega (Du_n, D\hat{u}_n - D\bar{u}_n)_{\mathbb{R}^N} \, dz = \|D\hat{u}_n\|_2^2 - \|D\bar{u}_n\|_2^2 = \int_\Omega f(z, u_n)(\hat{u}_n - \bar{u}_n) \, dz \leqslant \int_\Omega [\hat{\lambda}_{l+1}(2)\hat{u}_n^2 - \eta(z)\bar{u}_n^2] \, dz.$$

Hence, by (2.4) and Lemma 2.2 (a),

$$0 \leqslant \|D\hat{u}_n\|_2^2 - \hat{\lambda}_{l+1}(2)\|\hat{u}_n\|_2^2 \leqslant \|D\hat{u}_n\|_2^2 - \int_\Omega \eta(z)\bar{u}_n^2 \, dz \leqslant -c_{11}\|\bar{u}_n\|^2 \quad \text{for all } n \geqslant n_0,$$

for some $c_{11} > 0$. Therefore,

$$\bar{u}_n = 0 \quad \text{and} \quad \hat{u}_n \in E(\hat{\lambda}_{l+1}(2)) \quad \text{for all } n \in \mathbb{N}.$$

Then $u_n = \hat{u}_n$ for all $n \geqslant n_0$, and the UCP implies that

$$u_n(z) \neq 0 \quad \text{for almost all } z \in \Omega \text{ and all } n \geqslant n_0. \tag{3.47}$$

From (3.44) and (3.47) we have (see Hypothesis 3.6 (ii))

$$\hat{\lambda}_{l+1}(2)\|u_n\|_2^2 = \int_\Omega f(z, u_n)u_n \, dz < \hat{\lambda}_{l+1}(2)\|u_n\|_2^2 \quad \text{for all } n \geqslant n_0,$$

a contradiction. Therefore, $0 \in K_{\hat{\psi}}$ is isolated and we can conclude that $0 \in K_{h(t,\cdot)}$ is isolated for all $t \in [0, 1]$.

So, [12, Theorem 5.2] implies that

$$C_k(\hat{\psi}, 0) = C_k(\tau, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

and thus (see [23, Theorem 6.51])

$$C_k(\hat{\psi}, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that (see [25] and [8, p. 14])

$$C_k(\hat{\psi}, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

and thus

$$C_k(\psi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.48}$$

We have

$$|\varphi(u) - \psi(u)| \leqslant \frac{1}{p}\|u\|^p \tag{3.49}$$

and

$$\langle \varphi'(u) - \psi'(u), v\rangle| \leqslant c_{12}\|u\|^{p-1}\|v\| \quad \text{for all } v \in H_0^1(\Omega),$$

for some $c_{12} > 0$, which implies

$$\|\varphi'(u) - \psi'(u)\|_* \leqslant c_{12}\|u\|^{p-1}. \tag{3.50}$$

Then (3.49), (3.50) and the C^1-continuity of the critical groups (see [12, Theorem 5.1]), imply that

$$C_k(\varphi, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

and hence (see (3.48))

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

This completes the proof. □

Now we are ready for the existence theorem.

Theorem 3.8. *If Hypotheses 3.1 hold, then problem* (1.1) *admits a nontrivial solution* $u_0 \in C_0^1(\overline{\Omega})$.

Proof. As we have already mentioned, we can use two approaches.

In the first, we use Proposition 3.4 and have that $C_m(\varphi, \infty) \neq 0$. So, there exists $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_\varphi \quad \text{and} \quad C_m(\varphi, u_0) \neq 0. \tag{3.51}$$

On the other hand, from Proposition 3.7, we have

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.52}$$

Recalling that $d_l \neq m$ (see Hypothesis 3.1 (iv)) and comparing (3.51) and (3.52), we see that $u_0 \neq 0$.

In the second approach, we use Proposition 3.5. According to that result, we have $C_m(\varphi_0, \infty) \neq 0$. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_{\varphi_0} \quad \text{and} \quad C_m(\varphi_0, u_0) \neq 0. \tag{3.53}$$

Note that $\varphi_0|_{\bar{B}_r} = \varphi|_{\bar{B}_r}$ (see Proposition 3.5). So,

$$C_k(\varphi_0, 0) = C_k(\varphi, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

and thus (see Proposition 3.7)

$$C_k(\varphi_0, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{3.54}$$

Again, since $d_l \neq m$, from (3.53) and (3.54), it follows that (see Proposition 3.5)

$$u_0 \neq 0 \quad \text{and} \quad u_0 \in K_\varphi.$$

So, with both approaches we produced a nontrivial critical point u_0 of the functional φ. Then u_0 is a nontrivial solution of (1.1). Invoking [18, Theorem 7.1], we have $u_0 \in L^\infty(\Omega)$. So, we apply [20, Theorem 1] and conclude that $u_0 \in C^1(\overline{\Omega})$. □

4 Multiple Nontrivial Solutions

In this section we strengthen the conclusions on the reaction term $f(z, x)$ and prove a multiplicity theorem. More precisely, the new conditions on $f(z, x)$ are the following.

Hypotheses 4.1. $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the following properties:

(i) For every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leqslant a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant \rho.$$

(ii) There exists an integer $m \geqslant 1$ such that

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = \hat{\lambda}_m(p) \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) There exists $\tau \in (2, p)$ such that

$$0 < \beta_0 \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\tau} \quad \text{uniformly for almost all } z \in \Omega,$$

where $F(z,x) = \int_0^x f(z,s)\,ds$.

(iv) There exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and constants $c_\pm \in \mathbb{R}$ such that

$$\begin{aligned} w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) &\quad \text{for all } z \in \overline{\Omega},\\ f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) &\quad \text{for almost all } z \in \Omega,\\ A_p(w_-) + A(w_-) \leqslant 0 \leqslant A_p(w_+) + A(w_+) &\quad \text{in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*. \end{aligned}$$

(v) There exist an integer $l \geqslant 1$ with $d_l \neq m$ ($d_l = \dim \bar{H}_l$), $\delta > 0$ and $\eta \in L^\infty(\Omega)$ such that

$$\begin{aligned} \hat{\lambda}_l(2) \leqslant \eta(z) &\quad \text{for almost all } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_l(2),\\ \eta(z)x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_{l+1}(2)x^2 &\quad \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant \delta, \end{aligned}$$

and for $x \neq 0$, the second inequality is strict on a set of positive Lebesgue measure.

(vi) For every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$, the function $z \mapsto f(z,x) + \hat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

Remark 4.2. We see that in comparison to the Hypotheses 3.1, we have added Hypotheses 4.1 (iv), (vi). So, the problem remains resonant at both $\pm\infty$ and at zero. Hypothesis 3.1 (iv) is satisfied if, for example, we can find $c_- < 0 < c_+$ such that

$$f(z, c_+) \leqslant 0 \leqslant f(z, c_-) \quad \text{for almost all } z \in \Omega.$$

Therefore, this hypothesis implies that near zero $f(z,\cdot)$ exhibits an oscillatory behavior.

First, we produce two constant sign solutions.

Proposition 4.3. *If Hypotheses 4.1* (i), (iv), (v), (vi) *hold, then problem* (1.1) *admits two nontrivial smooth solutions of constant sign:*

$$\begin{aligned} u_0 \in \operatorname{int} C_+, &\quad \textit{with } u_0(z) < w_+(z) \textit{ for all } z \in \overline{\Omega},\\ v_0 \in -\operatorname{int} C_+, &\quad \textit{with } w_-(z) < v_0(z) \textit{ for all } z \in \overline{\Omega}. \end{aligned}$$

Proof. First, we produce the positive solution.

We introduce the following Carathéodory function:

$$\hat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0,\\ f(z,x) & \text{if } 0 \leqslant x \leqslant w_+(z),\\ f(z, w_+(z)) & \text{if } w_+(z) < x. \end{cases} \tag{4.1}$$

We set $\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)\,ds$ and consider the C^1-functional $\hat{\varphi}_+\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{F}_+(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (4.1) it is clear that $\hat{\varphi}_+$ is coercive. Also, using the Sobolev embedding theorem, we see that $\hat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(u_0) = \inf\{\hat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{4.2}$$

From (4.2) we have $\hat{\varphi}_+'(u_0) = 0$, and hence

$$\langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle = \int_\Omega \hat{f}_+(z, u_0)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{4.3}$$

In (4.3) we first choose $h = -u_0^- \in W_0^{1,p}(\Omega)$. Then $\|Du_0^-\|_p^p + \|Du_0^-\|_2^2 = 0$ (see (4.1)), and thus $u_0 \geqslant 0$. Also, in (4.3) we choose $h = (u_0 - w_+)^+ \in W_0^{1,p}(\Omega)$. Then, by (4.1) and Hypothesis 4.1 (iv),

$$\begin{aligned}
\int_\Omega |Du_0|^{p-2}(Du_0, D(u_0 - w_+)^+)_{\mathbb{R}^N}\,dz &+ \int_\Omega (Du_0, D(u_0 - w_+)^+)_{\mathbb{R}^N}\,dz \\
&= \int_\Omega f(z, w_+)(u_0 - w_+)^+\,dz \\
&\leqslant \int_\Omega |Dw_+|^{p-2}(Dw_+, D(u_0 - w_+)^+)_{\mathbb{R}^N}\,dz + \int_\Omega (Dw_+, D(u_0 - w_+)^+)_{\mathbb{R}^N}\,dz.
\end{aligned}$$

Thus,

$$\int_\Omega (|Du_0|^{p-2}Du_0 - |Dw_+|^{p-2}Dw_+, D(u_0 - w^+)^+)_{\mathbb{R}^N}\,dz + \|D(u_0 - w_+)^+\|_2^2 \leqslant 0,$$

and hence $u_0 \leqslant w_+$.

So, we have proved that

$$u_0 \in [0, w_+] = \{y \in W_0^{1,p}(\Omega) : 0 \leqslant y(z) \leqslant w_+(z) \text{ for almost all } z \in \Omega\}.$$

Then, on account of (4.1), equation (4.3) becomes

$$\langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle = \int_\Omega f(z, u_0)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

which implies

$$-\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) \quad \text{for almost all } z \in \Omega, \qquad u_0|_{\partial\Omega} = 0, \tag{4.4}$$

and hence $u_0 \in C_+$ (by the nonlinear regularity theory, see [20]).

Since $p > 2$, given $\epsilon > 0$, we can find $\delta_0 \in (0, \min\{\delta, C_+\})$ ($\delta > 0$ as in Hypothesis 4.1 (v)) such that

$$\frac{1}{p}|y|^p \leqslant \frac{\epsilon}{2}|y|^2 \quad \text{for all } y \in \mathbb{R}^N \text{ with } |y| \leqslant \delta_0. \tag{4.5}$$

Recall that $\hat{u}_1(2) \in \operatorname{int} C_+$. So, we can find small $t \in (0, 1)$ such that

$$\|t\hat{u}_1(2)\|_{C_0^1(\overline{\Omega})} \leqslant \delta_0.$$

By (4.5), (4.1) and Hypothesis 4.1 (v), since $\delta_0 \leqslant \delta$, we have

$$\begin{aligned}
\hat{\varphi}_+(t\hat{u}_1(2)) &\leqslant \frac{\epsilon + 1}{2}t^2\|D\hat{u}_1(2)\|_2^2 - \frac{1}{2}t^2\int_\Omega \eta(z)\hat{u}_1(2)^2\,dz \\
&= t^2\Big[\frac{\epsilon}{2}\hat{\lambda}_1(2)\|\hat{u}_1(2)\|_2^2 - \frac{1}{2}\int_\Omega (\eta(z) - \hat{\lambda}_1(2))\hat{u}_1(2)^2\,dz)\Big] \\
&< 0,
\end{aligned}$$

by choosing $\epsilon > 0$ small enough (see Lemma 2.2 (b)). Then $\hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0)$ (see (4.2)), and hence $u_0 \neq 0$.

Let $\rho = \|u_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by Hypothesis 4.1 (vi). Then, by (4.4), we have

$$\Delta_p u_0(z) + \Delta u_0(z) \leqslant \hat{\xi}_\rho u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega. \tag{4.6}$$

Let $V(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$. Evidently,

$$\operatorname{div}(V(Du)) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have $V \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\nabla V(y) = |y|^{p-2}\Big[I + (p-2)\frac{y \oplus y}{|y|^2}\Big] + I,$$

which implies

$$(\nabla V(y)\xi, \xi)_{\mathbb{R}^N} \geqslant |\xi|^2 \quad \text{for all } y \in \mathbb{R}^N \text{ and all } \xi \in \mathbb{R}^N. \tag{4.7}$$

Then (4.7), (4.6) and the tangency principle of Pucci and Serrin [33, Theorem 2.5.2] imply that

$$0 < u_0(z) \quad \text{for all } z \in \Omega.$$

Next, using the boundary point lemma (see [33, Theorem 5.5.1]), we obtain

$$u_0 \in \operatorname{int} C_+. \tag{4.8}$$

Also, Hypothesis 4.1 (iv) implies

$$A_p(u_0) + A(u_0) - N_f(u_0) = 0 \leqslant A_p(w_+) + A(w_+) - N_f(w_+) \quad \text{in } W^{-1,p'}(\Omega). \tag{4.9}$$

So, once more (4.7), (4.9) and the tangency principle of Pucci and Serrin [33, Theorem 2.5.2], imply that

$$u_0(z) < w_+(z) \quad \text{for all } z \in \Omega,$$

and, by Hypothesis 4.1 (iv), we have

$$u_0(z) < w_+(z) \quad \text{for all } z \in \overline{\Omega}.$$

Similarly, to produce the negative solution, we introduce the Carathéodory function

$$\hat{f}_-(z, x) = \begin{cases} f(z, w_-(z)) & \text{if } x < w_-(z), \\ f(z, x) & \text{if } w_-(z) \leqslant x \leqslant, 0 \\ 0 & \text{if } 0 < x. \end{cases}$$

We set $\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s)\,ds$ and consider the C^1-functional $\hat{\varphi}_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{F}_-(z, u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working with $\hat{\varphi}_-$ and using (4.8), we produce a solution of (1.1), $v_0 \in W_0^{1,p}(\Omega)$, such that

$$v_0 \in -\operatorname{int} C_+, \quad w_-(z) < v_0(z) \quad \text{for all } z \in \overline{\Omega}.$$

This completes the proof. □

In fact, we can show that we have extremal constant sign solutions in the order intervals $[0, w_+]$ and $[w_-, 0]$, that is, we can show that there is a smallest positive solution $u_* \in \operatorname{int} C_+$ in $[0, w_+]$ and a biggest negative solution $v_* \in -\operatorname{int} C_+$ in $[w_-, 0]$.

Proposition 4.4. *If Hypotheses 4.1* (i), (iv), (v), (vi) *hold, then problem* (1.1) *admits a smallest positive solution* $u_* \in \operatorname{int} C_+$ *in* $[0, w_+]$ *and a biggest negative solution* $v_* \in -\operatorname{int} C_+$ *in* $[w_-, 0]$.

Proof. First we produce the smallest positive solution in $[0, w_+]$. Let $\hat{S}_+$ be the set of positive solutions of problem (1.1) in the order interval $[0, w_+]$. From Proposition 4.3 and its proof, we have

$$\hat{S}_+ \neq \emptyset \quad \text{and} \quad \hat{S}_+ \subseteq [0, w_+] \cap \operatorname{int} C_+.$$

Invoking [17, Lemma 3.10, p. 178], we infer that we can find $\{u_n\}_{n\geqslant 1} \subseteq \hat{S}_+$ such that $\inf \hat{S}_+ = \inf_{n\geqslant 1} u_n$.

We have

$$A_p(u_n) + A(u_n) = N_f(u_n), \quad 0 \leqslant u_n \leqslant w_+ \text{ for all } n \in \mathbb{N}, \tag{4.10}$$

which implies that $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad u_n \to u_* \quad \text{in } L^p(\Omega) \text{ as } n \to \infty. \tag{4.11}$$

On (4.10) we act with $u_n - u_* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.11). Then

$$\lim_{n\to\infty} [\langle A_p(u_n), u_n - u_*\rangle + \langle A(u_n), u_n - u_*\rangle] = 0,$$

and since A is monotone, we have

$$\limsup_{n\to\infty} [\langle A_p(u_n), u_n - u_*\rangle + \langle A(u_*), u_n - u_*\rangle] \leqslant 0.$$

Thus, by (4.11), $\limsup_{n\to\infty}\langle A_p(u_n), u_n - u_*\rangle \leqslant 0$, which implies (see Proposition 2.3)

$$u_n \to u_* \quad \text{in } W_0^{1,p}(\Omega). \tag{4.12}$$

Passing to the limit as $n \to \infty$ in (4.10) and using (4.12), we obtain

$$A_p(u_*) + A(u_*) = N_f(u_*), \quad 0 \leqslant u_* \leqslant w_+.$$

Hence,

$$-\Delta_p u_*(z) - \Delta u_*(z) = f(z, u_*(z)) \quad \text{for almost all } z \in \Omega, \qquad u_*|_{\partial\Omega} = 0, \qquad 0 \leqslant u_* \leqslant w_+.$$

Then $u_* \in C_+$ (by the nonlinear regularity theory, see [20]) is a nonnegative solution of (1.1). If we can show that $u_* \neq 0$, then $u_* \in \hat{S}_+$ and $u_* = \inf \hat{S}_+$.

To this end, we proceed as follows. Hypotheses 4.1 (i), (v) imply that we can find $c_{13} > 0$ such that

$$f(z, x) \geqslant \eta(z)x - c_{13}x^{p-1} \quad \text{for almost all } z \in \Omega \text{ and all } 0 \leqslant x \leqslant w_+(z). \tag{4.13}$$

Let $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$g(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ \eta(z)x - c_{13}x^{p-1} & \text{if } 0 \leqslant x \leqslant w_+(z), \\ \eta(z)w_+(z) - c_{13}w_+(z)^{p-1} & \text{if } w_+(z) < x. \end{cases} \tag{4.14}$$

We consider the auxiliary Dirichlet problem

$$-\Delta_p u(z) - \Delta u(z) = g(z, u(z)) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{4.15}$$

We claim that this problem has a unique solution $\bar{u} \in \operatorname{int} C_+$. First, we show the existence of a nontrivial solution. So, let $\psi_+\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (4.15) defined by

$$\psi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G(z, u)\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $G(z, x) = \int_0^x g(z, s)\, ds$. Evidently, ψ_+ is coercive (see (4.14)) and sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi_+(\bar{u}) = \inf\{\psi_+(u) : u \in W_0^{1,p}(\Omega)\}.$$

As in the proof of Proposition 4.3, using Hypothesis 4.1 (v), we have (see (4.14))

$$\psi_+(\bar{u}) < 0 = \psi_+(0) \quad \text{and} \quad \bar{u} \in [0, w_+],$$

hence $\bar{u} \in K_{\psi_+} \subseteq [0, w_+] \cap \operatorname{int} C_+$.

Next, we show that this solution is unique. For this purpose, we consider the integral functional $j\colon L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|Du^{1/2}\|_p^p + \frac{1}{2}\|Du^{1/2}\|_2^2 & \text{if } u \geqslant 0,\ u^{1/2} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

By [7, Lemma 4] and [13, Lemma 1], we have that $j(\cdot)$ is convex. Suppose that $\bar{y} \in W_0^{1,p}(\Omega)$ is another nontrivial solution of (4.15). Then again we have $\bar{y} \in [0, w_+] \cap \operatorname{int} C_+$. Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of j). For every $h \in C_0^1(\overline{\Omega})$, we have

$$\bar{u}^2 + th \in \operatorname{dom} j \qquad \text{and} \qquad \bar{y}^2 + th \in \operatorname{dom} j \quad \text{for } |t| \leqslant 1 \text{ small.}$$

Then we can easily see that $j(\cdot)$ is Gâteaux differentiable at $\bar{u}^2$ and at $\bar{y}^2$ in the direction h. Moreover, using the nonlinear Green's identity (see, for example, [16, p. 211]), we have

$$j'(\bar{u}^2)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \bar{u} - \Delta \bar{u}}{\bar{u}} h\,dz = \frac{1}{2}\int_\Omega [\eta(z) - c_{13}\bar{u}^{p-2}]h\,dz,$$
$$j'(\bar{y}^2)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \bar{y} - \Delta \bar{y}}{\bar{y}} h\,dz = \frac{1}{2}\int_\Omega [\eta(z) - c_{13}\bar{y}^{p-2}]h\,dz,$$

see (4.15) and (4.14). The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence,

$$0 \leqslant \int_\Omega [\bar{y}^{p-2} - \bar{u}^{p-2}](\bar{u}^2 - \bar{y}^2)\,dz \implies \bar{u} = \bar{y}.$$

This proves the uniqueness of the nontrivial solution $\bar{u} \in [0, w_+] \cap \operatorname{int} C_+$ of the auxiliary problem (4.15).

Claim 4. $\bar{u} \leqslant u$ for all $u \in \hat{S}_+$.

Let $u \in \hat{S}_+$ and consider the Carathéodory function $k\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ \eta(z)x - c_{13}x^{p-1} & \text{if } 0 \leqslant x \leqslant u(z), \\ \eta(z)u(z) - c_{13}u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{4.16}$$

We set $K(z, x) = \int_0^x k(z, s)\,ds$ and consider the C^1-functional $\hat{\psi}_+\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega K(z, u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Again, $\hat{\psi}_+$ is coercive (see (4.16)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_+(\tilde{u}) = \inf\{\hat{\psi}_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{4.17}$$

Let $t \in (0, 1)$ be small such that $t\hat{u}_1(2) \leqslant u$ (see [22, Proposition 2.1] and recall that $u \in \operatorname{int} C_+$). Then, by taking $t \in (0, 1)$ even smaller if necessary and using Hypothesis 4.1 (v), we have $\hat{\psi}_+(t\hat{u}_1(2)) < 0$, which implies $\hat{\psi}_+(\tilde{u}) < \hat{\psi}_+(0) = 0$, hence $\tilde{u} \neq 0$. Using (4.13) and the fact that $u \in \hat{S}_+$, we can show that $K_{\hat{\psi}_+} \subseteq [0, u]$. From (4.17) we have $\tilde{u} \in K_{\hat{\psi}_+} \setminus \{0\} \subseteq [0, u] \setminus \{0\}$, which implies (see (4.16) and recall that $\bar{u}$ is the unique solution of (4.15)) $\tilde{u} = \bar{u}$. Thus,

$$\bar{u} \leqslant u \quad \text{for all } u \in \hat{S}_+.$$

This proves the claim.

On account of Claim 4, we have $\bar{u} \leqslant u_*$, and so

$$u_* \in \hat{S}_+, \quad u_* = \inf \hat{S}_+.$$

Similarly, if $\hat{S}_-$ is the set of negative solutions of (1.1) in $[w_-, 0]$, then

$$\hat{S}_- \neq \emptyset \quad \text{and} \quad \hat{S}_- \subseteq [w_-, 0] \cap (-\operatorname{int} C_+)$$

(see Proposition 4.3 and its proof). Reasoning as above, we can show that there exists $v_* \in [w_-, 0] \cap (-\operatorname{int} C_+)$ which is the biggest negative solution of (1.1) in $[w_-, 0]$. □

Using these extremal constant sign solutions of (1.1), we can generate a nodal (that is, sign changing) solution. To do this, we need a slightly stronger condition on $f(z,\cdot)$ near zero (see Hypothesis 4.1 (v)). The new hypotheses on the reaction $f(z,x)$ are the following.

Hypotheses 4.5. The conditions on the Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are the same as in Hypotheses 4.1, the only difference being that here we have $l \geqslant 2$.

Proposition 4.6. *If Hypotheses 4.5* (i), (iv), (v), (vi) *hold, then problem* (1.1) *admits a nodal solution y_0 in* $[v_*, u_*] \cap C_0^1(\bar{\Omega})$.

Proof. Let $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions of (1.1) produced in Proposition 4.4. Let $e\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$e(z,x) = \begin{cases} f(z, v_*(z)) & \text{if } x < v_*(z), \\ f(z,x) & \text{if } v_*(z) \leqslant x \leqslant u_*(z), \\ f(z, u_*(z)) & \text{if } u_*(z) < x. \end{cases} \tag{4.18}$$

We set $E(z,x) = \int_0^x e(z,s)\,ds$, and consider the C^1-functional $\tau\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tau(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega E(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also, we consider the positive and negative truncations of $e(z,\cdot)$, namely the Carathéodory functions

$$e_\pm(z,x) = e(z, \pm x^\pm).$$

We set $E_\pm(z,x) = \int_0^x e_\pm(z,s)\,ds$ and consider the C^1-functionals $\tau_\pm\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tau_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega E_\pm(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

As before (see the proof of Proposition 4.3), using (4.18), we can show that

$$K_\tau \subseteq [v_*, u_*], \quad K_{\tau_+} \subseteq [0, u_*], \quad K_{\tau_-} \subseteq [v_*, 0].$$

The extremality of $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ implies that

$$K_\tau \subseteq [v_*, u_*], \quad K_{\tau_+} = \{0, u_*\}, \quad K_{\tau_-} = \{0, v_*\}. \tag{4.19}$$

Claim 5. $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of τ.

The functional τ_+ is coercive (see (4.18)) and sequentially weakly lower semicontinuous. So, we can find $\hat{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\tau_+(\hat{u}_*) = \inf\{\tau_+(u) : u \in W_0^{1,p}(\Omega)\}.$$

As in the proof of Proposition 4.4 (see the part of the proof after (4.17)), we have $\tau_+(\hat{u}_*) < 0 = \tau_+(0)$, hence $\hat{u}_* \neq 0$. Since $\hat{u}_* \in K_{\tau_+} = \{0, u_*\}$, it follows that $\hat{u}_* = u_* \in \operatorname{int} C_+$ (see (4.19)). Note that $\tau|_{C_+} = \tau_+|_{C_+}$, which implies that $u_* \in \operatorname{int} C_+$ is a local $C_0^1(\overline{\Omega})$-minimizer of τ. Hence, by Proposition 2.4, $u_* \in \operatorname{int} C_+$ is a local $W_0^{1,p}(\Omega)$-minimizer of τ. Similar arguments apply for $v_* \in -\operatorname{int} C_+$, using this time the functional τ_-. This proves Claim 5.

We may assume that

$$\tau(v_*) \leqslant \tau(u_*).$$

The reasoning is similar if the opposite inequality holds. Also, we may assume that K_τ is finite. Indeed, if K_τ is infinite, then on account of (4.19), we see that we already have an infinity of nodal solutions, which belong to $C_0^1(\overline{\Omega})$ (nonlinear regularity theory). Then Claim 5 implies that we can find $\rho \in (0,1)$ small such that

$$\tau(v_*) \leqslant \tau(u_*) < \inf\{\tau(u) : \|u - u_*\| = \rho\} = m_\rho, \quad \|v_* - u_*\| > \rho \tag{4.20}$$

(see the proof of [1, Proposition 29]). The functional $\tau(\cdot)$ is coercive (see (4.18)) and so $\tau(\cdot)$ satisfies the C-condition (see [31]). Therefore, from (4.20), we see that we can apply Theorem 2.1 (the mountain pass theorem). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_\tau \quad \text{and} \quad m_\rho \leqslant \tau(y_0). \tag{4.21}$$

From (4.19), (4.20), (4.21) and the nonlinear regularity theory (see [20]), we infer that

$$y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega}), \quad y_0 \notin \{v_*, u_*\}.$$

Also, from [23, Corollary 6.81], we have

$$C_1(\tau, y_0) \neq 0. \tag{4.22}$$

Let $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$\hat{f}(z,x) = \begin{cases} f(z, w_-(z)) & \text{if } x < w_-(z), \\ f(z,x) & \text{if } w_-(z) \leqslant x \leqslant w_+(z), \\ f(z, w_+(z)) & \text{if } w_+(z) < x. \end{cases}$$

We set $\hat{F}(t,x) = \int_0^x \hat{f}(z,s)\,ds$ and consider the C^1-functional $\hat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{F}(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From Proposition 3.7, we know that (recall that $d_l = \dim \bar{H}_l$)

$$C_k(\hat{\varphi}, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.23}$$

Claim 6. $C_k(\tau, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.

We consider the homotopy $h(t,u)$ defined by

$$h(t,u) = (1-t)\hat{\varphi}(u) + t\tau(u) \quad \text{for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Suppose we can find $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t, \qquad u_n \to 0 \quad \text{in } W_0^{1,p}(\Omega), \qquad h'_u(t_n, u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.24}$$

From the equality in (4.24) we have

$$A_p(u_n) + A(u_n) = (1-t_n)N_{\hat{f}}(u_n) + t_n N_\tau(u_n) \quad \text{for all } n \in \mathbb{N},$$

which implies

$$-\Delta_p u_n(z) - \Delta u_n(z) = (1-t_n)\hat{f}(z, u_n(z)) + t_n e(z, u_n(z)) \quad \text{for almost all } z \in \Omega, \qquad u_n|_{\partial\Omega} = 0. \tag{4.25}$$

By (4.24), (4.25) and [18, Theorem 7.1] (see also [23, Corollary 8.7]), we can find $c_{14} > 0$ such that

$$\|u_n\|_\infty \leqslant c_{14} \quad \text{for all } n \in \mathbb{N}. \tag{4.26}$$

Then, from (4.26) and [20, Theorem 1], we infer that there exist $\alpha \in (0, 1)$ and $c_{15} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_{15} \quad \text{for all } n \in \mathbb{N}.$$

Since $C_0^{1,\alpha}(\overline{\Omega})$ is embedding compactly in $C_0^1(\overline{\Omega})$, it follows that (see (4.24))

$$u_n \to 0 \quad \text{in } C_0^1(\overline{\Omega}),$$

which implies

$$u_n \in [v_*, u_*] \quad \text{for all } n \geqslant n_0,$$

and thus (see (4.19)) $\{u_n\}_{n \geqslant n_0} \subseteq K_\tau$, a contradiction to our hypothesis that K_τ is finite.

So, (4.24) cannot happen and this shows that $0 \in K_{h(t,\cdot)}$ is isolated uniformly in $t \in [0, 1]$. Hence, the homotopy invariance of critical groups, [12, Theorem 5.2], implies that

$$C_k(h(0, \cdot), 0) = C_k(h(1, \cdot), 0) \quad \text{for all } k \in \mathbb{N}_0,$$

and thus

$$C_k(\hat{\varphi}, 0) = C_k(\tau, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

Therefore, by (4.23),

$$C_k(\tau, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

This proves Claim 6.

Since $l \geqslant 2$ (see Hypotheses 4.5), we have $d_l \geqslant 2$. So, from Claim 6 and (4.22), it follows that $y_0 \neq 0$. Therefore, $y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega}) \setminus \{0\}$ is nodal. □

So far we have not used the asymptotic conditions at $\pm\infty$ (that is, Hypotheses 4.5 (ii), (iii)). Next, by using them, we will generate two more nontrivial smooth solutions of constant sign, for a total of five nontrivial smooth solutions all with sign information and ordered.

Theorem 4.7. *If Hypotheses 4.5 hold, then problem* (1.1) *admits the following five nontrivial smooth solutions:*

$$\begin{aligned} &u_0, \hat{u} \in \operatorname{int} C_+, && \hat{u} - u_0 \in C_+ \setminus \{0\}, \\ &v_0, \hat{v} \in -\operatorname{int} C_+, && v_0 - \hat{v} \in C_+ \setminus \{0\}, \\ &y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega}) && \textit{(nodal)}. \end{aligned}$$

Proof. Propositions 4.3 and 4.6 provide the following three nontrivial smooth solutions:

$$\begin{aligned} &u_0 \in [0, w_+] \cap \operatorname{int} C_+, && \text{with } (w_+ - u_0)(z) > 0 \text{ for all } z \in \overline{\Omega}, \\ &v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+), && \text{with } (u_0 - w_-)(z) > 0 \text{ for all } z \in \overline{\Omega}, \\ &y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega}) && \text{(nodal)}. \end{aligned}$$

On account of Proposition 4.4, we may assume that u_0 and v_0 are extremal constant sign solutions (that is, $u_0 = u_*$ and $v_0 = v_*$).

We consider the Carathéodory function $\gamma_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\gamma_+(z, x) = \begin{cases} f(z, u_0(z)) & \text{if } x \leqslant u_0(z), \\ f(z, x) & \text{if } u_0(z) < x, \end{cases} \tag{4.27}$$

and set $\Gamma_+(z, x) = \int_0^x \gamma_+(z, s)\, ds$. We consider the C^1-functional $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \Gamma_+(z, u)\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using (4.27), we can easily show that

$$K_{\sigma_+} \subseteq [u_0] = \{u \in W_0^{1,p}(\Omega) : u_0(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}. \tag{4.28}$$

Note that $u_0 \in K_{\sigma_+}$. We may assume that

$$K_{\sigma_+} \cap [u_0, w_+] = \{u_0\}. \tag{4.29}$$

Otherwise, we already have a second positive solution $\hat{u} \geqslant u_0$, $\hat{u} \neq u_0$, $\hat{u} \in C_0^1(\overline{\Omega})$. Consider the following Carathéodory function:

$$\hat{\gamma}_+(z,x) = \begin{cases} \gamma_+(z,x) & \text{if } x \leqslant w_+(z), \\ \gamma_+(z, w_+(z)) & \text{if } w_+(z) < x. \end{cases} \tag{4.30}$$

We set $\hat{\Gamma}_+(z,x) = \int_0^x \hat{\gamma}_+(z,s)\,ds$ and consider the C^1-functional $\hat{\sigma}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{\Gamma}_+(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (4.30) it is clear that $\hat{\sigma}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}_+(\tilde{u}_0) = \inf\{\hat{\sigma}_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{4.31}$$

Using (4.30), we can show that (see also (4.28))

$$K_{\hat{\sigma}_+} \subseteq [u_0, w_+]. \tag{4.32}$$

Then (4.29), (4.31) and (4.32) imply that

$$\tilde{u}_0 = u_0 \in [0, w_+], \quad (w_+ - u_0)(z) > 0 \quad \text{for all } z \in \overline{\Omega}. \tag{4.33}$$

From (4.30) we see that $\sigma_+|_{[0,w_+]} = \hat{\sigma}_+|_{[0,w_+]}$, which, in view of (4.33), implies that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of σ_+. Hence, u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of σ_+ (see Proposition 2.4).

Because of (4.28), we see that we may assume that K_{σ_+} is finite or otherwise we already have infinite positive and smooth (by the nonlinear regularity theory) solutions of (1.1), all bigger than u_0. Hence, we can find small $\rho \in (0,1)$ such that

$$\sigma_+(u_0) < \inf\{\sigma_+(u) : \|u - u_0\| = \rho\} = m_\rho^+. \tag{4.34}$$

Reasoning as in the proof of Proposition 3.3, we can establish that

$$\sigma_+ \text{ satisfies the C-condition.} \tag{4.35}$$

Note that in this case, due to (4.27), for any Cerami sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$, we have automatically that $\{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

Hypotheses 4.5 (i), (ii) imply that we can find $\vartheta > \hat{\lambda}_m(p)$ and $c_{16} > 0$ such that

$$F(z,x) \leqslant \frac{\vartheta}{p}x^p + c_{16} \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{4.36}$$

Since $\hat{u}_1(p) \in \operatorname{int} C_+$, we can find $t \geqslant 1$ big such that $t\hat{u}_1(p) \geqslant u_0$ (see [22, Proposition 2.1]). Then (see (4.36) and recall that $\|\hat{u}_1(p)\|_p = 1$)

$$\sigma(t\hat{u}_1(p)) \leqslant \frac{t^p}{p}\hat{\lambda}_1(p) + \frac{t^2}{2}\|D\hat{u}_1(p)\|_2^2 - \frac{t^p}{p}\vartheta + c_{17} = \frac{t^p}{p}[\hat{\lambda}_1(p) - \vartheta] + \frac{t^2}{2}\|D\hat{u}_1(p)\|_2^2 + c_{17}, \tag{4.37}$$

for some $c_{17} > 0$. Since $\vartheta > \hat{\lambda}_1(p)$ and $p > 2$, from (4.37), it follows that

$$\sigma(t\hat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{4.38}$$

Then (4.34), (4.35) and (4.38) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\sigma_+} \quad \text{and} \quad m_\rho^+ \leqslant \sigma_+(\hat{u}). \tag{4.39}$$

From (4.27), (4.28), (4.34) and (4.39), it follows that $u_0 \leqslant \hat{u}$, $\hat{u} \neq u_0$, and $\hat{u} \in \operatorname{int} C_+$ is a solution of (1.1).

Similarly, by working with $v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+)$ on the negative semiaxis as above, we produce $\hat{v} \in -\operatorname{int} C_+$, $\hat{v} \leqslant v_0$, $\hat{v} \neq v_0$, a second negative solution for problem (1.1). □

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