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Resolutions of 2-Polyhedra by Fake Surfaces and Embeddings into $\mathbb{R}^4$

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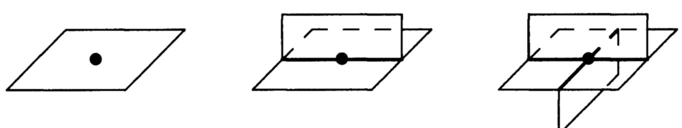
ABSTRACT. We present several corollaries (concerning embeddings of 2-polyhedra into $\mathbb{R}^4$) of the following result due to the second and the third author: *for every 2-polyhedron P there exists a fake surface Q and an onto map $f : Q \to P$ with contractible preimages.* We show that the class of fake surface in this theorem cannot be replaced by a certain interesting smaller class of 2-polyhedra. We announce (with indication of proof) that *any non-locally flat PL sphere $S^2 \subset \mathbb{R}^4$ has a regular neighborhood, non-homeomorphic to $S^2 \times D^2$.*

A finite 2-polyhedron Q is called a *fake surface* if each of its points has a neighbourhood homeomorphic to one of the following: D^2, (triod)$\times I$ or the cone over the complete graph with four vertices (see the figure below). We shall refer to these points as points of type 1, 2 and 3, respectively. Soap films in $\mathbb{R}^3$ exhibit singularities precisely of type 2 and 3. This notion of a soap film from differential geometry has proven to be an important tool and object of investigation in algebraic and geometric topology.



(1) (2) (3)

By Q' we shall denote the *intrinsic 1-skeleton* of a fake surface Q, i.e. the set of points of type 2 or 3. Obviously, Q' is a graph whose vertices have degree 1, 2 or 4. By Q'' we shall denote the finite set of points of Q, which have type 3. A fake surface Q is called a *special 2-polyhedron*, if both $Q - Q'$ and $Q' - Q''$ are *trivial*, i.e. they are a disjoint union of open 2- and 1- disks, respectively.

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The concept of fake surfaces and special polyhedra as 'general position' 2-polyhedra is formalized by the following result:

Theorem 0. *a) Every 3-manifold has a spine which is a special polyhedron* [HAMS93; Theorem I.3.1.a].

b) If two 3-manifolds have the same special spine, then they are homeomorphic [Cas65].

c) Every finite 2-polyhedron is homotopy equivalent to a special polyhedron [Wri77; Proposition 1].

d) For every 2-polyhedron P there exists a fake surface Q and an onto map $f : Q \to P$ with contractible preimages (they are actually either points or arcs or 2-disks). Moreover, if P is dimensionally homogeneous and has a trivial manifold set, then we can make Q to be special [ReSk00; Theorem 1].

A polyhedron P in a 3-manifold M is called a *spine* of M, if M (or $M - \text{int}D^3$, if M is closed) is a regular neighborhood of P. The *manifold set* of a 2-polyhedron is the set of points having a neighborhood homeomorphic to D^2. A 2-polyhedron P is said to be *dimensionaly homogeneous* if every point of P has an arbitrarily small 2-dimensional neighbourhood. Theorem 0.c can be derived from [Wri77; Proposition 1] (or from Theorem 0.d) by applying the construction from [HAMS93; p.37].

The purpose of this paper is to announce several new results related to Theorem 0. For most of them we give a complete proof, for some we present only the main idea of the proof. Our first result was motivated (like Theorem 0.b) by the following problem [Hor85] [Rep88] [HAMS93; I.49].

Problem. *Find conditions on a compact polyhedron P and a PL manifold M under which regular neighborhoods in M of each two homotopic PL embeddings $P \to M$ are PL homeomorphic.*

This problem can be regarded as a PL version of the Massey problem on uniqueness of the normal bundle for distinct embeddings of a manifold [Mas59].

Theorem 1. *a) There exists a special 2-polyhedron Q and two PL embeddings $Q \to \mathbb{R}^4$ with non-homeomorphic regular neighborhoods.*

b) Any non-locally flat PL sphere $S^2 \subset \mathbb{R}^4$ has a regular neighborhood, non-homeomorphic to $S^2 \times D^2$.

Theorem 1.b is true even for all $k \geq 2$ and any PL embedding $S^k \subset S^{k+2}$ having an isolated singularity $S^{k-1} \subset S^{k+1}$ such that $\pi_1(S^{k+1} - S^{k-1}) \not\cong \mathbb{Z}$. Theorem 1 is sharp in the sense that for $m \neq 4$, every special 2-polyhedron (or even fake surface) Q and every two homotopic PL embeddings of Q into an m-manifold, their regular neighborhoods are homeomorphic [LiSi69] [RBS99] [ReSk]. Also, every PL sphere $S^n \subset \mathbb{R}^m$ has a PL standard regular neighborhood if either $m - n \neq 2$ or $S^n \subset \mathbb{R}^m$ is locally flat [Wal67] [Zee63].

Theorem 1.a is a corollary of the ambient version of Theorem 0.d – for details see below. We shall illustrate the idea of proof of Theorem 1.b by proving the same result for *some* sphere $S^2 \subset S^4$, which already gives a counterexample to the above PL analogue of the Massey problem (for the complete proof see [ReSk]).

Proof of Theorem 1.a. Let P be the Dunce hat, i.e. the quotient space of the 2-dimensional triangle ABC obtained by the identification $AB = AC = BC$ (in this direction). By [Zee63] [Hor85] there are two PL embeddings $f, g : P \to \mathbb{R}^4$ with distinct regular neighborhoods M_1 and M_2. Let Q be the special resolution of P given by Theorem 2.a below. Since Q is a spine of both M_1 and M_2, these two embeddings $Q \subset M_i \subset \mathbb{R}^4$ are as desired.

Proof of Theorem 1.b (for *some* sphere $S^2 \subset S^4$). Take the suspension of a non-trivial knot $S^1 \subset S^3$ such that $\pi_1(S^3 - S^1) \neq \mathbb{Z}$ (in fact, any non-trivial knot satisfies this assumption). For a polyhedron $P \subset M$ we denote by $R_M(P)$ the regular neighborhood of P in M. Consider the decomposition $S^4 = D^4_+ \cup D^4_-$ such that $D^4_+ \cap D^4_- = \partial D^4_\pm = S^3$. Let $B^2_\pm = D^4_\pm \cap S^2$ be the cone over S^1. Let $B^4_\pm = R_{D^4_\pm}(B^2_\pm)$ so that $B^4_\pm \cap \partial D^4_\pm = R_{S^3}(S^1)$.

Denote $C_\pm = \partial B^4_\pm - \mathrm{Int}(R_{\partial B^4_\pm}(S^1))$. Then $R_{S^4}(S^2) = B^4_+ \cup B^4_-$ and $\partial R_{S^4}(S^2) = C_+ \cup C_-$. Since $B^2_\pm$ is a cone it follows that $D^4_\pm \searrow B^2_\pm$, therefore $D^4_\pm$ is also a regular neighborhood of $B^2_\pm$ in $D^4_\pm$. Hence by the uniqueness of regular neighborhoods, $(\partial B^4_\pm, S^1) \cong (S^3, S^1)$ and so $\pi_1(C_\pm) \cong \pi_1(S^3 - S^1)$.

The composition $C_+ \to C_+ \cup C_- \xrightarrow{r} C_+$ of the inclusion and the 'symmetric' retraction is a homeomorphism. So the induced composition $\pi_1(S^3 - S^1) \to \pi_1(C_+ \cup C_-) \xrightarrow{r_*} \pi_1(S^3 - S^1)$ is an isomorphism, hence r_* is an epimorphism. Therefore $\pi_1(\partial R_{S^4}(S^2)) \ncong \mathbb{Z}$ and we are done. □

Our second result (already used in the proof of Theorem 1.a) concerns the ambient version of Theorem 0.d.

Theorem 2. *a) For each $m \geq 4$ and every 2-polyhedron P there is an onto map $f : Q \to P$ with contractible preimages such that Q is a fake surface and any m-manifold M having P as the spine, also possesses a spine homeomorphic to Q.*

b) There is a 2-polyhedron P such that for each contractible resolution $f : Q \to P$ by a fake surface Q there is a 3-manifold M which has P as the spine, but possesses no spine homeomorphic to Q.

Theorem 2.a is proved analogously as Theorem 0.d, by using general position. Theorem 2.b follows by taking P to be a common spine of non-homeomorphic manifolds M_1 and M_2 (see examples in [Rep88] [HAMS93] [MPR89] [CLR97]). Since M_1 and M_2 are not homeomorphic, it follows by [RBS99] that they cannot have the same spine, which would be a fake surface.

Our third result is motivated by the well-known fact that every 2-manifold embeds into $\mathbb{R}^4$.

Theorem 3. *There exists a fake surface (even a special 2-polyhedron) Q which does not embed into $\mathbb{R}^4$.*

Proof. Let Q be a resolution, given by Theorem 0.d, of the 2-skeleton P of the standard 6-simplex. Suppose that Q embedded into $\mathbb{R}^4$. It is clear from the proof of Theorem 0.d that the non-trivial preimages of the resolution are those of the points of the 1-skeleton of P. Hence by contracting in $\mathbb{R}^4$ the preimages of the resolution we would obtain $\mathbb{R}^4$ (cf. [BDVW; the 1-LCC shrinking theorems for ANR's and the remark at the bottom of p.2], in which P is embedded. The latter is well-known to be impossible. Contradiction. □

Our fourth result shows that in Theorem 0.c-d the class of fake surfaces cannot be replaced by a certain smaller class of 2-polyhedra. Clearly, there exists a 2-polyhedron P which is not homotopy equivalent to any surface. Indeed, we can take any polyhedron P with $\pi_1(P) \cong \mathbb{Z}_3$, because $\mathbb{Z}_3$ is not the fundamental group of any surface.

Also, there exists a 2-polyhedron P which is not homotopy equivalent to any fake surface without points of type 3. Indeed, we can take as P any 2-dimensional spine of any homology 3-sphere, because [La00; Proposition 1.1] proved that the fundamental group of any fake surface without points of type 3 can never be a non-trivial perfect group. All homology groups used in this paper are assumed to have $\mathbb{Z}$-coefficients.

Theorem 4. *a) For each k there exists a connected 2-polyhedron P which is not homotopy equivalent to any fake surface with at most k vertices of type 3.*

b) If Q is a fake surface such that $H_1(Q) \cong 0$ (e.g. a fake surface spine of a homology 3-sphere), then $\operatorname{rk} \pi_1(Q) \leq 20|Q''|$.

Proof. Take $P = K_1 \vee K_2 \vee \cdots \vee K_{20k+1}$, where K_i is a 2-spine of a non-trivial homology 3-sphere. By [LySh77; Chapter 4, Corolary 1.9],

$$\pi_1(P) \cong \pi_1(K_1) * \pi_1(K_2) * \cdots * \pi_1(K_{20k+1})$$

has at least $20k + 1$ generators, so (a) follows from (b).

Our proof of (b) is a reduction to the case $|Q''| = 0$ proved in [La00; Proposition 1.1]. For a space X consisting of connected components $X_1, \ldots, X_n$ we denote $\pi_1(X) := \oplus \pi_1(X_i)$. The following inequality holds for finite polyhedra X and Y (neither X nor Y nor $X \cap Y$ are assumed to be connected):

$$(*) \qquad \operatorname{rk} \pi_1(X \cup Y) \leq \operatorname{rk} \pi_1(X) + \operatorname{rk} \pi_1(Y) + 2 \operatorname{rk} H_0(X \cap Y).$$

The inequality is proved by adding to both X and Y a point and the union of arcs joining this point to each connected component of $X \cap Y$, and then applying the van Kampen theorem to the modified X, Y and (now connected) $X \cap Y$.

Let G be the subgraph of Q' obtained by deleting isolated circles. Then $Q' - G$ does not contain any points of type 3. Let N be a regular neighbourhood of G in Q. Then $\partial N = N \cap \operatorname{Cl}(Q - N)$ is homeomorphic to a disjoint union of n circles. Denote $v = |Q''|$. By (*), it suffices to prove that:

(1) $\operatorname{rk} \pi_1(N) = \operatorname{rk} \pi_1(G) \leq 2v$;

(2) $n = \operatorname{rk} H_0(\partial N) \leq 6v$; and

(3) $\operatorname{rk} \pi_1(\operatorname{Cl}(Q - N)) \leq n$.

In order to prove (1), observe that all vertices of G have degree 4, hence G has $2v$ edges. Since G does not have any isolated circles, it follows that the number of connected components of G is at most v. Therefore $\operatorname{rk} \pi_1(G) \leq 2v - v + v = 2v$.

In order to prove (2), observe that n does not exceed the number of pairs (e, s) where e is an edge of G and s is a boundary circle of ∂N, going close to e. Since for each of $2v$ edges e of G there are at most three circles s close to it, there are $6v$ such pairs, and (3) follows.

In order to prove (3), let $D = D_1 \sqcup \cdots \sqcup D_n$, identify ∂D and ∂N and let $X = \operatorname{Cl}(Q - N) \cup D$. Since $Q \cup D$ is obtained from Q by attaching disks and $H_1(Q) = 0$, it follows that $H_1(Q \cup D) = 0$. Then the Mayer-Vietoris sequence for $Q \cup D = (N \cup D) \cup X$ with $(N \cup D) \cap X = D$ implies that $H_1(X) = 0$. Since the fake surface X does not contain any points of type 3, it follows by [La00; Proposition 1.1] that $\pi_1(X) = 0$. Hence $\operatorname{rk} \pi_1(\operatorname{Cl}(Q - N)) \leq n$. □

Note that the estimate in Theorem 4.b can be improved to $\operatorname{rk} \pi_1(Q) \leq 8|Q''|$.

We conclude this paper by a conjecture on a higher-dimensional generalization of Theorem 0.d. Note that the class of 'resolving' polyhedra from our conjecture does not coincide with the class of higher-dimensional special polyhedra [Mat73]. Let Θ^k be the union of S^k with $k + 1$ disks D^k attached to S^k along the main equator $S^{k-1} \subset S^k$.

Conjecture. *For every n-polyhedron P there is an onto map $f : Q \to P$ with contractible preimages and such that every point $x \in Q$ has a regular neighbourhood homeomorphic to the product $I^{n-k-1} \times \mathrm{Con}\,\Theta^k$.*

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