Spherical fibrations and L-groups

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Let $X \subset Y$ be a submanifold of codimension q in an n-dimensional manifold Y, and let U be a tubular neighbourhood of X in Y. The obstruction groups $LS_{n-q}(F)$ to splitting and the obstruction groups $LP_{n-1}(F)$ to surgery with respect to the pair of manifolds are defined, and depend functorially on the universal push-out square of oriented fundamental groups

$$F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(Y \setminus X) \\ \downarrow & & \downarrow \\ \pi_1(X) & \longrightarrow & \pi_1(Y) \end{pmatrix} = \begin{pmatrix} A & \xrightarrow{f} & C \\ \downarrow i & & \downarrow j \\ B^{\xi} & \xrightarrow{g} & D \end{pmatrix}, \tag{1}$$

and on the value of $n-q \mod 4$ ([1], [2]).

The groups $LS_*(F)$ and $LP_*(F)$ are closely connected with the Wall groups of the square F by means of diagrams of exact sequences (see [1], [2]). If the horizontal maps in F are isomorphisms, then the groups $LS_*(F)$ are denoted by $LN_*(A \to B)$. When X is a one-sided submanifold in Y ($q=1$) and the horizontal maps in F are epimorphisms, then there are in addition a number of diagrams of exact sequences in which the groups $LS_*(F)$ and $LP_*(F)$ appear (see [3]–[5]).

In this paper we give similar results for arbitrary codimension q under certain natural restrictions on the square F. If $q \geq 3$ the surgery problem inside the manifold coincides with the abstract surgery problem ([1], [2]), so that the results become trivial when $q \geq 3$. We emphasize that in the case $q = 1$ the results obtained hold for both two-sided and one-sided manifolds.

The map i in the square (1) is the map of fundamental groups from the exact sequence of the spherical fibration $S^{q-1} \to \partial U \to X$. According to [1], there is the exact sequence

$$\to L_{n+1}(B^{\xi}) \to L_{n+q+1}(A \to B) \to LN_n(A \to B) \to L_n(B^{\xi}) \to . \tag{2}$$

We shall assume that the pair of maps (f, g) in (1) is induced by a morphism of spherical fibrations with fibre S^{q-1}, and that g is compatible with the orientation of the base spaces. We shall call such a diagram *geometric* by analogy with [3]–[6]. In particular, we may regard j as the map of fundamental groups from the exact sequence of a fibration, by analogy with i. Let Ψ denote the universal push-out square in which the vertical columns coincide with the left-hand column of the square F, and let Φ denote the analogous square for the right-hand column of F. The left-hand group in Ψ will be denoted by B^{ξ}, and the right-hand one by B, since the orientation of these groups may be different (see [1], [2]). Thus, for example, it is always different in the case of a one-sided submanifold of codimension 1. We denote the orientation of the group D in Φ similarly. There is an exact sequence similar to (2) for Φ. The natural maps of the universal push-out squares of oriented groups are defined,

$$\begin{pmatrix} A & \longrightarrow & A \\ \downarrow i & & \downarrow i \\ B^{\xi} & \longrightarrow & B \end{pmatrix} \to \begin{pmatrix} A & \xrightarrow{f} & C \\ \downarrow i & & \downarrow j \\ B^{\xi} & \xrightarrow{g} & D \end{pmatrix} \to \begin{pmatrix} C & \longrightarrow & C \\ \downarrow j & & \downarrow j \\ D^{\xi} & \longrightarrow & D \end{pmatrix},$$

inducing maps of the corresponding LS_*- and LP_*-groups.

There exist simplicial Ω-spectra whose homotopy groups are isomorphic to the corresponding L-, LN-groups (see [6], [7]). The map $L_{n+1}(B^{\xi}) \to L_{n+3}(A \to B)$ in the exact sequence (2) can

The work of the first author was carried out with the support of the Russian Foundation for Fundamental Research (grant no. 96-01-01166a) and grant no. 96-15-96841 of the President of the Russian Federation.

The work of the second author was carried out with the support of the Ministry of Science and Technology of the Republic of Slovenia (grant no. J1-0885-0101-98).

1991 *Mathematics Subject Classification*. Primary 57R67.

be realized at the spectral level (see [1]). The natural map of relative groups can also be realized at the spectral level (see [6]). The composite gives a map of spectra $s\colon \Omega\mathbb{L}(B^\xi) \to \Omega^{q+1}\mathbb{L}(C \to D)$, the homotopy cofibre of which we denote by $\mathbb{LS}(F)$. The spectrum of $\mathbb{LP}(F)$ is defined similarly as the homotopy cofibre of the map $p\colon \Omega\mathbb{L}(B^\xi) \to \Omega^q\mathbb{L}(C)$. For the spectra defined in this way we have isomorphisms $\pi_n(\mathbb{LS}(F)) = LS_n(F)$, $\pi_n(\mathbb{LP}(F)) = LP_n(F)$.

Theorem 1. *Let F be a geometric diagram of groups for a pair of manifolds of codimension $q = 1, 2$. Then we have the following universal squares of spectra*:

$$\begin{array}{ccc} \Omega\mathbb{L}(D^\xi) & \longrightarrow & \Omega^{q+1}\mathbb{L}(C \to D) \\ \downarrow & & \downarrow \\ \Omega\mathbb{L}(B^\xi \to D^\xi) & \longrightarrow & \mathbb{LS}(F), \end{array} \qquad \begin{array}{ccc} \Omega\mathbb{LN}^{\mathrm{rel}} & \longrightarrow & \mathbb{LN}(A \to B) \\ \downarrow & & \downarrow \\ \Omega\mathbb{L}(B^\xi \to D^\xi) & \longrightarrow & \mathbb{LS}(F) \end{array},$$

where $\mathbb{LN}^{\mathrm{rel}}$ denotes the homotopy cofibre of the map $\mathbb{LN}(A \to B) \to \mathbb{LN}(C \to D)$.

All the maps in the squares can be considered as fibrations (or cofibrations). The homotopy long exact sequences of these fibrations together with the universality property give commutative diagrams of exact sequences analogous to [3]. We mention the most interesting ones.

Corollary. *Under the conditions of Theorem* 1 *there are the exact sequences*

$$\to L_{n+1}(B^\xi \to D^\xi) \to LS_n(F) \to LN_n(C \to D) \to,$$

$$\to L^{\mathrm{rel}}_{n+1} \to L_{n+1}(B^\xi \to D^\xi) \to L_{n+q+1}(F) \to .$$

We note that there are also results analogous to [5] for the groups LP_* in any codimension. Here we shall only mention the exact sequences obtained in this way.

Theorem 2. *Under the conditions of Theorem* 1 *there are the exact sequences*

$$\to LP_n(\Psi) \to LP_n(F) \to L_{n+q}(A \to C) \to,$$

$$\to L_{n+1}(B^\xi \to D^\xi) \to LP_n(F) \to LP_n(\Phi) \to,$$

$$\to L_{n+q+1}(A \to C) \to LP^{\mathrm{rel}}_{n+1} \to L_{n+1}(B^\xi \to D^\xi) \to,$$

where LP^{rel}_* are the relative groups appearing in the exact sequence

$$\to LP_*(\Psi) \to LP_*(\Phi) \to LP^{\mathrm{rel}}_* \to .$$

In the case $q \geq 3$ the vertical maps in the square F become isomorphisms, and all the relative Wall groups are trivial. Hence the corresponding exact sequences become degenerate.

We note also that these arguments enable us to define the groups LS_* and LP_* for an arbitrary morphism of S^q-fibrations. The groups thus defined will appear not only in the diagrams of the groups constructed by Wall (see [1]) but also in the diagrams of the groups obtained in the present paper.

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Received 26/JAN/99