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LIFTINGS OF NORMAL FUNCTORS IN THE CATEGORY OF COMPACTA TO CATEGORIES OF TOPOLOGICAL ALGEBRA AND ANALYSIS

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Abstract: We prove that the liftings of a normal functor F in the category of compact Hausdorff spaces to the categories of (abelian) compact semigroups (monoids) are determined by natural transformations $F(-)\times F(-)\to F(-\times -)$ satisfying requirements that correspond to associativity, commutativity, and the existence of a unity. In particular, the tensor products for normal monads satisfy (not necessarily all) these requirements.

It is proved that the power functor in the category of compacta is the only normal functor that admits a natural lifting to the category of convex compacta and their continuous affine mappings.

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Introduction

The article is devoted to the extension of continuous algebraic operations from compact Hausdorff spaces to the images of these spaces under functorial topological constructions. First of all, we mention functor-semigroups common in various areas of mathematics. The most known (and probably most important) of these are convolution semigroups, the spaces $P(G)$ of Borel probability measures on a compact topological group G endowed with the operation of convolution of measures. Convolution semigroups are the main objects and tools of harmonic analysis [1].

Well known is also another functor-semigroup, the global semigroup (or the hypersemigroup) $\exp(G)$ over a compact topological group G (see [2]). This is the hyperspace of nonempty closed subsets in G endowed with the semigroup operation $A\cdot B=\{ab : a\in A,\ b\in B\}$.

Convolution semigroups and global semigroups are particular cases of topological semigroups whose spaces are the results of applying a functor F to a compact topological group (or even a semigroup) G. Teleiko and Zarichnyi observed that the operations on these semigroups are naturally defined by tensor products related to the structure of the monad for the functor F [3, 4]. This led to the introduction of functor-semigroups. These authors have shown that, for a functor F in the category $\mathscr{C}omp$ of compact Hausdorff spaces that is the functorial part of a monad and belongs to the class of normal functors introduced by Shchepin [5], the semigroup operation of every compact topological semigroup X extends functorially to a continuous semigroup operation on $F(X)$. Thus they defined a lifting of a functor $F:\mathscr{C}omp\to\mathscr{C}omp$ to a functor $\overline{F}:\mathscr{CSG}r\to\mathscr{CSG}r$ in the category of compact topological semigroups.

This, in particular, made it possible to use in [2, 6] a general approach for solving a sequence of problems of the following form: Characterize the topological semigroups embeddable in functor-semigroups for a weakly normal functor in the category of compacta. These studies led naturally to the following question: Is there a way to construct liftings for functors in the category of compact spaces on the category of compacta to the category of (abelian) compact semigroups (monoids) under conditions less

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restrictive than the existence of monads? In this article we show that every such lifting is determined by a unique natural transformation satisfying certain conditions. In particular, the tensor product is such a natural transformation. At the same time, we construct an example of a natural lifting for a normal functor not defined by the tensor product.

Another main result of this work is an analog of a theorem by Zarichnyi [7] that, of all normal natural functors, only the power functors, i.e., the functors of taking a finite or countable power, admit natural liftings to the category of compact topological *groups*. We show that, for a normal functor, there is a natural lifting to the category of convex compacta and continuous affine mappings if and only if the functor is a power functor. This means that, for a normal functor $F : \mathscr{C}omp \to \mathscr{C}omp$ different form a power functor, it is impossible to extend the convex structures from each convex compactum X to the compactum FX so that the mappings $Ff : FX \to FY$ be affine for all affine $f : X \to Y$.

§ 1. Necessary Notions and Facts

In this article by a *compactum* we mean a not necessarily metrizable compact Hausdorff topological space [8]. For the concepts of *category, morphism, mono- and epimorphism, diagram, limit,* (*covariant*) *functor, inverse system*, the reader is referred to [9, 10]. Denote by $\mathscr{C}omp$ [10] the category consisting of compacta and their continuous mappings. Recall that the mono- and epimorphisms in $\mathscr{C}omp$ are injective and surjective mappings respectively. The category whose objects are compact semigroups and whose arrows are continuous homomorphisms is denoted by $\mathscr{C}\mathscr{S}\mathscr{G}r$. Similar notation $\mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$, $\mathscr{C}\mathscr{M}on$, and $\mathscr{C}\mathscr{A}b\mathscr{M}on$ is also used for categories of compact abelian semigroups, compact monoids, and compact abelian monoids. The objects of the category of convex compacta $\mathscr{C}onv$ are compacta whose convex combinations are defined by embeddings as convex sets in locally convex topological vector spaces. Morphisms of this category are continuous affine mappings, that is, they preserve convex combinations. We use the common symbol U for the forgetful functors from $\mathscr{C}\mathscr{S}\mathscr{G}r$, $\mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$, $\mathscr{C}\mathscr{M}on$, $\mathscr{C}\mathscr{A}b\mathscr{M}on$, and $\mathscr{C}onv$ into $\mathscr{C}omp$.

Given a functor $F : \mathscr{C}omp \to \mathscr{C}omp$, a compactum X, an element $a \in FX$, and a closed subset $A \subset X$, we put $a \in FA$ if $a \in Fi(FA)$, where $i : A \hookrightarrow X$ is the inclusion.

A functor F in $\mathscr{C}omp$ is called *normal* [5] if the following conditions are fulfilled:

(a) preservation of mono- and epimorphisms (the corresponding functor is called *mono-* or *epimorphic*);

(b) preservation of preimages (for every morphism $f : X \to Y$ in $\mathscr{C}omp$, $a \in FX$, $Y_0 \subset Y$, $Ff(a) \in FY_0$, we have $a \in F(f^{-1}(Y_0)))$;

(c) preservation of intersections (if X_α, $\alpha \in \mathscr{A}$, is a family of closed subsets in a compactum X then

$$\bigcap_{\alpha\in\mathscr{A}} F(X_\alpha) = F(\bigcap_{\alpha\in\mathscr{A}} X_\alpha));$$

(d) preservation of the limits of inverse systems (*continuity* of the functor) [10];

(e) preservation of the empty set ($F\varnothing = \varnothing$);

(f) preservation of singletons (FX is a singleton if so is X);

(g) preservation of the weight ($w(FX) = w(X)$ for every compactum X).

For a monomorphic (for example, normal) functor $F : \mathscr{C}omp \to \mathscr{C}omp$ and a closed subset A in a compactum X, identify the homeomorphic spaces FA and $Fi(FA) \subset FX$, where $i : A \hookrightarrow X$ is the inclusion.

For arbitrary $a \in FX$ and a functor F preserving intersections, the set $\operatorname{supp}_F a = \bigcap\{A \mid A \subset X \text{ is closed}, a \in FA\}$ is the least of the closed subsets A in X for which $a \in FA$ and it is called the *support* of a. If F preserves mono- and epimorphisms then the preservation of preimages is equivalent to the preservation of supports: $\operatorname{supp}_F Ff(a) = f(\operatorname{supp}_F a)$ for every $a \in FX$ and every mapping of compacta $f : X \to Y$.

The axioms of a normal functor are not independent. In particular [11], for a monomorphic and epimorphic functor in $\mathscr{C}omp$ the preservation of intersections implies continuity.

§ 2. The Main Results

Let $\mathscr{C}$ be one of the categories $\mathscr{CSG}r$, $\mathscr{CA}b\mathscr{SG}r$, $\mathscr{CM}on$, and $\mathscr{CA}b\mathscr{M}on$. Call a functor $\overline{F} : \mathscr{C} \to \mathscr{C}$ a *lifting* of a functor $F : \mathscr{C}omp \to \mathscr{C}omp$ to $\mathscr{C}$ if the diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\overline{F}} & \mathscr{C} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathscr{C}omp & \xrightarrow[F]{} & \mathscr{C}omp \end{array}$$

commutes, i.e., $U\overline{F} = FU$. This means that, for every compact (abelian) semigroup (or monoid) $(X, \odot)$ on the compactum FX, there is defined a continuous associative operation $\bar{\odot}$ turning FX into an (abelian) semigroup (respectively, monoid) and $\overline{F}(X, \odot) = (FX, \bar{\odot})$. Moreover, for every continuous homomorphism $f : (X, \odot) \to (Y, \odot)$, i.e., every arrow in the corresponding category, the mapping Ff is also a homomorphism $(FX, \bar{\odot}) \to (FY, \bar{\odot})$.

Given an arbitrary functor $F : \mathscr{C}omp \to \mathscr{C}omp$, it is easy to construct at least two liftings $\overline{F} : \mathscr{CSG}r \to \mathscr{CSG}r$ by putting the product $a \,\bar{\odot}\, b$ of elements $a, b \in FX$ to be always equal to a (or b). To "cut off" these trivial cases, impose a natural extra condition.

In the sequel, we assume always that the functor F preserves monomorphisms, preimages, intersections, the empty set, and singletons. Let $\eta X : X \to FX$ be the mapping assigning to each x in X a single element $F\{x\} \subset FX$. T. Banakh proved that ηX is a topological embedding and a component of a unique natural transformation $\eta : \mathbf{1}_{\mathscr{C}omp} \to F$ [10]. Therefore, we regard X as a subset in FX. If ηX is a morphism $(X, \odot) \to (FX, \bar{\odot})$ in the corresponding category, i.e., the operation $\bar{\odot}$ on $\overline{F}X$ is an extension of $\bar{\odot}$ to X, and, for monoids, the unity in X is also the unity of FX, then we call the lifting $\overline{F}$ *natural* [4].

We prove that every natural lifting of a normal functor F in $\mathscr{C}omp$ to the above-mentioned categories is defined by a unique natural transformation $t : F(-) \times F(-) \to F(- \times -)$, i.e., by a family of continuous mappings $t(X, Y) : FX \times FY \to F(X, Y)$ for all compacta X and Y such that, for all continuous mappings of compacta $f : X \to X'$, $g : Y \to Y'$ and elements $a \in FX$, $b \in FY$, we have

$$t(X', Y')(Ff(a), Fg(b)) = F(f \times g)(t(X, Y)(a, b)).$$

Below the mappings $p : X \times Y \to Y \times X$ and $\bar{p} : FX \times FY \to FY \times FX$ are defined for all $x \in X$, $y \in Y$, $a \in FX$, and $b \in FY$ as $p(x, y) = (y, x)$ and $\bar{p}(a, b) = (b, a)$.

Consider the following possible properties of a natural transformation t:

$$t(X \times Y, Z)(t(X, Y)(a, b), c) = t(X, Y \times Z)(a, t(Y, Z)(b, c)) \qquad (*)$$

for $a \in FX$, $b \in FY$, and $c \in FZ$ ("the associative law");

$$F\,\mathrm{pr}_1 \circ t(X, \{y\})(a, \eta\{y\}(y)) = a, F\,\mathrm{pr}_2 \circ t(\{x\}, Y)(\eta\{x\}(x), b) = b \qquad (**)$$

for $a \in FX$, $b \in FY$, $x \in X$, and $y \in Y$ ("a double-sided unity");

$$Fp \circ t(X, Y)(a, b) = t(Y, X)(b, a) \qquad (***)$$

for all $a \in FX$ and $b \in FY$ ("the commutative law").

It is convenient to represent these properties as the commutative diagrams:

$$\begin{array}{ccc} FX\times FY\times FZ & \xrightarrow{t(X,Y)\times \mathbf{1}_{FZ}} & F(X\times Y)\times FZ \\ \Big\downarrow{\scriptstyle \mathbf{1}_{FX}\times t(Y,Z)} & & \Big\downarrow{\scriptstyle t(X\times Y,Z)} \\ FX\times F(Y\times Z) & \xrightarrow[t(X,Y\times Z)]{} & F(X\times Y\times Z) \end{array} \qquad (*)$$

$$\begin{array}{ccc} FX\times F\{y\} & \xrightarrow{t(X,\{y\})} & F(X\times\{y\}) \\ & \searrow{\scriptstyle \mathrm{pr}_1} & \Big\downarrow{\scriptstyle F\,\mathrm{pr}_1} \\ & & FX \end{array} \qquad \begin{array}{ccc} F\{x\}\times FY & \xrightarrow{t(\{x\},Y)} & F(\{x\}\times Y) \\ & \searrow{\scriptstyle \mathrm{pr}_2} & \Big\downarrow{\scriptstyle F\,\mathrm{pr}_2} \\ & & FY \end{array} \qquad (**)$$

$$\begin{array}{ccc} FX\times FY & \xrightarrow{t(X,Y)} & F(X\times Y) \\ \Big\downarrow{\scriptstyle \bar{p}} & & \Big\downarrow{\scriptstyle Fp} \\ FY\times FX & \xrightarrow[t(Y,X)]{} & F(Y\times X) \end{array} \qquad (***)$$

Theorem 1. *Let $\mathscr{C}$ be one of the categories $\mathscr{CSG}r$, $\mathscr{CM}on$, $\mathscr{CA}b\mathscr{SG}r$, and $\mathscr{CA}b\mathscr{M}on$. For every natural lifting $\overline{F}:\mathscr{C}\to\mathscr{C}$ of a functor $F:\mathscr{C}omp\to\mathscr{C}omp$ preserving monomorphisms, preimages, the empty set, and singletons, there exists a unique natural transformation $t(-,-):F(-)\times F(-)\to F(-\times -)$ such that the operation $\bar{\odot}$ on $\overline{F}X$ is defined by the formula $a\,\bar{\odot}\,b = F\odot\circ t(X,X)(a,b)$ for every compact semigroup $(X,\odot)$. This natural transformation satisfies: $(*)$ for $\mathscr{C}=\mathscr{CSG}r$, $(*)$ and $(**)$ for $\mathscr{C}=\mathscr{CM}on$, $(*)$ and $(***)$ for $\mathscr{C}=\mathscr{CA}b\mathscr{SG}r$, and $(*)$, $(**)$, and $(***)$ for $\mathscr{C}=\mathscr{CA}b\mathscr{M}on$. Conversely, every natural transformation $t(-,-):F(-)\times F(-)\to F(-\times -)$ with the indicated properties defines a natural lifting of F to $\mathscr{C}$ by the above formula.*

Thus, the problem of the existence of natural liftings for a functor F in the category of compacta that preserves monomorphisms, intersections, preimages, the empty set, and singletons is reduced to the question of the existence of natural transformations with certain properties. Moreover, it can be seen that the existence of "good" natural transformations is *sufficient* for the existence of natural liftings of a functor that preserves monomorphisms, the empty set, and singletons but not necessarily intersections and preimages.

The answer to this question is positive if we want to lift the functor $F:\mathscr{C}omp\to\mathscr{C}omp$ that is the functorial part of a monad to $\mathscr{CSG}r$ or $\mathscr{CM}on$. A method for constructing a lifting (and a sufficient condition for the existence of a natural transformation satisfying the conditions of Theorem 1) was proposed by Teleiko in [4].

A triple $\mathbb{F}=(F,\eta,\mu)$ is called a *monad* [9] in a category $\mathscr{C}$ if F is a functor in $\mathscr{C}$, $\eta:\mathbf{1}_{\mathscr{C}}\to F$ and $\mu:F^2\equiv FF\to F$ are natural transformations, and the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\eta F} & F^2 \\ \Big\downarrow{\scriptstyle F\eta} & \searrow{\scriptstyle \mathbf{1}_F} & \Big\downarrow{\scriptstyle \mu} \\ F^2 & \xrightarrow[\mu]{} & F, \end{array} \qquad \begin{array}{ccc} F^3 & \xrightarrow{\mu F} & F^2 \\ \Big\downarrow{\scriptstyle F\mu} & & \Big\downarrow{\scriptstyle \mu} \\ F^2 & \xrightarrow[\mu]{} & F \end{array}$$

commute. Here F is called the *functorial part* of $\mathbb{F}$, η is the *unity* of $\mathbb{P}$, and μ is the *multiplication* of $\mathbb{F}$.

Recall that, for a functor $F:\mathscr{C}omp\to\mathscr{C}omp$ preserving monomorphisms, intersections, preimages, the empty set, and singletons, there exists a unique natural transformation $\eta:\mathbf{1}_{\mathscr{C}omp}\to F$. Then the tensor products [3] $\otimes,\tilde{\otimes}:FX\times FY\to F(X\times Y)$ for compacta X and Y and a monad $\mathbb{F}$ are defined as follows: Given $x\in X$ (respectively, $y\in Y$), define the embedding $i_x:Y\to X\times Y$ ($\tilde{i}_y:X\to X\times Y$)

by the formula $i_x(y) = (x, y)$ (respectively, $\tilde{i}_y(x) = (x, y)$). The correspondences $x \mapsto i_x$ and $y \mapsto \tilde{i}_y$ are continuous mappings $X \to C(Y, X \times Y)$ and $Y \to C(X, X \times Y)$ into spaces of continuous mappings with compact open topology. Since F preserves intersections, the mappings $C(Y, X \times Y) \to C(FY, F(X \times Y))$ and $C(X, X \times Y) \to C(FX, F(X \times Y))$, induced by F, are continuous [11]. Therefore, the mappings $j : X \times FY \to F(X \times Y)$ and $\tilde{j} : FX \times Y \to F(X \times Y)$, defined by $j(x, b) = Fi_x(b)$ and $\tilde{j}(a, y) = F\tilde{i}_y(a)$, $a \in FX$, $b \in FY$, are continuous too. Put $j_b(x) = j(x, b)$ and $\tilde{j}_a(y) = j(a, y)$ and define the *tensor products* $a \otimes b$ and $a \tilde{\otimes} b$ by the formulas $a \otimes b = \mu(X \times Y) \circ Fj_b(a)$ and $a \tilde{\otimes} b = \mu(X \times Y) \circ F\tilde{j}_a(b)$. Both tensor products are continuous, natural with respect to both arguments in the following sense: for all $a \in FX$ and $b \in FY$ and all continuous mappings $f : X \to X'$ and $g : Y \to Y'$, we have $(Ff(a)) \otimes (Fg(b)) = F(f \times g)(a \otimes b)$, $(Ff(a)) \tilde{\otimes} (Fg(b)) = F(f \times g)(a \tilde{\otimes} b)$, and the associative law: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$, $(a \tilde{\otimes} b) \tilde{\otimes} c = a \tilde{\otimes} (b \tilde{\otimes} c)$ for all $a \in FX$, $b \in FY$, and $c \in FZ$, which enables us to simply write $a \otimes b \otimes c$ etc. It is easy to check that $F\,\mathrm{pr}_1(a \otimes b) = F\,\mathrm{pr}_1(a \tilde{\otimes} b) = a$ and $F\,\mathrm{pr}_2(a \otimes b) = F\,\mathrm{pr}_2(a \tilde{\otimes} b) = b$ for $a \in FX$ and $b \in FY$.

As was proved in [4], the two natural transformations $\otimes, \tilde{\otimes} : F(-) \times F(-) \to F(- \times -)$ meet $(*)$ and $(***)$ and so define liftings of a functor $F : \mathscr{C}omp \to \mathscr{C}omp$ preserving monomorphisms, the empty set, and singletons that is the functorial part of a monad to $\mathscr{C}\mathscr{S}\mathscr{G}r$ and $\mathscr{C}\mathscr{M}on$. For a compact semigroup (monoid) $(S, \odot)$, these liftings are given by the formulas $a \mathbin{\bar{\odot}} b = F \odot (a \otimes b)$ and $a \mathbin{\bar{\odot}} b = F \odot (a \tilde{\otimes} b)$ for $a, b \in FS$.

Both tensor products coincide, that is, $a \otimes b = a \tilde{\otimes} b$ for all $a, b \in FX$, if and only if $(**)$ holds for one of the products $\otimes$ and $\tilde{\otimes}$. In this case the tensor product is called *symmetric.* The symmetry of the tensor product is a necessary and sufficient condition for it to define (in the above sense) a lifting of a functor F to the categories of compact abelian semigroups and compact abelian monoids.

Despite the fact that the tensor products coincide for the hyperspace monad, the monad of probability measures, and the power monad [10] (hence, the corresponding functors exp, P, $(-)^\alpha$ are lifted to $\mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$ and $\mathscr{C}\mathscr{A}b\mathscr{M}on$); they can in general differ, for example, for the inclusion hyperspace monad and the superextension monad, which was proved in [4]. Recall that the inclusion hyperspace functor and the superextension functor are not normal. Unfortunately, the tensor product is not always symmetric even for monads with normal functorial part. In [12], Radul defined a normal analog $\widetilde{\mathbb{G}}$ for the hyperspace inclusion monad. Given a compactum X, the space $\widetilde{G}X$ consists of all families $\mathscr{F} \in \exp^2 X$ for which from $\mathscr{F} \ni A \subset B \subset \bigcup \mathscr{F}$, $B \in \exp F$, it follows that $B \in \mathscr{F}$. If $\mathscr{F}_1 \in \widetilde{G}X$, $\mathscr{F}_2 \in \widetilde{G}Y$, $F_1 = \bigcup \mathscr{F}_1$, and $F_2 = \bigcup \mathscr{F}_2$, then the tensor products for $\widetilde{\mathbb{G}}$ are defined by the formulas

$$\mathscr{F}_1 \otimes \mathscr{F}_2 = \{F \subset F_1 \times F_2 \mid \exists A \in \mathscr{F}_1\ \forall x \in A\ \ \mathrm{pr}_2(F \cap \{x\} \times Y) \in \mathscr{F}_2\},$$

$$\mathscr{F}_1 \tilde{\otimes} \mathscr{F}_2 = \{F \subset F_1 \times F_2 \mid \exists B \in \mathscr{F}_2\ \forall x \in B\ \ \mathrm{pr}_1(F \cap X \times \{y\}) \in \mathscr{F}_1\}.$$

Putting $X = Y = \{a, b\}$, $\mathscr{F}_1 = \{\{a\}, \{b\}, \{a, b\}\}$, and $\mathscr{F}_2 = \{\{a, b\}\}$, we can easily prove that $\mathscr{F}_1 \otimes \mathscr{F}_2 \neq \mathscr{F}_1 \tilde{\otimes} \mathscr{F}_2$. Moreover, the formula

$$t(X, Y)(\mathscr{F}_1, \mathscr{F}_2) = \{F \in \exp(F_1 \times F_2) \mid \exists A \in \mathscr{F}_1\ \exists B \in \mathscr{F}_2\ F \supset A \times B\}$$

defines a natural transformation $t(-, -) : \widetilde{G}(-) \times \widetilde{G}(-) \to \widetilde{G}(- \times -)$ which differs from $\otimes$ as well as from $\tilde{\otimes}$ and satisfies $(*)$, $(**)$, and $(***)$, and, therefore, defines liftings of $\widetilde{G}$ not only to $\mathscr{C}\mathscr{S}\mathscr{G}r$ and $\mathscr{C}\mathscr{M}on$ but also to $\mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$ and $\mathscr{C}\mathscr{A}b\mathscr{M}on$.

By analogy to the above, by a *lifting of a functor $F : \mathscr{C}omp \to \mathscr{C}omp$ to $\mathscr{C}onv$* we mean a functor $\overline{F} : \mathscr{C}onv \to \mathscr{C}onv$ such that $U\overline{F} = FU$ for the forgetting functor $U : \mathscr{C}onv \to \mathscr{C}omp$. If F preserves monomorphisms, preimages, intersections, the empty set, and singletons and $\eta : \mathbf{1}_{\mathscr{C}omp} \to F$ is the corresponding unique natural transformation then call $\overline{F}$ *natural* if, for every convex compactum X, the component $\eta X : X \to FX$ is an affine embedding.

In [7], Zarichnyi proved that, for a normal functor, a natural lifting to the category of compact topological groups exists if and only if the functor is a power functor, i.e., the functor $(-)^\tau$, where τ is an at most countable cardinal. We prove a similar result for the category of convex compacta.

Theorem 2. *For a normal functor $F : \mathscr{C}omp \to \mathscr{C}omp$, a natural lifting to $\mathscr{C}onv$ exists if and only if the functor is a power functor.*

Analysis of the proof of Theorem 2 given below shows the importance of the identity $(1-\lambda)x+\lambda x = x$ fulfilled in every convex compactum. This property is the main obstacle to the existence of natural extensions. Its rejection leads to the class of *semiconvex compacta* which was introduced and studied in [13, 14]; i.e., the spaces in which points can be "joined by segments" but a segment whose beginning coincides with the end need not consist of a single point. Application to the class of semiconvex compacta (which includes all convex compacta), as well as to the class of their affine continuous mappings (that is, mappings preserving a semiconvex combination) of normal and weakly normal functors exp, G, P, M, etc. again gives semiconvex compacta and affine continuous mappings. Formally speaking, these functors are lifted from the category of compacta $\mathscr{C}omp$ to the category of semiconvex compacta $\mathscr{S}\mathscr{C}onv$. These liftings will be considered in a subsequent article.

§ 3. Proofs

Each of the categories $\mathscr{C}\mathscr{S}\mathscr{G}r$, $\mathscr{C}\mathscr{M}on$, $\mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$, and $\mathscr{C}\mathscr{A}b\mathscr{M}on$ requires a separate consideration, although the scheme of the proof remains the same in all cases. Below F is a functor $F : \mathscr{C}omp \to \mathscr{C}omp$ preserving monomorphisms, preimages, intersections, the empty sets, and singletons; $\overline{F}$ is its natural lifting, $\overline{F}(X, \odot) = (FX, \bar{\odot})$.

Lemma 1. *For a natural lifting $\overline{F}$ of F to $\mathscr{C}\mathscr{S}\mathscr{G}r$ and $a, b \in FX$, we have $\operatorname{supp}_F(a \mathbin{\bar{\odot}} b) \subset \operatorname{supp}_F a \odot \operatorname{supp}_F b = \{x \odot y \mid x \in \operatorname{supp}_F a,\ y \in \operatorname{supp}_F b\}$.*

Proof. Define the associative binary operation $\odot$ on $\{0, 1, 2, 3\}$ as follows: $0 \odot 1 = 2$, and for $\alpha \neq 0$ or $\beta \neq 1$ let $\alpha \odot \beta = 3$. Putting $X' \subset X \times \{0, 1, 2, 3\}$ and $X' = \operatorname{supp}_F a \times \{0\} \sqcup \operatorname{supp}_F b \times \{1\} \sqcup (\operatorname{supp}_F a \odot \operatorname{supp}_F b) \times \{2\} \sqcup X \times \{3\}$ and defining $\odot$ on X' by the formula $(x, \alpha) \odot (y, \beta) = (x \odot y, \alpha \odot \beta)$ makes X' with the sum topology into a compact semigroup; then the projections $\mathrm{pr}_1 : X' \to X$ and $\mathrm{pr}_2 : X' \to \{0, 1, 2, 3\}$ become continuous semigroup homomorphisms. Put $a' = Fi_0(a)$ and $b' = Fi_1(b)$, where the embeddings $i_0, i_1 : X \hookrightarrow X'$ are defined as follows: $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. Then $\mathrm{pr}_1 \circ i_0 = \mathrm{pr}_1 \circ i_1 = \mathbf{1}_X$, which implies that $F\,\mathrm{pr}_1(a') = a$ and $F\,\mathrm{pr}_1(b') = b$. Obviously, $F\,\mathrm{pr}_2(a') = \eta\{0, 1, 2, 3\}(0)$ and $F\,\mathrm{pr}_2(b') = \eta\{0, 1, 2, 3\}(1)$; therefore,

$$\begin{aligned} F\,\mathrm{pr}_2(a' \mathbin{\bar{\odot}} b') &= F\,\mathrm{pr}_2(a') \mathbin{\bar{\odot}} F\,\mathrm{pr}_2(b') = \eta\{0,1,2,3\}(0) \mathbin{\bar{\odot}} \eta\{0,1,2,3\}(1) \\ &= \eta\{0,1,2,3\}(0 \odot 1) = \eta\{0,1,2,3\}(2) \in F\{2\}. \end{aligned}$$

The preservation of preimages implies $a' \mathbin{\bar{\odot}} b' \in F({\mathrm{pr}_2}^{-1}(2)) = (\operatorname{supp}_F a \odot \operatorname{supp}_F b) \times \{2\}$. Hence,

$$\begin{aligned} a \mathbin{\bar{\odot}} b = F\,\mathrm{pr}_1(a') \mathbin{\bar{\odot}} F\,\mathrm{pr}_1(b') &= F\,\mathrm{pr}_1(a' \mathbin{\bar{\odot}} b') \in F(\mathrm{pr}_1((\operatorname{supp}_F a \odot \operatorname{supp}_F b) \times \{2\}) \\ &= F(\operatorname{supp}_F a \odot \operatorname{supp}_F b). \end{aligned}$$

Proof of Theorem 1 for $\mathscr{C} = \mathscr{C}\mathscr{S}\mathscr{G}r$. Assume that a natural lifting $\overline{F}$ to $\mathscr{C}\mathscr{S}\mathscr{G}r$ for a functor F exists. Given compacta X and Y, denote by $s(X, Y)$ the topological sum $X \sqcup Y \sqcup (X \times Y) \sqcup \{*\}$ with the operation $\odot$ defined as follows: $a \odot b = (x, y)$ if $a = x \in X$ and $b = y \in Y$; otherwise, $a \odot b = *$. Then $(s(X, Y), \odot)$ is a compact semigroup and, for continuous mappings of compacta $f : X \to X'$ and $g : Y \to Y'$, the mapping $s(f, g) : s(X, Y) \to s(X', Y')$ equal to $s(f, g)(x) = f(x)$ for $x \in X$, $s(f, g)(y) = g(y)$, for $y \in Y$, $s(f, g)(x, y) = (f(x), g(y))$, for $(x, y) \in X \times Y$, and $s(f, g)(*) = *$, is a continuous homomorphism. Consider X, Y, $X \times Y \subset s(X, Y)$ and denote by $i_X : X \hookrightarrow s(X, Y)$, $i_Y : Y \hookrightarrow s(X, Y)$, and $i_{X\times Y} : X \times Y \hookrightarrow s(X, Y)$ the corresponding inclusions. Then $FX, FY, F(X \times Y) \subset Fs(X, Y)$. By the last lemma, $a \mathbin{\bar{\odot}} b \in F(X \times Y) \subset \overline{F}s(X, Y)$ for arbitrary $a \in FX \subset \overline{F}s(X, Y)$, $b \in FY \subset \overline{F}s(X, Y)$. Denote by $t(X, Y) : FX \times FY \to F(X \times Y)$ the correspondence $(a, b) \mapsto a \mathbin{\bar{\odot}} b$ (i.e., $(a, b) \mapsto {i_{X\times Y}}^{-1}(Fi_X(a) \mathbin{\bar{\odot}} Fi_Y(b)))$. Since $\overline{F}s(f, g)$ is also a homomorphism for all $f : X \to X'$ and $g : Y \to Y'$, we infer that

$$t(X', Y')(Ff(a), Fg(b)) = F(f \times g)(t(X, Y)(a, b))$$

for all $a \in FX$, $b \in FY$, i.e., $t(-,-)$ is a natural transformation of the functors $F(-)\times F(-) \to F(-\times-)$.

This natural transformation has some property following from the associativity of multiplication in a semigroup. Consider the topological sum

$$s(X,Y,Z) = X \sqcup Y \sqcup Z \sqcup (X\times Y) \sqcup (Y \times Z) \sqcup (X\times Y\times Z) \sqcup \{*\}.$$

Turn $s(X,Y,Z)$ into a compact semigroup by putting

$$x \odot y = (x,y), \quad y \odot z = (y,z), \quad x \odot (y,z) = (x,y,z), \quad (x,y)\odot z = (x,y,z)$$

for $x \in X$, $y \in Y$, $z \in Z$ and $a \odot b = *$ otherwise. The obvious inclusions[1] $i_1 : s(X\times Y, Z) \hookrightarrow s(X,Y,Z)$ and $i_2 : s(X, Y\times Z) \hookrightarrow s(X,Y,Z)$ are continuous semigroup homomorphisms. Consequently, $\overline{F}i_1$ and $\overline{F}i_2$ are also homomorphisms; therefore, $A \mathbin{\bar\odot} c = t(X\times Y, Z)(A,c)$ for any $A \in F(X\times Y) \subset \overline{F}s(X,Y,Z)$, $A \in FZ \subset \overline{F}s(X,Y,Z)$ and $a \mathbin{\bar\odot} C = t(X, Y\times Z)(a,C)$ for any $a \in FX \subset \overline{F}s(X,Y,Z)$, $C \in F(Y\times Z) \subset \overline{F}s(X,Y,Z)$. Choose arbitrary $a\in FX$, $b \in FY$, $c\in FZ$, and, by the associative law $(a\bar\odot b)\bar\odot c = a\bar\odot(b\bar\odot c)$, we have

$$t(X\times Y, Z)(t(X,Y)(a,b),c) = t(X, Y\times Z)(a, t(Y,Z)(b,c)),$$

i.e., we obtain $(*)$.

Show how to define the operation $\bar\odot$ on $\overline{F}X$ from the natural transformation $t(-,-)$ for a compact semigroup $(X,\odot)$. The semigroup $s(X,X)$ has the form $X\sqcup X' \sqcup (X\times X)\sqcup\{*\}$, where X' is "another copy" of X. Denote by $s(X)$ the topological sum $X\sqcup X'\sqcup X\times X \sqcup X''$, where X' and X'' are two copies of X. Let $i_X, i_{X'} : X \hookrightarrow s(X)$ and $i_{X\times X} : X\times X \hookrightarrow s(X)$ be the corresponding embeddings. Define the mapping $p : S(X) \to X$ by the formulas $p(x) = x$ and $p((x,y)) = x\odot y$ for all $x,y\in X$, i.e., q multiplies the elements of each pair and glues three elements of X to one. Define the operation $\odot$ on $s(X)$ as follows: $x\odot y = (x,y)$ for $x\in X$, $y \in X'$, and $a\odot b = p(a)\odot p(b) \in X''$ otherwise. Obviously, $(s(X),\odot)$ is a compact semigroup and $q : S(X) \to X$ is a semigroup homomorphism; therefore, $\overline{F}q : \overline{F}s(X) \to \overline{F}s(X,X)$ and $\overline{F}p : \overline{F}s(X) \to \overline{F}X$ is also a homeomorphism.

Given $a,b\in FX$, put $a' = Fi_X(a)$ and $b' = Fi_{X'}(b)$. Then $a\mathbin{\bar\odot} b = \overline{F}p(a')\mathbin{\bar\odot}\overline{F}p(b') = \overline{F}p(a'\mathbin{\bar\odot} b')$. By Lemma 1, $\operatorname{supp}_F a' \subset X$ and $\operatorname{supp}_F b' \subset X'$, which implies the inclusion $\operatorname{supp}_F(a'\odot b') \subset X\times X$. The space $F(X\sqcup X'\sqcup X\times X)$ can be regarded as a subspace in $s(X)$ and $s(X,X)$ simultaneously; we may thus assume that $a', b', a'\mathbin{\bar\odot} b' \in s(X,X)$.

The mapping $q : s(X) \to s(X,X)$, which is the identity on $X\sqcup X'\sqcup X\times X$ and equals $*$ on X'', as well as $\overline{F}q : \overline{F}s(X) \to \overline{F}s(X,X)$, is a homomorphism. The last mapping preserves the elements of $F(X\sqcup X'\sqcup X\times X)$, and hence both in $s(X)$ and $s(X,X)$, we obtain $a'\mathbin{\bar\odot} b' = Fi_{X\times X}\circ t(X,X)(a,b)$. Consequently,

$$a\mathbin{\bar\odot} b = Fp\circ Fi_{X\times X}\circ t(X,X)(a,b) = F(p\circ i_{X\times X})\circ t(X,X)(a,b).$$

Taking into account that, on $X\times X\subset s(X)$, the mapping p coincides with the multiplication $\odot : X\times X\to X$, we show that, in $\overline{F}X$, the operation is defined by the equality $a\mathbin{\bar\odot} b = F\odot\circ\, t(X,X)(a,b)$.

Above we have described how to construct a natural transformation $(*)$ starting from a lifting and how a natural transformation defines a continuous binary operation on FX for a compact semigroup $(X,\odot)$. Show that this operation is associative. Let $a,b,c\in FX$. Then

$$\begin{aligned}(a\mathbin{\bar\odot} b)\mathbin{\bar\odot} c &= F\odot\circ\, t(X,X)(F\odot\circ\, t(X,X)(a,b),c)\\ &= F\odot\circ\, t(X,X)(F\odot\circ\, t(X,X)(a,b), F\mathbf{1}_X c)\\ &= F\odot\circ\, F(\odot\times\mathbf{1}_X)\circ t(X\times X, X)(t(X,X)(a,b),c)\\ &= F(\odot\circ\ (\odot\times\mathbf{1}_X))\circ t(X\times X,X)(t(X,X)(a,b),c)\\ &= F\odot_3\circ\, t(X\times X,X)(t(X,X)(a,b),c),\end{aligned}$$

[1]Identify $X\times Y\times Z$, $(X\times Y)\times Z$, and $X\times(Y\times Z)$.

where $\odot_3 : X \times X \times X \to X$ is the multiplication of three factors: $\odot_3(x,y,z) = x \odot y \odot z$. Similarly,

$$(a \mathbin{\bar\odot} b) \mathbin{\bar\odot} c = F\odot_3 \circ t(X, X \times X)(a, t(X,X)(b,c)).$$

Then, by $(*)$, we infer $(a \mathbin{\bar\odot} b) \mathbin{\bar\odot} c = a \mathbin{\bar\odot} (b \mathbin{\bar\odot} c)$.

If $f : (X, \odot) \to (X', \odot)$ is a homomorphism of compact semigroups and $a, b \in \overline{F}X$ then

$$\overline{F}f(a \mathbin{\bar\odot} b) = Ff \circ F\odot \circ t(X,X)(a,b) = F\odot \circ F(f \times f) \circ t(X,X)(a,b)$$
$$= F\odot \circ t(X',X') \circ F(f \times f)(a,b) = F\odot \circ t(X,X)(Ff(a), Ff(b)) = \overline{F}f(a) \mathbin{\bar\odot} \overline{F}f(b).$$

Consequently, $\overline{F}f$ is also a homomorphism and $\overline{F}$ is a lifting of F from $\mathscr{C}omp$ to $\mathscr{C}\mathscr{S}\mathscr{G}r$. Show that it is a natural homomorphism. Given $x, y \in X$, put $a = \eta X(x)$ and $b = \eta X(y)$. Since $t(-,-)$ is a natural transformation, the following diagram commutes: ($i_x : \{x\} \hookrightarrow X$ and $i_y : \{y\} \hookrightarrow X$ are the inclusions):

$$\begin{array}{ccc} F\{x\} \times F\{y\} & \xrightarrow{Fi_x \times Fi_y} & FX \times FX \\ \downarrow{\scriptstyle t(\{x\},\{y\})} & & \downarrow{\scriptstyle t(X,X)} \\ F(\{x\} \times \{y\}) & \xrightarrow[F(i_x \times i_y)]{} & F(X \times X) \end{array}$$

Hence, $t(X,X)(a,b) \in F(\{x\} \times \{y\})$; therefore, $a \mathbin{\bar\odot} b = F\odot \circ t(X,X)(a,b) \in F\{x \odot y\}$; consequently, $a \mathbin{\bar\odot} b = \eta X(x \odot y)$. It is readily checked that the lifting constructed from the given natural transformation t defines exactly the initial natural transformation t.

Lemma 2. *Given a natural lifting $\overline{F}$ of a functor F in $\mathscr{C}omp$ to $\mathscr{C}\mathscr{M}on$ and elements $a, b \in FX$, we have*

$$\operatorname{supp}_F(a \mathbin{\bar\odot} b) \subset \operatorname{supp}_F a \odot \operatorname{supp}_F b = \{x \odot y \mid x \in \operatorname{supp}_F a,\ y \in \operatorname{supp}_F b\}.$$

Proof. Define the associative operation $\odot$ on $\{0,1,2,3,e\}$ as follows: $0 \odot 1 = 2$, $\alpha \odot e = e \odot \alpha = \alpha$ if $\alpha \in \{0,1,2,3,e\}$, and $\alpha \odot \beta = 3$ otherwise. Put

$$X' = \operatorname{supp}_F a \times \{0\} \sqcup \operatorname{supp}_F b \times \{1\} \sqcup (\operatorname{supp}_F a \odot \operatorname{supp}_F b) \times \{2\} \sqcup X \times \{3\} \sqcup \{e\}$$

and define $\odot$ on X' by the equalities

$$(x,\alpha) \odot (y,\beta) = (x \odot y, \alpha \odot \beta), \quad (x,\alpha) \odot e = e \odot (x,\alpha) = (x,\alpha), \quad e \odot e = e.$$

Then X' with the sum topology is a compact monoid, and the mappings $p_1 : X' \to X$ and $p_2 : X' \to \{0,1,2,3,e\}$ coinciding with the projections pr_1 and pr_2 for the elements of $X \times \{0,1,2,3\}$ and satisfying the equalities $p_1(e) = e_X$ (the unity of the monoid X), $p_2(e) = e$, are monoid homomorphisms. Further arguments repeat the corresponding part of the proof of Lemma 1.

Proof of Theorem 1 for $\mathscr{C} = \mathscr{C}\mathscr{M}on$. Suppose that $\overline{F}$ is a natural lifting. Given compacta X and Y, denote by $m(X,Y)$ the topological sum $X \sqcup Y \sqcup (X \times Y) \sqcup \{*, e\}$ and denote by $i_X : X \hookrightarrow m(X,Y)$, $i_Y : Y \hookrightarrow m(X,Y)$, and $i_{X \times Y} : X \times Y \hookrightarrow m(X,Y)$ the corresponding inclusions. Define the operation $\odot$ as follows: $a \odot e = e \odot a = a$ for all $a \in m(X,Y)$, $x \odot y = (x,y)$, $x \in X$, $y \in Y$, and $a \odot b = *$ for $a, b \neq e$ or $a \notin X$, or $b \notin Y$. For $a \in FX \subset Fm(X,Y)$, $b \in FY \subset Fm(X,Y)$, denote by $t(X,Y)(a,b)$ the element $a \mathbin{\bar\odot} b \in F(X \times Y) \subset Fm(X,Y)$. By analogy to the previous case, prove that $t(-,-) : F(-) \times F(-) \to F(- \times -)$ is a natural transformation defining an operation $\bar\odot$ on $\overline{F}X$, where $(X, \odot)$ is a compact monoid, by the formula $a \mathbin{\bar\odot} b = F\odot \circ t(X,X)(a,b)$. The associativity of the operation in a monoid implies the fulfillment of $(*)$ for $t(-,-)$. The existence of a unity in $\overline{F}X$ imposes some extra constraints on $t(-,-)$. If e_X is the unity of the monoid $(X, \odot)$ then the inclusion $\{e_X\} \hookrightarrow X$ is a homomorphism as well as $\overline{F}\{e_X\} \hookrightarrow \overline{F}X$. Therefore, the unity of $\overline{F}X$ lies in $F\{e_X\}$ and equals to $\eta X(e_X)$.

For an arbitrary compactum X, turn $X'' = X \sqcup \{y\}$ into a compact monoid by defining a binary operation $\odot$ by the equalities: $x \odot y = y \odot x = x$ for all $x \in X$, $x_1 \odot x_2 = x_1$, $x_1, x_2 \in X$, and $y \odot y = y$, i.e., by declaring y the unity. Then for $a \in FX \subset \overline{F}X''$ and $b = \eta X''(y) \in F\{y\} \subset \overline{F}X''$ we obtain $a = a \mathbin{\bar{\odot}} b = F\odot \circ\, t(X'', X'')(a, b)$. Since t is a natural transformation, we have

$$t(X'', X'')(a, b) \in t(X, \{y\})(FX \times F\{y\}) \subset F(X \times \{y\}).$$

The operation $\odot$ acts on $X \times \{y\}$ as the projection $\mathrm{pr}_1 : X \times \{y\} \cong X$. So $a = F\,\mathrm{pr}_1 \circ t(X, \{y\})(a, \eta\{y\}(y))$ and $F\,\mathrm{pr}_1 \circ t(X, \{y\}) = \mathrm{pr}_1$. Analogously, we prove that $F\,\mathrm{pr}_2 \circ t(\{x\}, Y) = \mathrm{pr}_2$ for a one-point compactum $\{x\}$ and an arbitrary compactum Y. Thus, $(**)$ holds.

The remaining arguments are an easy modification of the proof for $\mathscr{C} = \mathscr{C}\mathscr{S}\mathscr{G}r$.

Lemma 3. *For a natural lifting $\overline{F}$ of a functor F to $\mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$ and $a, b \in FX$, we have*

$$\mathrm{supp}_F(a \mathbin{\bar{\odot}} b) \subset \mathrm{supp}_F\, a \odot \mathrm{supp}_F\, b = \{x \odot y \mid x \in \mathrm{supp}_F\, a,\ y \in \mathrm{supp}_F\, b\}.$$

Proof. Consider the abelian semigroup $\{0, 1, 2, 3\}$ with the operation $\odot$ defined as follows: $0 \odot 1 = 1 \odot 0 = 2$, and $\alpha \odot \beta = 3$ otherwise. Put

$$X' \subset X \times \{0, 1, 2, 3\},$$
$$X' = \mathrm{supp}_F\, a \times \{0\} \sqcup \mathrm{supp}_F\, b \times \{1\} \sqcup (\mathrm{supp}_F\, a \odot \mathrm{supp}_F\, b) \times \{2\} \sqcup X \times \{3\}$$

and define $\odot$ on X' by the formula $(x, \alpha) \odot (y, \beta) = (x \odot y, \alpha \odot \beta)$. Then X' with the sum topology is a compact abelian semigroup and the projections $\mathrm{pr}_1 : X' \to X$ and $\mathrm{pr}_2 : X' \to \{0, 1, 2, 3\}$ are continuous homomorphisms. The rest of the proof verbatim repeats the proof of the previous similar lemma.

Proof of Theorem 1 for $\mathscr{C} = \mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$. Let $\overline{F}$ be a natural lifting. Given compacta X and Y, denote by $s(X, Y)$ the topological sum $X \sqcup Y \sqcup (X \times Y) \sqcup \{*\}$ with the operation $\odot$ defined as $a \odot b = b \odot a = (x, y)$ for $a = x \in X$, $b = y \in Y$, and $a \odot b = *$ otherwise. Denote by $i_X : X \hookrightarrow s(X, Y)$, $i_Y : Y \hookrightarrow s(X, Y)$, and $i_{X\times Y} : X \times Y \hookrightarrow s(X, Y)$ the corresponding embeddings and put $t(X, Y)(a, b)$ equal to the unique element $F(X, Y)$ for which $Fi_{X\times Y}(t(X, Y)(a, b)) = Fi_X(a) \mathbin{\bar{\odot}} Fi_Y(b)$, $a \in FX$, $b \in FY$. Obviously, $t(-, -)$ is a natural transformation $F(-) \times F(-) \to F(- \times -)$.

To prove $(*)$ for t consider the topological sum

$$s(X, Y, Z) = X \sqcup Y \sqcup Z \sqcup (X \times Y) \sqcup (Y \times Z) \sqcup (X \times Z) \sqcup (X \times Y \times Z) \sqcup \{*\}$$

and define the operation $\odot$ as follows: given $x \in X$, $y \in Y$, and $z \in Z$, put

$$x \odot y = y \odot x = (x, y), \quad y \odot z = z \odot y = (y, z),$$
$$x \odot (y, z) = (y, z) \odot x = y \odot (x, z) = (x, z) \odot y = z \odot (x, y) = (x, y) \odot z = (x, y, z),$$

and let $a \odot b = *$ otherwise. The embeddings $s(X \times Y, Z) \hookrightarrow s(X, Y, Z)$, $s(X, Y \times Z) \hookrightarrow s(X, Y, Z)$ are also homomorphisms. The remaining part of the proof is standard in this article.

We also omit the easy proof of the “commutative law” $(***)$.

If the natural transformation $t(-, -)$ is known then reconstruct the operation $\odot$ on $\overline{F}X$ for an arbitrary compact abelian semigroup $(X, \odot)$. The abelian semigroup $s(X, X)$ has the form $X \sqcup X' \sqcup X \times X \sqcup \{*\}$, where X' is a copy of X. As in the case of the category of compact semigroups, denote by $s(X)$ the topological sum $X \sqcup X' \sqcup X \times X \sqcup X''$, where X' and X'' are two copies of X. The embeddings $i_X, i_{X'} : X \hookrightarrow s(X)$ and $i_{X\times X} : X \times X \hookrightarrow s(X)$ and the mappings $p : s(X) \to X$ and $q : s(X) \to s(X, Y)$ are defined as for the case of $\mathscr{C} = \mathscr{C}\mathscr{S}\mathscr{G}r$. The only difference is in the definition of $\odot$ on $s(X)$; namely, $x \odot y = y \odot x = (x, y)$ for $x \in X$, $y \in X'$, and $a \odot b = p(a) \odot p(b) \in X''$ otherwise. Obviously, $(s(X), \odot)$ is a compact abelian semigroup, while q and p are semigroup homomorphisms. The remaining part of the proof of the fact that, in $\overline{F}X$, the operation is defined by the equality $a \mathbin{\bar{\odot}} b = F\odot \circ\, t(X, X)(a, b)$ is carried out as above.

Proof of Theorem 1 for $\mathscr{C} = \mathscr{C}\mathscr{A}b\mathscr{M}on$ is obtained by the "crossing" of the proofs for $\mathscr{C} = \mathscr{C}\mathscr{M}on$ and $\mathscr{C} = \mathscr{C}\mathscr{A}b\mathscr{S}\mathscr{G}r$.

Pass to the study of the liftings of normal liftings to the category of convex compacta.

We will use joins only for topological spaces that are convex compacta. Suppose that some convex compacta X and Y lie in topological vector spaces L_0 and L_1 respectively and contain the zero vectors of these spaces designated as $\mathbf{0}$. Call the convex hull of $X \times \{0\} \times \{\mathbf{0}\} \cup \{\mathbf{0}\} \times \{1\} \times Y$ in the product $L_0 \times \mathbb{R} \times L_1$ the *join* of X and Y and denote it by $X \bowtie Y$. Obviously, $X \bowtie Y$ is a convex compactum, and the convex structure on it does not depend on the embeddings of X and Y in L_0 and L_1, and, hence, on the location of $\mathbf{0}$ inside X and Y. The mappings $e_0 : X \to X \bowtie Y$ and $e_1 : Y \to X \bowtie Y$, defined as $e_0(x) = (x, 0, \mathbf{0})$ and $e_1(y) = (\mathbf{0}, 1, y)$ for all $x \in X$ and $y \in Y$, are affine embeddings.

Our realization of the join is homeomorphic to the standard one but has the advantage that $X \bowtie Y$ is the sum (coproduct) of X and Y in the category of convex compacta. In practice this means that, for any affine continuous mappings $f : X \to Z$, $g : Y \to Z$ into a convex compactum Z, there exists a unique affine continuous extension $f \triangledown g : X \bowtie Y \to Z$, i.e., a mapping satisfying the equalities $(f \triangledown g) \circ e_0 = f$, $(f \triangledown g) \circ e_1 = g$. Every element in the join is represented as $\lambda e_0(x) + \mu e_1(y)$, where $x \in X$, $y \in Y$, $0 \le \lambda$, $\mu \le 1$, $\lambda + \mu = 1$. Then the desired extension $f \triangledown g$ maps this element to $\lambda f(x) + \mu g(y)$.

As a consequence, for every continuous affine mappings $f : X \to Z$, $g : Y \to T$ of convex compacta, there exists a unique common affine extension $f \bowtie g : X \bowtie Y \to Z \bowtie T$, and we obtain the functor $\bowtie : \mathscr{C}onv \times \mathscr{C}onv \to \mathscr{C}onv$.

The existence of a natural lifting to $\mathscr{C}onv$ for a power functor in $\mathscr{C}omp$ is obvious. Partition the proof of the necessity of the fact that F is a power functor for the existence of a natural lifting for a normal functor F into a sequence of lemmas. Henceforth we assume that $\overline{F}$ is a natural lifting for F and $0 \le \lambda \le 1$.

Note that the intersection of $X \bowtie Y$ and ${\mathrm{pr}_2}^{-1}(\lambda) = L_0 \times \{\lambda\} \times L_1$ (where $\mathrm{pr}_2 : L_0 \times \mathbb{R} \times L_1 \to \mathbb{R}$ is the projection onto the second factor) is equal to $(1-\lambda)X \times \{\lambda\} \times \lambda Y$. The mapping $i_\lambda : X \times Y \to (1-\lambda)X \times \{\lambda\} \times \lambda Y$, defined as $i_\lambda(x, y) = \big((1-\lambda)x, \lambda, \lambda y\big)$ for all $x \in X$ and $y \in Y$, is a homeomorphism for all $0 < \lambda < 1$. Denote by $j_\lambda : (1-\lambda)X \times \{\lambda\} \times \lambda Y \to X \times Y$ the mapping inverse to i_λ.

Lemma 4. *Suppose that X and Y are convex compacta, while $a \in FX$ and $b \in FY$. Then the convex combination $(1-\lambda)a_0 + \lambda b_1$ of the elements $a_0 = Fe_0(a)$ and $b_1 = Fe_1(b)$ in $\overline{F}(X \bowtie Y)$ lies in $F((1-\lambda)X \times \{\lambda\} \times \lambda Y))$.*

Proof. It suffices to notice that $F\,\mathrm{pr}_2(a_0)$ and $F\,\mathrm{pr}_2(b_1)$ are the only elements of the subsets $F\{0\}$ and $F\{1\}$ in FI respectively. These elements can be identified with $0, 1 \in I$; therefore, their convex combination with the coefficients $1-\lambda$ and λ is identified with λ, and so it is the only element in $F\{\lambda\}$. The preservation of preimages implies that $(1-\lambda)a_0 + \lambda b_1 \in F(\mathrm{pr}_2^{-1}(\lambda))$.

Lemma 5. *For $a \in FX$, where X is a convex compactum, the convex combination $a_\lambda = (1 - \lambda)a_0 + \lambda a_1$ of $a_0 = Fe_0(a)$ and $a_1 = Fe_1(a)$ in $\overline{F}(X \bowtie X)$ is equal to $Fe_\lambda(a)$, where the embedding $e_\lambda : X \to X \bowtie X$ maps each $x \in X$ to $((1-\lambda)x, \lambda, \lambda x)$.*

Proof. Recall that, for every compactum X, there exists an embedding $\eta_P X$ into the space PX of probability measures on X endowed with the weak* topology [10]. This embedding assigns to each $x \in X$ the Dirac measure δ_x concentrated at x. Note that PX is a convex compactum but the embedding $\eta_P X$ is not affine. Moreover, its image lies in PX freely (in the categorical sense); in particular, no nontrivial convex combination of this image belongs to it. On the other hand, the barycenter mapping $bX : PX \to X$ [10] is affine and left inverse to $\eta_P X$.

Consider the embeddings $e_0', e_1' : PX \to PX \bowtie PX$ defined similarly to e_0 and e_1. Put $a' = F\eta_P X(a)$, $a_0' = Fe_0'(a')$, and $a_1' = Fe_1'(a')$. By the previous lemma, the element $a_\lambda' = (1-\lambda)a_0' + \lambda a_1'$ is contained in $F((1-\lambda)PX \times \{\lambda\} \times \lambda PX)$. On the other hand, the mapping $p = \mathbf{1}_{PX} \triangledown \mathbf{1}_{PX} : PX \bowtie PX \to PX$ is affine and $Fp(a_0') = Fp(a_1') = a'$; hence,

$$Pf(a_\lambda') = (1-\lambda)a' + \lambda a' = a' \in F(\eta_P X(X)).$$

The preservation of preimages implies that

$$a_\lambda \in F((1-\lambda)PX \times \{\lambda\} \times \lambda PX \cap p^{-1}(\eta_P X(X))).$$

The last intersection in parentheses is equal to $\{((1-\lambda)\eta_P X(x), \lambda, \lambda\eta_P X(x)) \mid x \in X\}$ and is mapped by p onto $\eta_P X(X)$ bijectively. This implies that $a'_\lambda = Fe'_\lambda(a)$, where $e'_\lambda(x) = ((1-\lambda)\eta_P X(x), \lambda, \lambda\eta_P X(x))$ for every $x \in X$.

It remains to observe that

$$a = FbX(a'), \quad a_0 = F(bX \bowtie bX)(a'_0), \quad a_1 = F(bX \bowtie bX)(a'_1);$$

therefore,

$$a_\lambda = F(bX \bowtie bX)(a'_\lambda) = F(bX \bowtie bX) \circ Fe'_\lambda(a) = Fe_\lambda(a).$$

Recall that a functor $F : \mathscr{C}omp \to \mathscr{C}omp$ is called *weakly bicommutative* if, for given compacta X and Y together with $a \in FX$ and $b \in FY$, there is $c \in F(X \times Y)$ such that $F\,\mathrm{pr}_1(c) = a$ and $F\,\mathrm{pr}_2(c) = b$. If such an element c is unique for all a and b then F is called *multiplicative*.

Lemma 6. *The functor F is multiplicative.*

PROOF. Let us first check weak bicommutativity. By the preservation of embeddings and preimages, we may assume without loss of generality that $X = Y$ are coinciding convex compacta. Suppose that $a, b \in FX$, the embeddings $e_0, e_1 : X \to X \bowtie X$, and the mapping $j_\lambda : (1-\lambda)X \times \{\lambda\} \times \lambda X \to X \times X$ are defined above, $q : X \to X$ is the mapping identically equal to $\mathbf{0}$, and z is the only element in $F\{\mathbf{0}\}$. Put $a_0 = Fe_0(a)$, $a_1 = Fe_1(a)$, $b_0 = Fe_0(b)$, and $b_1 = Fe_1(b)$. Fix any $0 < \lambda < 1$ and put $c' = (1-\lambda)a_0 + \lambda b_1$ and $c = Fj_\lambda(c') \in F(X \times X)$. Then

$$\begin{aligned} F(q \bowtie \mathbf{1}_X)(c') &= (1-\lambda)F(q \bowtie \mathbf{1}_X) \circ Fe_0(a) + \lambda F(q \bowtie \mathbf{1}_X) \circ Fe_1(b) \\ &= (1-\lambda)Fe_0(z) + \lambda F(q \bowtie \mathbf{1}_X) \circ Fe_1(b) \\ &= (1-\lambda)F(q \bowtie \mathbf{1}_X) \circ Fe_0(b) + \lambda F(q \bowtie \mathbf{1}_X) \circ Fe_1(b) \\ &= F(q \bowtie \mathbf{1}_X)\big((1-\lambda)Fe_0(b) + \lambda Fe_1(b)\big) = F(q \bowtie \mathbf{1}_X) \circ Fe_\lambda(b), \end{aligned}$$

whence

$$\begin{aligned} F\,\mathrm{pr}_2(c) &= F\,\mathrm{pr}_2 \circ Fj_\lambda(c') = F\,\mathrm{pr}_2 \circ Fj_\lambda \circ F(q \bowtie \mathbf{1}_X)(c') \\ &= F\,\mathrm{pr}_2 \circ Fj_\lambda \circ F(q \bowtie \mathbf{1}_X) \circ Fe_\lambda(b). \end{aligned}$$

Since $\mathrm{pr}_2 \circ j_\lambda \circ (q \bowtie \mathbf{1}_X) \circ e_\lambda = \mathbf{1}_X$, we have $F\,\mathrm{pr}_2(c) = b$. The equality $F\,\mathrm{pr}_1(c) = a$ is proved similarly.

Suppose that $a \in FX$, $b \in FY$, $c \in F(X \times Y)$, $F\,\mathrm{pr}_1(c) = a$, and $F\,\mathrm{pr}_2(c) = b$. To prove the uniqueness of such c we may assume that X and Y are convex compacta. Along with the join $X \bowtie Y$ and the embeddings $e_0 : X \to X \bowtie Y$, $e_1 : Y \to X \bowtie Y$, consider the join $(X \times Y) \bowtie (X \times Y)$ and the analogous embeddings $e'_0, e'_1 : X \times Y \to (X \times Y) \bowtie (X \times Y)$. Put $c'_0 = Fe'_0(c)$, $c'_1 = Fe'_1(c)$, and $c' = (1-\lambda)c'_0 + \lambda c'_1$. By Lemma 5 $c' = Fe'_\lambda(c)$, where $e_\lambda(x, y) = ((1-\lambda)(x, y), \lambda, \lambda(x, y))$. Since, under the affine mapping $F(\mathrm{pr}_1 \bowtie \mathrm{pr}_1) : F((X \times Y) \bowtie (X \times Y)) \to F(X \bowtie Y)$, the elements c'_0, c'_1, and c' are taken to $Fe_0(a)$, $Fe_1(b)$, and $F(\mathrm{pr}_1 \bowtie \mathrm{pr}_2) \circ Fe'_\lambda(c) = Fi_\lambda(c)$ respectively, we obtain the equality

$$(1-\lambda)Fe_0(a) + \lambda Fe_1(b) = Fi_\lambda(c)$$

in $\overline{F}(X \bowtie Y)$. Since i_λ is an embedding, c is uniquely determined by a and b. This proves the multiplicativity of F.

Since the only multiplicative normal functor is the power functor [7], Theorem 2 is proved.

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