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Surgery on triples of manifolds

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Abstract. The surgery obstruction groups for a manifold pair were introduced by Wall for the study of the surgery problem on a manifold with a submanifold. These groups are closely related to the problem of splitting a homotopy equivalence along a submanifold and have been used in many geometric and topological applications.

In the present paper the concept of surgery on a triple of manifolds is introduced and algebraic and geometric properties of the corresponding obstruction groups are described. It is then shown that these groups are closely related to the normal invariants and the classical splitting and surgery obstruction groups, respectively, of the manifold in question. In the particular case of one-sided submanifolds relations between the newly introduced groups and the surgery spectral sequence constructed by Hambleton and Kharshiladze are obtained.

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§ 1. Introduction

Let X^n be a closed topological n-manifold with fundamental group $\pi_1(X)$ and orientation homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$. The main question in algebraic and geometric topology is the description of all possible closed topological (piecewise linear, smooth) manifolds that are (simply) homotopy equivalent to X (see, for example, [1]–[3]).

Throughout the present paper we shall consider topological manifolds and simple homotopy equivalences between them (see [1], [4]).

Two simple homotopy equivalences $f_i\colon M_i \to X$, $i = 0, 1$, are said to be *equivalent* if there exists an orientation-preserving homeomorphism $g\colon M_0 \to M_1$ such that the composite map $f_1 \circ g$ is homotopic to f_0. Let $\mathcal{S}^{TOP}(X)$ be the set of such equivalence classes. Elements of this set are called *homotopy triangulations* (*s-triangulations*) of the manifold X. Ranicki's total surgery obstruction theory (see [1], [4]) yields a map of spectra

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\pi_1(X)), \tag{1}$$

where $\mathbb{L}(\pi_1(X))$ is the surgery L-spectrum of the fundamental group $\pi_1(X)$ with

$$\pi_n(\mathbb{L}(\pi_1(X))) \cong L_n(\pi_1(X))$$

and $\mathbf{L}_\bullet$ is the 1-connected cover of the Ω-spectrum $\mathbb{L}(\mathbb{Z})$ such that $\mathbf{L}_{\bullet 0} \simeq G/TOP$.

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Let $\mathbb{S}$ be a homotopy cofibre of the map in (1). For the homotopy groups $\mathcal{S}_m(X) = \pi_m(\mathbb{S})$ we have an isomorphism $\mathcal{S}_{n+1}(X) \cong \mathcal{S}^{TOP}(X)$. The algebraic surgery exact sequence of the manifold X (see [1] and [4])

$$\cdots \to L_{m+1}(\pi_1(X)) \to \mathcal{S}_{m+1}(X) \to H_m(X; \mathbf{L}_\bullet) \xrightarrow{\sigma} L_m(\pi_1(X)) \to \cdots \tag{2}$$

is the homotopy long exact sequence of the fibration in (1). Here we have an isomorphism $H_n(X_+; \mathbf{L}_\bullet) \cong [X, G/TOP]$. The set $[X, G/TOP]$ of normal invariants consists of the classes of normal cobordisms of normal maps into the manifold X, and the assembly map σ is defined by taking the surgery obstruction of the normal map (see [2]–[4]). The study of the assembly map is closely related to many problems in the topology of manifolds (see, for example, [3], [5]).

Let $Y \subset X$ be a submanifold of codimension q of X. In this case the map σ factors into the composite of maps

$$[X, G/TOP] \to LP_{n-q}(F) \to L_n(\pi_1(X)),$$

where $LP_{n-q}(F)$ is the surgery obstruction group for the manifold pair (see [1], [3]). The group $LP_{n-q}(F)$ depends functorially on the square of fundamental groups with orientation

$$F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X) \end{pmatrix}, \tag{3}$$

where ∂U is the boundary of a tubular neighbourhood of Y in X. The groups $LP_{n-q}(F)$ are closely related to other algebraic objects occurring in L-theory by means of various types of diagrams of exact sequences (see [1], [3], and [6]). In particular, these groups fit in the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{S}_{n+1}(X) & \longrightarrow & H_n(X; \mathbf{L}_\bullet) & \xrightarrow{\sigma} & L_n(\pi_1(X)) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle =} & & \\ \cdots & \longrightarrow & LS_{n-q}(F) & \longrightarrow & LP_{n-q}(F) & \xrightarrow{p} & L_n(\pi_1(X)) & \longrightarrow & \cdots \end{array}, \tag{4}$$

in which all rows are exact sequences and $LS_{n-q}(F)$ is the splitting obstruction group. The lower exact sequence provides some additional information about the assembly map. In some particular cases this sequence was applied to the investigation of the oozing problem and for the construction of the surgery spectral sequence of Hambleton and Kharshiladze (see, for example, [7]–[9]). In the case of one-sided manifolds there exist many results on the relation of LS_*- and LP_*-groups to classical L_*-groups [6], [10]–[16].

Let $Z \subset Y \subset X$ be a triple of closed topological manifolds such that the dimension of X is n, the codimension of Y in X is q, and the codimension of Z in Y is q'. We shall assume that for each pair of manifolds we have fixed a normal fibration of the submanifold (see [1], § 7.2). In the present paper we introduce certain groups $LT_{n-q-q'}(X, Y, Z)$ which contain the obstructions to surgery on a triple of manifolds. Assume that $n-q-q' \geqslant 5$. For an arbitrary normal map $(f, b) \in [X, G/TOP]$, where $f\colon M \to X$, we define an obstruction $\Theta_*(f,b) \in LT_{n-q-q'}(X,Y,Z)$, which is trivial if and only if (f,b) is normally cobordant to an s-triangulation of the triple (X,Y,Z). The concept of s-triangulation for a triple of manifolds (see § 3 below) is a natural generalization of a similar concept for a manifold pair introduced in [1].

The map Θ_* fits in the following natural decomposition of the maps σ_1 and σ:

$$[X, G/TOP] \to LT_{n-q-q'}(X,Y,Z) \to LP_{n-q}(F) \to L_n(\pi_1(X)).$$

The groups LT_* are closely related to the surgery exact sequence (2) and the surgery and the splitting obstruction groups for the manifold pairs $Y \subset X$ and $Z \subset Y$. These connections are revealed by certain braids of exact sequences containing the classical obstruction groups. Let (X,Y,Z) be a triple of manifolds in which (X,Y) and (Y,Z) are Browder–Livesay pairs (see [7], [17], [18]). For such a triple Hambleton and Pedersen [19] introduced LNS_*-groups, which are related to the splitting obstruction groups and the L_*-groups in this case. LT_*-groups are also connected with iterated transfers and LNS_*-groups (see [20]).

For the proof of our main results we shall use L-spectra and realizations on the spectral level of natural maps in L-theory (see [3], [4], [6], [21]–[24]).

The paper is organized as follows. In § 2 we recall some basic definitions and facts on L-spectra and their relations to surgery theory. Furthermore, we prove a couple of technical lemmas. In § 3 we define a spectrum $\mathbb{L}T$ with homotopy groups $\pi_n(\mathbb{L}T) = LT_n(X,Y,Z)$ and prove the main theorem of this paper. After that we obtain some braids of exact sequences containing the groups $LT_n(X,Y,Z)$ as well as the classical surgery and splitting obstruction groups. In § 4 we describe applications of the spectra $\mathbb{L}T$ and the groups $LT_n(X,Y,Z)$ to the case of one-sided manifolds and their relations to the surgery exact sequence constructed by Hambleton and Kharshiladze in [8].

§ 2. Spectra in *L*-theory

In this section we recall several basic definitions and known facts about the realization of algebraic objects and natural maps in surgery theory on the spectral level (see [1], [3], [4], [11], [21]–[25]). After that, we prove two technical lemmas.

A topological normal map $(f,b)\colon M \to X$ into a closed topological n-manifold X (a *t-triangulation of* X) is defined by the following conditions (see [1] and [4]):

(i) a closed topological n-manifold M with normal topological block bundle

$$\nu_M = \nu_{M\subset S^{n+k}}\colon M \to BTOP(k),$$
$$\rho_M\colon S^{n+k} \to S^{n+k}/\overline{S^{n+k} \setminus E(\nu_M)} = T(\nu_M);$$

(ii) a normal topological block bundle over the manifold X

$$\nu_X\colon X \to BTOP(k),$$
$$\rho_X\colon S^{n+k} \to T(\nu_X);$$

(iii) a map $f\colon M \to X$ of degree 1 together with a map of topological block bundles $b\colon \nu_M \to \nu_X$ covering f. Furthermore,

$$T(b)_*(\rho_M) = \rho_X \in \pi_{n+k}(T(\nu_X)).$$

The set of topological normal invariants of the manifold X consists of all concordance classes of topological normal maps $(f,b)\colon M \to X$. According to [1], for each $n \geqslant 5$ this set coincides with $[X, G/TOP]$.

Let Y be a closed submanifold of codimension q of a closed topological manifold X. In the present paper we shall consider only *manifold pairs* (X,Y,ξ) in the sense of Ranicki (see [1], § 7.2). This means that Y is a locally flat submanifold and for the normal fibration

$$\xi = \xi_{Y\subset X}\colon Y \to \widetilde{BTOP}(q)$$

one has

$$X = E(\xi) \bigcup_{S(\xi)} \overline{X \setminus E(\xi)}.$$

In particular, the associated (D^q, S^{q-1}) fibration

$$(D^q, S^{q-1}) \to (E(\xi), S(\xi)) \to Y$$

is well defined.

A topological normal map (t-*triangulation of* (X,Y,ξ)) is defined by the following object (see [1] for details):

$$((f,b),(g,c))\colon (M,N) \to (X,Y).$$

Here (f,b) is a t-triangulation of the manifold X such that f is transversal to Y with $N = f^{-1}(Y)$ and (M,N) is a topological manifold pair with normal fibration

$$\nu\colon N \xrightarrow{f|_N} Y \xrightarrow{\xi} \widetilde{BTOP}(q).$$

The restriction

$$(f,b)\big|_N = (g,c)\colon N \to Y$$

is a t-triangulation of Y. The restriction

$$(f,b)\big|_P = (h,d)\colon (P,S(\nu)) \to (Z,S(\xi))$$

is a t-triangulation of $(Z,S(\xi))$, where

$$P = \overline{M \setminus E(\nu)} \quad \text{and} Z = \overline{X \setminus E(\xi)}.$$

The restriction

$$(h,d)\big|_{S(\nu)}\colon S(\nu) \to S(\xi)$$

coincides with the induced map

$$(g,c)^!\colon S(\nu) \to S(\xi)$$

and $(f,b) = (g,c)^! \cup (h,d)$.

By [1] the set of concordance classes of t-triangulations of (X, Y, ξ) coincides with the set of classes of t-triangulations of the manifold X.

An s-triangulation of a manifold pair (X, Y, ξ) (see [1]) is a t-triangulation

$$((f, b), (g, c))\colon (M, N) \to (X, Y)$$

of this pair such that the maps

$$f\colon M \to X, \qquad g\colon N \to Y, \qquad h\colon (P, S(\nu)) \to (Z, S(\xi)) \tag{5}$$

are s-triangulations.

From now on we shall assume that $n - q \geqslant 5$. A simple homotopy equivalence $f\colon M \to X$ splits along the submanifold Y if it is homotopy equivalent to a map g with properties (5). This means that the map g is an s-triangulation of (X, Y, ξ). There exists a group $LS_{n-q}(F)$ of obstructions to the splitting of a simple homotopy equivalence. This group depends only on $n-q \pmod 4$ and on the push-out square F of fundamental groups with orientations (3) (see [1]). We consider now a normal map $(f, b) \in [X, G/TOP]$ with $f\colon M \to X$. By [1] there exists a group $LP_{n-q}(F)$ of obstructions to surgery on the pair of manifolds (X, Y), which also depends only on $n - q \pmod 4$ and the square F. The surgery obstruction $\sigma_1(f, b) \in LP_{n-q}(F)$ for a normal map into a pair (X, Y) is trivial if and only if the concordance class of (f, b) contains an s-triangulation of (X, Y, ξ).

The upper row in the diagram (4) is the surgery exact sequence for the manifold X, and the lower row is an exact sequence containing the groups $LS_*(F)$, $LP_*(F)$, and $L_*(\pi_1(X))$. Deeper relations between these groups are revealed by the following braid of exact sequences (see [1] and [3]):

$$\begin{array}{ccccccccccc}
\to & L_{n+1}(C) & & \to & & L_{n+1}(D) & & \stackrel{\partial}{\to} & & LS_{n-q}(F) & \to \\
 & \nearrow \quad \searrow & & & \nearrow & & \searrow & & \nearrow & & \searrow \\
 & & & LP_{n-q+1}(F) & & & & L_{n+1}(C \to D) & & & \\
 & \searrow \quad \nearrow & & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LS_{n-q+1}(F) & & \to & & L_{n-q+1}(B) & & \to & & L_n(C) & \to
\end{array}, \tag{6}$$

where $A = \pi_1(\partial U)$, $B = \pi_1(Y)$, $C = \pi_1(X \setminus Y)$, and $D = \pi_1(X)$. The map

$$\partial\colon L_{n+1}(\pi_1(X)) \to LS_{n-q}(F)$$

in the diagram (6) coincides with the composite map

$$L_{n+1}(\pi_1(X)) \to \mathcal{S}_{n+1}(X) \to LS_{n-q}(F)$$

arising from the diagram (4).

Let $\mathcal{S}_{n+1}(X, Y, \xi)$ be the set of concordance classes of s-triangulations of the manifold pair (X, Y, ξ) (see [1]).

It follows from [17] that there exists a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathcal{S}_{n+1}(X, Y, \xi) & \longrightarrow & H_n(X; \mathbf{L}_\bullet) & \xrightarrow{\sigma_1} & LP_{n-q}(F) & \longrightarrow \cdots \\
 & \big\downarrow & & \big\downarrow & & \big\downarrow & \\
\cdots \longrightarrow & \mathcal{S}_{n-q+1}(Y) & \longrightarrow & H_{n-q}(Y; \mathbf{L}_\bullet) & \longrightarrow & L_{n-q}(Y) & \longrightarrow \cdots
\end{array}, \tag{7}$$

the rows of which are exact sequences. The lower row is the algebraic surgery exact sequence for the manifold Y. Note that the set $\mathbb{S}_{n+1}(X, Y, \xi)$ has a group structure.

We now recall the requisite definitions and properties relating to the homotopy category of spectra (see, for example, [25]). A *spectrum* $\mathbb{E}$ consists of a family of CW-complexes $\{(E_n, *)\}$, $n \in \mathbb{Z}$, and a family of cellular maps $\{\varepsilon_n : SE_n \to E_{n+1}\}$, where SE_n is the suspension of the space E_n.

For each ε_n there exists an adjoint map $\varepsilon_n' : E_n \to \Omega E_{n+1}$. A spectrum $\mathbb{E}$ is called an Ω-spectrum if all adjoint maps are homotopy equivalences.

We also recall the definition of the spectrum $\Sigma\mathbb{E}$ with $\{\Sigma\mathbb{E}\}_n = \mathbb{E}_{n+1}$ and $\{\Sigma\varepsilon\}_n = \varepsilon_{n+1}$. It is easily seen that the functor Σ has an inverse functor Σ^{-1}. Thus, it is possible to define the iterated functors Σ^k, $k \in \mathbb{Z}$, on the category of spectra. We have an isomorphism of homotopy groups $\pi_n(\mathbb{E}) = \pi_{n+k}(\Sigma^k\mathbb{E})$ for each spectrum $\mathbb{E}$ (see [25]).

In the homotopy category of spectra the concepts of pull-back and push-out squares are equivalent. Thus, a homotopy commutative square of spectra

$$\begin{array}{ccc} \mathbb{G} & \longrightarrow & \mathbb{H} \\ \downarrow & & \downarrow \\ \mathbb{E} & \longrightarrow & \mathbb{F} \end{array}$$

is a *pull-back* (*push-out*) *square* if and only if the fibres (the cofibres) of the two horizontal or the two vertical maps are naturally homotopy equivalent (see, for example, [8]).

Let $f : \pi \to \pi'$ be a homomorphism of oriented groups. Then one can define a cofibration of Ω-spectra

$$\mathbb{L}(\pi) \to \mathbb{L}(\pi') \to \mathbb{L}(f), \tag{8}$$

as shown, for example, in [21]. Here $\pi_n(\mathbb{L}(\pi)) = L_n(\pi)$ and similar relations hold for the other spectra. The homotopy long exact sequence of the cofibration (8) yields the relative exact sequence of L-groups for the map f:

$$\cdots \to L_n(\pi) \to L_n(\pi') \to L_n(f) \to L_{n-1}(\pi) \to \cdots .$$

Let $p : E \to X$ be a fibration over a closed topological n-manifold X with closed topological m-manifold M^m as a fibre. Then a transfer map

$$p^* : L_n(\pi_1(X)) \to L_{n+m}(\pi_1(E))$$

can be defined (see [3], [23], [24]). This map can be realized on the spectral level by a map of Ω-spectra

$$p^! : \mathbb{L}(\pi_1(X)) \to \Sigma^{-m}\mathbb{L}(\pi_1(E)).$$

For a manifold pair (X, Y) we consider now the homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \overset{p_1^!}{\to} & \Sigma^{-q}\mathbb{L}(\pi_1(\partial U) \to \pi_1(U)) & \overset{\alpha}{\to} & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\ & {\scriptstyle p^!} \searrow & \downarrow {\scriptstyle \delta} & & \downarrow {\scriptstyle \delta_1} \\ & & \Sigma^{1-q}\mathbb{L}(\pi_1(\partial U)) & \overset{\beta}{\to} & \Sigma^{1-q}\mathbb{L}(\pi_1(X \setminus Y)) \end{array}, \tag{9}$$

where the left-hand maps are transfer maps and the right-hand horizontal maps are induced by the horizontal maps in the square F.

We define the spectrum $\mathbb{L}S(F)$ as the homotopy cofibre of the map

$$\Sigma^{-1}(\alpha p_1^!)\colon \Sigma\mathbb{L}(\pi_1(Y)) \to \Sigma^{-q-1}\mathbb{L}(\pi_1(X\setminus Y)\to\pi_1(X))$$

and the spectrum $\mathbb{L}P(F)$ as the homotopy cofibre of the map

$$\Sigma^{-1}(\beta p^!)\colon \Sigma^{-1}\mathbb{L}(\pi_1(Y)) \to \Sigma^{-q}\mathbb{L}(\pi_1(X\setminus Y))$$

(see [10], [11], [14], [22] for further details). Let $LS_n(F)$ and $LP_n(F)$ be the splitting and the surgery obstruction groups, respectively, of the pair (X,Y). Then we have the isomorphisms

$$\pi_n(\mathbb{L}S(F))\cong LS_n(F), \qquad \pi_n(\mathbb{L}P(F))\cong LP_n(F).$$

Using these definitions and the homotopy commutative diagram of spectra (9) we obtain the following homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \Sigma^{-1}\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Sigma^{-q-1}\mathbb{L}(\pi_1(X\setminus Y)\to\pi_1(X)) & \longrightarrow & \mathbb{L}S(F) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}L(\pi_1(Y)) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(X\setminus Y)) & \longrightarrow & \mathbb{L}P(F) \end{array}, \qquad (10)$$

in which the right-hand square is a push-out square. The homotopy long exact sequences of the maps in the right-hand square give rise to the commutative diagram (6).

Let Y^{n-q} be a closed submanifold of codimension q of the closed n-dimensional topological manifold X. Then the commutative diagram (4) is defined.

Lemma 1. *With the above notation there exists a homotopy commutative square of spectra*

$$\begin{array}{ccc} X_+\wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\pi_1(X)) \\ \downarrow & & \downarrow{\scriptstyle =} \\ \Sigma^q\mathbb{L}P(F) & \longrightarrow & \mathbb{L}(\pi_1(X)) \end{array} \qquad (11)$$

such that the commutative diagram (4) *is obtained by means of the induced map of the homotopy long exact sequence of the top map in* (11) *into that of the bottom map.*

Proof. We consider the commutative diagram

$$\begin{array}{cccc} H_{n-q}(Y;\mathbb{L}_\bullet) & \overset{\cong}{\to} & H_n(X,X\setminus Y;\mathbb{L}_\bullet) & \\ & \searrow & \downarrow & \\ & & H_{n-1}(X\setminus Y;\mathbb{L}_\bullet) & \to \quad H_{n-1}(X;\mathbb{L}_\bullet) \end{array}, \qquad (12)$$

where the upper horizontal map is the transfer map. The vertical map and the bottom horizontal map arise from the homotopy long exact sequence of the pair $(X, X\setminus Y)$. Thus, the diagram (12) is generated by the following homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \Sigma^{-1}(Y_+\wedge \mathbb{L}_\bullet) & \longrightarrow & \Sigma^{-q-1}(\operatorname{Cof}\mathbf{j}) & \longrightarrow & * \\ =\downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}(Y_+\wedge \mathbb{L}_\bullet) & \longrightarrow & \Sigma^{-q}((X\setminus Y)_+\wedge \mathbb{L}_\bullet) & \longrightarrow & \Sigma^{-q}(X_+\wedge \mathbb{L}_\bullet) \end{array}, \qquad (13)$$

where Cof $\mathbf{j}$ s a homotopy cofibre of the natural inclusion

$$\mathbf{j}\colon (X\setminus Y)_+\wedge \mathbf{L}_\bullet \to X_+\wedge \mathbf{L}_\bullet.$$

The upper row and the vertical maps in the diagram (13) are cofibrations. The cofibres of the two right-hand vertical maps in (13) are homotopy equivalent to $\Sigma^{-q}(X_+\wedge \mathbb{L}_\bullet)$. The cofibres of the two right-hand vertical maps in (10) are homotopy equivalent to $\Sigma^{-q}\mathbb{L}(\pi_1(X))$. The assembly map in (1) (see [1]) induces a map from the left-hand square in the diagram (13) into the left-hand square in (10). By [25] we obtain maps from the right-hand column of the diagram (13) into the right-hand column of (10). Thus, we have a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma^{-q}(X_+\wedge \mathbf{L}_\bullet) & \longrightarrow & \mathbb{L}P(F) \\ \downarrow & & \downarrow \\ \Sigma^{-q}(X_+\wedge \mathbf{L}_\bullet) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(X)) \end{array}.$$

The application of the functor Σ^q to this diagram yields the required result.

Lemma 2. *There exists a homotopy commutative square of spectra*

$$\begin{array}{ccc} X_+\wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q\mathbb{L}P(F) \\ \downarrow & & \downarrow \\ \Sigma^q(Y_+\wedge \mathbf{L}_\bullet) & \longrightarrow & \Sigma^q(\mathbb{L}(\pi_1(Y))) \end{array}, \qquad (14)$$

such that the commutative diagram (7) *is obtained by means of the induced map of the homotopy long exact sequence of the top map in* (14) *into that of the bottom map.*

The proof is similar to the proof of Lemma 1.

§ 3. Surgery on triples of manifolds

Let $Z\subset Y\subset X$ be a triple of closed topological manifolds and let n be the dimension of X, q the codimension of Y in X, and q' the codimension of Z in Y. In this section we assume that for a fixed triple we have manifold pairs (X, Y, ξ) and (Y, Z, ξ_1). Sometimes we shall not indicate the fibrations in our notations if this will not lead to confusion. We assume that the dimension k of the manifold Z is at least 5: $k = n-q-q'\geqslant 5$.

Definition 1. A topological normal map (a *t-triangulation*)

$$((f,b),(g,c),(h,d))\colon (M,N,K)\to (X,Y,Z)$$

of a manifold triple (X,Y,Z) consists of t-triangulations of manifold pairs

$$((f,b),(g,c))\colon (M,N)\to (X,Y)$$

and

$$((g,c),(h,d))\colon (N,K)\to (Y,Z).$$

In particular, the topological normal map

$$(f,b)\colon M\to X$$

is topologically transversal to Y with respect to ξ with $N=f^{-1}(Y)$, and the topologically normal map

$$(g,c)\colon N\to Y$$

is topologically transversal to Z with respect to ξ_1 with $K=g^{-1}(Z)$.

Definition 2. An *s-triangulation* of a triple (X,Y,Z) is a t-triangulation

$$((f,b),(g,c),(h,d))\colon (M,N,K)\to (X,Y,Z)$$

for which the associated t-triangulations

$$((f,b),(g,c))\colon (M,N)\to (X,Y),$$
$$((g,c),(h,d))\colon (N,K)\to (Y,Z)$$

are s-triangulations.

Consider now a topological normal map $(f,b)\in [X,G/TOP]$ with $f\colon M\to X$. It follows from the topological transversality theorem (see [1]) that (f,b) defines a t-triangulation

$$((f,b),(g,c),(h,d))\colon (M,N,K)\to (X,Y,Z)$$

of the triple (X,Y,Z). We say that there exists a surgery on the manifold M yielding an s-triangulation of the triple (X,Y,Z) if this t-triangulation is normally cobordant to an s-triangulation of the triple (X,Y,Z). Recall that the set of classes of normal cobordisms of normal maps into a fixed closed topological manifold X coincides with the set $[X,G/TOP]=H_n(X;\mathbf{L}_\bullet)$ (see [1], [4]).

The square F in (3) and the obstruction groups $LP_*(F)$ and $LS_*(F)$ are well defined for the pair of manifolds $Y\subset X$. Let

$$\Psi=\begin{pmatrix}\pi_1(\partial V) & \longrightarrow & \pi_1(Y\setminus Z)\\ \Big\downarrow & & \Big\downarrow\\ \pi_1(Z) & \longrightarrow & \pi_1(Y)\end{pmatrix} \tag{15}$$

be a square of fundamental groups with orientations, where ∂V is the boundary of a tubular neighbourhood of Z in Y. Then the surgery obstruction groups $LP_*(\Psi)$ and the splitting obstruction groups $LS_*(\Psi)$ are defined for the manifold pair (Y, Z).

We consider now the map

$$v\colon LP_{n-q+1}(F) \to LS_{n-q-q'}(\Psi)$$

defined as the composite

$$LP_{n-q+1}(F) \to \mathcal{S}_{n+1}(X, Y, \xi) \to \mathcal{S}_{n-q+1}(Y) \to LS_{n-q-q'}(\Psi). \tag{16}$$

The first two maps in the sequence (16) lie in the diagram (7), while the last map $\mathcal{S}_{n-q+1}(Y) \to LS_{n-q-q'}(\Psi)$ lies in a diagram similar to (4) written for the pair of manifolds (Y, Z).

Proposition 1. *The map v can be realized by a map of Ω-spectra*

$$\mathbf{v}\colon \mathbb{L}P(F) \to \Sigma^{q'+1}\mathbb{L}S(\Psi)$$

so that the induced homomorphism of homotopy groups

$$\mathbf{v}_*\colon \pi_{n-q+1}(\mathbb{L}P(F)) = LP_{n-q+1}(F) \to LS_{n-q-q'}(\Psi) = \pi_{n-q+1}(\Sigma^{q'+1}\mathbb{L}S(\Psi))$$

coincides with v.

Proof. Let $\mathbb{S}(X, Y, \xi)$ be a homotopy cofibre of the map

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^q \mathbb{L}P(F)$$

considered in the statement of Lemma 2. Then we have

$$\pi_{n+1}(\mathbb{S}(X, Y, \xi)) = \mathcal{S}_{n+1}(X, Y, \xi).$$

The composite

$$LP_{n-q+1}(F) \to \mathcal{S}_{n+1}(X, Y, \xi) \to \mathcal{S}_{n-q+1}(Y)$$

of maps arising from the diagram (7) is realized by Lemma 2 and [25] by the following composite map of spectra:

$$\mathbb{L}P(F) \to \Sigma^{-q}\mathcal{S}_{n+1}(X, Y, \xi) \to \mathbb{S}(Y).$$

The map $\mathcal{S}_{n-q+1}(Y) \to LS_{n-q-q'}(\Psi)$ can be realized on the spectral level by Lemma 1 for the pair of manifolds (Y, Z). Thus, the composite map v can be realized on the spectral level as required.

Let $\mathbb{L}T(X, Y, Z)$ be a homotopy cofibre of the map $\Sigma^{-q'-1}\mathbf{v}$, and

$$LT_n(X, Y, Z) = \pi_n(\mathbb{L}T(X, Y, Z))$$

the homotopy groups of the corresponding spectra. In particular, we have a cofibration of Ω-spectra:

$$\Sigma^{-q'-1}\mathbb{L}P(F) \to \mathbb{L}S(\Psi) \to \mathbb{L}T(X, Y, Z). \tag{17}$$

The homotopy long exact sequence of the cofibration (17) yields the following exact sequence:

$$\cdots \to LP_{n-q+1}(F) \to LS_{n-q-q'}(\Psi) \to LT_{n-q-q'}(X, Y, Z) \to \cdots. \tag{18}$$

Theorem 1. *There exists a map of spectra*

$$\Theta\colon X_+ \wedge \mathbf{L}_\bullet \to \Sigma^{q+q'}\mathbb{L}T(X,Y,Z)$$

inducing the homomorphism

$$\Theta_*\colon H_n(X;\mathbf{L}_\bullet) \to LT_{n-q-q'}(X,Y,Z)$$

of homotopy groups. Let $(f,b) \in [X, G/TOP] = H_n(X;\mathbf{L}_\bullet)$ be a topological normal map with $f\colon M \to X$. Then (f,b) is normally cobordant to an s-triangulation of the triple (X,Y,Z) if and only if $\Theta_*(f,b) = 0$.

Proof. We consider the diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}P(F) & \longrightarrow & \Sigma^{-q}\mathbb{S}(X,Y,\xi) & \longrightarrow & \Sigma^{-q+1}(X_+ \wedge \mathbf{L}_\bullet) \\ \big\downarrow = & & \big\downarrow & & \big\downarrow \\ \mathbb{L}P(F) & \longrightarrow & \Sigma^{q'+1}\mathbb{L}S(\Psi) & \longrightarrow & \Sigma^{q'+1}\mathbb{L}T(X,Y,Z) \end{array}, \tag{19}$$

in which the left-hand square exists by Proposition 1 and the horizontal rows are cofibrations. Hence there exists a right-hand vertical map (see [25]) such that the diagram (19) is homotopy commutative. The application of the functor Σ^{q-1} to the right-hand vertical map yields the map Θ. We point out here that the right-hand square of (19) is a push-out (and a pull-back) square since the fibres of the horizontal maps are naturally homotopy equivalent. Taking the homotopy long exact sequences of the rows of the homotopy commutative diagram (19) one obtains the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{S}_{n+1}(X,Y,\xi) & \xrightarrow{\mu} & H_n(X;\mathbf{L}_\bullet) & \xrightarrow{\sigma_1} & LP_{n-q}(F) & \longrightarrow & \cdots \\ & & \big\downarrow \omega & & \big\downarrow \Theta_* & & \big\downarrow = & & \\ \cdots & \longrightarrow & LS_{n-q-q'}(\Psi) & \xrightarrow{\nu} & LT_{n-q-q'}(X,Y,Z) & \xrightarrow{\gamma} & LP_{n-q}(F) & \longrightarrow & \cdots . \end{array} \tag{20}$$

Assume that the normal map (f,b) is normally cobordant to an s-triangulation

$$((f',b'),(g',c'),(h',d'))\colon (M,N,K) \to (X,Y,Z).$$

The homomorphism Θ_* depends only on the normal cobordism class $x = [(f,b)] = [(f',b')] \in H_n(X;\mathbf{L}_\bullet)$. By the definition of an s-triangulation of the triple (X,Y,Z) the representative $((f',b'),(g',c'))$ of the class x lies in the set $\mathcal{S}_{n+1}(X,Y,\xi)$ of concordance classes of s-triangulations of the manifold pair (X,Y,ξ) [1]. Let y be the class $((f',b'),(g',c'))$ in $\mathcal{S}_{n+1}(X,Y,\xi)$. Then $x = \mu(y)$. By the definition of the s-triangulation $((f',b'),(g',c'),(h',d'))$ the simple homotopy equivalence $g'\colon N \to Y$ is already split along the submanifold Z. Hence by the definition of the map ω we obtain $\omega(y) = 0 \in LS_{n-q-q'}(\Psi)$ and $\nu\omega(y) = 0$. The commutativity of the diagram (20) now shows that

$$\Theta_*(x) = \Theta_*\mu(y) = \nu\omega(y) = 0.$$

This proves the first implication.

Conversely, let $\Theta_*(x) = 0$ for an element $x = [(f, b)] \in H_n(X; \mathbf{L}_\bullet)$. We consider the following commutative diagram:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
A_n & \xrightarrow{=} & A_n \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \lambda} \\
\mathcal{S}_{n+1}(X, Y, \xi) & \xrightarrow{\mu} & H_n(X; \mathbf{L}_\bullet) \\
\downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \Theta_*} \\
LS_{n-q-q'}(\Psi) & \xrightarrow{\nu} & LT_{n-q-q'}(X, Y, Z) \\
\downarrow & & \downarrow \\
A_{n-1} & \xrightarrow{=} & A_{n-1} \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}, \qquad (21)$$

in which the vertical columns are exact sequences. The diagram (21) follows from the right-hand push-out square in (19). Here the A_i are the homotopy groups of the homotopy cofibre of the vertical map in the right-hand square of the diagram (19). Since $\Theta_*(x) = 0$, there exists $a \in A_n$ such that $\lambda(a) = x$ and therefore $x = \mu\tau(a)$. The class $\tau(a) \in \mathcal{S}_{n+1}(X, Y, \xi)$ is represented by an s-triangulation

$$((f, b), (g, c)) \colon (M, N) \to (X, Y).$$

Since $\omega \circ \tau = 0$, the s-triangulation $(g, c) \colon N \to Y$ is homotopic to an s-triangulation

$$((g_1, c_1), (h, d)) \colon (N, K) \to (Y, Z).$$

Let $F \colon N \times I \to Y$ be a homotopy with $F\big|_{N\times 0} = g$ and $F\big|_{N\times 1} = g_1$. By definition we have a tubular neighbourhood of Y in X with normal fibration ξ such that $g^*(\xi)$ is a tubular neighbourhood of N in M. We can express the induced fibration $F^*(\xi)$ over $N \times I$ as $g^*(\xi) \times I$. Hence we can continue the homotopy F by means of a homotopy $G \colon M \times I \to X$ such that $G(x, t) = g(x)$ for the points $x \in M$ not lying in the tubular neighbourhood of the submanifold N. The restriction $G\big|_{M\times 1}$ yields an s-triangulation of the triple (X, Y, Z).

Theorem 2. *With the above notation there exists a commutative diagram of exact sequences*

$$\begin{array}{ccccccccccc}
\to & L_n(C) & & \to & LP_{n-q}(F) & & \to & LS_{k-1}(\Psi) & & \to \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & LT_k(X, Y, Z) & & & & L_{n-q}(\pi_1(Y)) & & & & \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LS_k(\Psi) & & \to & LP_k(\Psi) & & \to & L_{n-1}(C) & & \to
\end{array}, \qquad (22)$$

where $k = n - q - q'$ is the dimension of the submanifold Z and $C = \pi_1(X \setminus Y)$.

Proof. The diagram (6) for the pair of manifolds (X, Y) is generated by the following push-out square of spectra [25] (see also [6], [8], and[11] for the case of one-sided submanifolds):

$$\begin{array}{ccc} \Sigma^q \mathbb{L}P(F) & \longrightarrow & \mathbb{L}(\pi_1(X)) \\ \downarrow & & \downarrow \\ \Sigma^q \mathbb{L}(\pi_1(Y)) & \longrightarrow & \mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \end{array}, \tag{23}$$

where the top horizontal map arises from the diagram (11) and the left-hand vertical map is the map in the diagram (14).

For the pair of manifolds (Y, Z) there exists a similar push-out square

$$\begin{array}{ccc} \Sigma^{q+q'} \mathbb{L}P(\Psi) & \longrightarrow & \Sigma^q \mathbb{L}(\pi_1(Y)) \\ \downarrow & & \downarrow \\ \Sigma^{q+q'} \mathbb{L}(\pi_1(Z)) & \longrightarrow & \Sigma^q \mathbb{L}(\pi_1(Y \setminus Z) \to \pi_1(Y)) \end{array}, \tag{24}$$

which we write shifting the dimensions. From the diagrams (23) and (24), using the definition of the spectrum $\mathbb{L}T(X, Y, Z)$ we obtain the homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \Sigma^{q+q'} \mathbb{L}T(X, Y, Z) & \longrightarrow & \Sigma^q \mathbb{L}P(F) & \longrightarrow & \Sigma^{q+q'+1} \mathbb{L}S(\Psi) \\ \downarrow & & \downarrow & & \downarrow = \\ \Sigma^{q+q'} \mathbb{L}P(\Psi) & \longrightarrow & \Sigma^q \mathbb{L}(\pi_1(Y)) & \longrightarrow & \Sigma^{q+q'+1} \mathbb{L}S(\Psi) \end{array}, \tag{25}$$

where the existence of the left-hand map follows from [25]. The left-hand square in the diagram (25) is a push-out square since the homotopy cofibres of the left-hand horizontal maps are naturally homotopy equivalent. A homotopy cofibre of the middle vertical map is $\Sigma^1 \mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X))$, as follows from the diagrams (6) and (23). Considering the homotopy long exact sequences of the maps in the left-hand square of (25) we obtain the diagram in the statement of the theorem.

We observe that the exact sequence (18) lies in the commutative diagram (22).

Theorem 3. *Let q, $q \geqslant 3$, be the codimension of a submanifold Y in a manifold X and let q', $q' \geqslant 3$, be the codimension of a submanifold Z in the manifold Y. Then there exists an isomorphism*

$$LT_{n-q-q'}(X, Y, Z) = L_n(\pi_1(X)) \oplus L_{n-q}(\pi_1(Y)) \oplus L_{n-q-q'}(\pi_1(Z)).$$

Proof. By [1] we obtain the following isomorphisms:

$$LP_{n-q}(F) \cong L_n(\pi_1(X)) \oplus L_{n-q}(\pi_1(Y))$$

for $q \geqslant 3$ and

$$LP_k(\Psi) \cong L_{n-q}(\pi_1(Y)) \oplus L_k(\pi_1(Z)),$$
$$LS_k(\Psi) \cong L_k(Z)$$

for $q' \geqslant 3$. For $q \geqslant 3$ we also have an isomorphism $C = \pi_1(X \setminus Y) \cong \pi_1(X)$ of fundamental groups. Using (6) we can write a part of the diagram (22) in the following form:

$$\begin{array}{ccccccc}
L_n(\pi_1(X)) & & \overset{\text{mono}}{\to} & & L_n(\pi_1(X)) \oplus L_{n-q}(\pi_1(Y)) & & \\
& \searrow & & \nearrow & & {}^{\text{epi}}\searrow & \\
& & LT_k(X,Y,Z) & & & & L_{n-q}(\pi_1(Y)), \\
& \nearrow & & \searrow & & {}^{\text{epi}}\nearrow & \\
L_k(\pi_1(Z)) & & \overset{\text{mono}}{\to} & & L_k(\pi_1(Z)) \oplus L_{n-q}(\pi_1(Y)) & &
\end{array} \tag{26}$$

where $k = n - q - q'$. In particular, the diagram (26) contains the short exact sequence

$$0 \to L_n(\pi_1(X)) \to LT_k(X,Y,Z) \to L_k(\pi_1(Z)) \oplus L_{n-q}(\pi_1(Y)) \to 0.$$

The map $L_n(\pi_1(X)) \to LT_k(X,Y,Z)$ has a left inverse since the top horizontal map in the diagram (26) has a left inverse, and the map

$$LT_k(X,Y,Z) \to L_n(\pi_1(X)) \oplus L_{n-q}(\pi_1(Y))$$

is an epimorphism. This yields the required isomorphism.

Let $\mathbb{S}(X,Y,Z)$ be a homotopy cofibre of the map Θ obtained in Theorem 1. Then the homotopy groups

$$\mathcal{S}_n(X,Y,Z) = \pi_n(\mathbb{S}(X,Y,Z))$$

are natural generalizations of the structural groups $\mathcal{S}_n(X)$ and $\mathcal{S}_n(X,Y,\xi)$. By the cofibration

$$X_+ \wedge \mathbf{L}_\bullet \xrightarrow{\ \Theta\ } \Sigma^{q+q'}\mathbb{L}T(X,Y,Z) \longrightarrow \mathbb{S}(X,Y,Z)$$

we obtain the homotopy long exact sequence

$$\cdots \to LT_{k+1}(X,Y,Z) \to \mathcal{S}_{n+1}(X,Y,Z) \to H_n(X;\mathbf{L}_\bullet) \to LT_k(X,Y,Z) \to \cdots, \tag{27}$$

where $k=n-q-q'$ is the dimension of the submanifold Z. The exact sequence (27) for the triple of manifolds (X,Y,Z) generalizes both the surgery exact sequence (2) for the manifold X and the top exact sequence in the diagram (7) for the manifold pair (X,Y).

Theorem 4. *With the above notation there exists a commutative diagram of exact sequences*

$$\begin{array}{ccccccccccccc}
\to & \mathcal{S}_{n+1}(X,Y,Z) & & \to & & H_n(X;\mathbf{L}_\bullet) & & \to & & LP_{n-q}(F) & & \to & \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
& & & \mathcal{S}_{n+1}(X,Y,\xi) & & & & LT_k(X,Y,Z) & & & & & . \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
\to & LP_{n-q+1}(F) & & \to & & LS_k(\Psi) & & \to & & \mathcal{S}_n(X,Y,Z) & & \to &
\end{array} \tag{28}$$

Proof. The right-hand square in (19) is a push-out. Hence the cofibres of the two right-hand vertical maps are naturally homotopy equivalent to $\Sigma^{-q+1}\mathbb{S}(X,Y,Z)$. The homotopy long exact sequences of the maps in that square now give rise to the diagram (26). The proof of the theorem is thus complete.

We observe that the diagram (28) is a relative version of the diagram described in [1], Proposition 7.2.6(iv).

Theorem 5. *With the above notation there exists a commutative diagram of exact sequences*

$$\begin{array}{ccccccccccc}
\to & \mathcal{S}_n(X\setminus Y) & & \to & & \mathcal{S}_n(X,Y) & & \to & & LS_{k-1}(\Psi) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & \mathcal{S}_n(X,Y,Z) & & & & \mathcal{S}_{n-q}(Y) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LS_k(\Psi) & & \to & & \mathcal{S}_{n-q}(Y,Z) & & \to & & \mathcal{S}_{n-1}(X\setminus Y) & \to
\end{array} \quad . \tag{29}$$

Proof. By Lemma 2 we can write down a commutative square of spectra

$$\begin{array}{ccc}
X_+\wedge \mathbf{L}_\bullet & \longrightarrow & X_+\wedge \mathbf{L}_\bullet \\
\downarrow & & \downarrow \\
\Sigma^q(Y_+\wedge \mathbf{L}_\bullet) & \longrightarrow & \Sigma^q(Y_+\wedge \mathbf{L}_\bullet)
\end{array} , \tag{30}$$

in which the horizontal maps are identical. The diagram (25) yields the homotopy commutative square of spectra

$$\begin{array}{ccc}
\Sigma^{q+q'}\mathbb{L}T(X,Y,Z) & \longrightarrow & \Sigma^q\mathbb{L}P(F) \\
\downarrow & & \downarrow \\
\Sigma^{q+q'}\mathbb{L}P(\Psi) & \longrightarrow & \Sigma^q\mathbb{L}(\pi_1(Y))
\end{array} . \tag{31}$$

We consider now the natural maps of the spectra in (30) into the corresponding spectra in (31) yielding a homotopy commutative diagram in the form of a cube. Its commutativity follows by the definition of the spectrum $\mathbb{L}T(X,Y,Z)$ similarly to Lemmas 1 and 2. The squares (30) and (31) are push-outs. Hence the cofibres of the maps of spectra in the square (30) into spectra in the square (31) yield the push-out square

$$\begin{array}{ccc}
\mathbb{S}(X,Y,Z) & \longrightarrow & \mathbb{S}(X,Y,\xi) \\
\downarrow & & \downarrow \\
\Sigma^q\mathbb{S}(Y,Z,\xi_1) & \longrightarrow & \Sigma^q\mathbb{S}(Y)
\end{array} . \tag{32}$$

The homotopy long exact sequences of the maps from the push-out square (32) give one the diagram (29).

Corollary. *With the above notation there exists a commutative diagram*

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{S}_{n+1}(X,Y,Z) & \longrightarrow & H_n(X;\mathbf{L}_\bullet) & \longrightarrow & LT_k(X,Y,Z) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{S}_{n-q+1}(Y,Z) & \longrightarrow & H_{n-q}(Y;\mathbf{L}_\bullet) & \longrightarrow & LP_k(\Psi) & \longrightarrow & \cdots, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{S}_n(X\setminus Y) & \longrightarrow & H_{n-q}(X\setminus Y;\mathbf{L}_\bullet) & \longrightarrow & L_{n-1}(X\setminus Y) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{33}$$

in which all rows and columns are exact sequences.

Proof. The homotopy long exact sequences of the maps of the homotopy commutative square of spectra

$$\begin{array}{ccc} X_+\wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^{q+q'}\mathbb{L}T(X,Y,Z) \\ \downarrow & & \downarrow \\ \Sigma^q(Y_+\wedge\mathbf{L}_\bullet) & \longrightarrow & \Sigma^{q+q'}\mathbb{L}P(\Psi) \end{array}$$

yield the commutative diagram (33).

§ 4. Applications to one-sided manifolds

In this section we consider triples of manifolds $Z\subset Y\subset X$, where $Y\subset X$ and $Z\subset Y$ are Browder–Livesay pairs. In this case (see [17] and [18]) Y is a one-sided submanifold of codimension 1 in the manifold X, and Z is a one-sided submanifold of codimension 1 in the manifold Y.

The square F of fundamental groups in (3) has the following form:

$$F=\begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(X\setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(X) \end{pmatrix} = \begin{pmatrix} A & \xrightarrow{\cong} & A \\ \downarrow{\scriptstyle i_-} & & \downarrow{\scriptstyle i_+} \\ B^- & \xrightarrow{\cong} & B^+ \end{pmatrix}. \tag{34}$$

All the groups in (34) are oriented. Furthermore, the orientation of the group B^- is distinct from that of B^+ outside the ranges of the vertical maps (which are inclusions of index 2). All maps in the square F in (34), except for the lower horizontal map, preserve orientation. The lower isomorphism preserves the orientation on the range of i_- and reverses the orientation outside it. In this case the splitting obstruction groups $LS_*(F)$ are called the *Browder–Livesay groups* and are denoted by $LN_*(A \to B^+) = LN_*(A \to B)$. We also have the isomorphism [15]

$$LP_n(F) \cong L_{n+1}(i_-^*),$$

where $i^* \colon L_{n+1}(B^-) \to L_{n+1}(A)$ is the transfer map.

The square Ψ (15) of fundamental groups for the manifold pair $Z \subset Y$ has the following form:

$$\Psi = F^- = \begin{pmatrix} A & \xrightarrow{\cong} & A \\ \downarrow{\scriptstyle i_+} & & \downarrow{\scriptstyle i_-} \\ B^+ & \xrightarrow{\cong} & B^- \end{pmatrix}. \tag{35}$$

The squares (35) and (34) coincide as commutative squares of groups.

Similarly to § 3, for the manifold triple (X, Y, Z) with Browder–Livesay pairs (X, Y) and (Y, Z) we can define the spectrum $\mathbb{L}T(X, Y, Z)$ and the obstruction groups $LT_*(X, Y, Z)$ to surgery on triples on manifolds.

Proposition 2. *The spectrum $\mathbb{L}T(X, Y, Z)$ and the groups $LT_n(X, Y, Z)$ depend functorially on the inclusion $A \to B$ of index 2 of oriented groups and the integer $n \pmod 4$.*

Proof. We consider the left homotopy push-out square

$$\begin{array}{ccc} \Sigma^2\mathbb{L}T(X, Y, Z) & \longrightarrow & \mathbb{L}(i_-^*) \\ \downarrow & & \downarrow \\ \Sigma^1\mathbb{L}(i_+^*) & \longrightarrow & \Sigma^1\mathbb{L}(B^-) \end{array}, \tag{36}$$

arising from the diagram (25). The right-hand vertical and the lower horizontal maps depend functorially on the inclusion $A \to B$ of oriented groups since the realization of the diagram (6) on the spectral level is functorial (see [15], [21]). Using these maps we can obtain the spectrum $\Sigma^2\mathbb{L}T(X, Y, Z)$ in (36) with the help of the pull-back construction, which is functorial. The required result is now immediate.

By Proposition 2, in the present case we can use the following notation:

$$\mathbb{L}T(F) = \mathbb{L}T(F^+) = \mathbb{L}T(X, Y, Z),$$
$$LT_*(F) = LT_*(F^+) = LT_*(A \to B) = LT_*(A \to B^+) = LT_*(X, Y, Z).$$

Theorem 6. *With the above notation there exists a commutative diagram of exact sequences*

$$\begin{array}{ccccccccccccc} \to & & L_{n+1}(A) & & \to & & L_{n+1}(i_-^*) & & \to & & LN_{n-2}(A \to B^-) & & \to \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & & & LT_{n-1} & & & & L_n(B^-) & & & & \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ \to & & LN_{n-1}(A \to B^-) & & \to & & L_n(i_+^*) & & \to & & L_n(A) & & \to \end{array}, \tag{37}$$

where $LT_{n-1} = LT_{n-1}(A \to B^+)$.

Proof. The proof is actually a reformulation of Theorem 2 in the notation of this section.

In [8] the authors constructed a spectral sequence of surgery theory. The main step of their construction is the consideration of the push-out squares (23) and (24) for the Browder–Livesay pairs (X, Y) and (Y, Z), respectively. After that one can consider the filtration of spectra from [8]

$$\cdots \to \mathbb{X}_{3,0} \to \mathbb{X}_{2,0} \to \mathbb{X}_{1,0} \to \mathbb{X}_{0,0} \to \mathbb{X}_{-1,0} \to \cdots, \tag{38}$$

which was used for the construction of the above-mentioned spectral sequence. Here $\mathbb{X}_{0,0} = \mathbb{L}(B^+)$ and $\mathbb{X}_{1,0} = \Sigma\mathbb{L}P(F) = \mathbb{L}(i_-^*)$. In accordance with [8], the spectrum $\mathbb{X}_{2,0}$ is obtained by a pull-back construction from the maps of spectra in (36)

$$\mathbb{L}(i_-^*) \to \Sigma^1\mathbb{L}(B^-) \leftarrow \Sigma^1\mathbb{L}(i_+^*). \tag{39}$$

The other spectra $\mathbb{X}_{k,0}$ are obtained by iterations of this procedure (see [8]).

By [8] we obtain the isomorphisms

$$E_1^{p,s} = \pi_{s-p}(\mathbb{X}_{p,0}, \mathbb{X}_{p+1,0}) = LN_{s+2}(A \to B),$$

and the first differential

$$d_1^{p,s} \colon E_1^{p,s} \to E_1^{p+1,s}$$

coincides with the composite

$$LN_{s-2p-2}(A \to B^{(-)^p}) \to L_{s-2p-2}(B^{(-)^{p+1}}) \to LN_{s-2p}(A \to B^{(-)^{p+1}}). \tag{40}$$

The maps in the diagram (40) arise from the diagram (6) considered in the case of one-sided submanifolds with squares F (34) and F^- (35).

Theorem 7. *The spectrum* $\mathbb{X}_{2,0}$ *in the filtration* (38) *coincides with the spectrum* $\Sigma^2\mathbb{L}T(F)$ *for the surgery problem on manifold triples as defined in this section. The map* $\mathbb{X}_{2,0} \to \mathbb{X}_{1,0}$ *in the filtration* (38) *coincides with the map*

$$\Sigma^2\mathbb{L}T(A \to B) \to \mathbb{L}(i_-^*) = \Sigma^1\mathbb{L}P(F)$$

in the diagram (36).

Proof. In view of (39), the definition of the spectrum $\mathbb{X}_{2,0}$ coincides with the definition of the spectrum $\Sigma^2\mathbb{L}T(F)$ by means of the pull-back square (36).

Remark 1. From the algebraic point of view the spectrum $\mathbb{L}T(A \to B^+)$ can be constructed starting from the inclusion $A \to B^+$ of index 2 of oriented groups. In fact, there exists a realization of an algebraic version of the diagram (6) on the spectral level (see [11], [12], [15]). In particular, we can define the spectrum $\mathbb{L}T(A \to B^-)$ fitting in the following pull-back square of spectra:

$$\begin{array}{ccc} \Sigma^2\mathbb{L}T(A \to B^-) & \longrightarrow & \mathbb{L}(i_+^*) \\ \downarrow & & \downarrow \\ \Sigma^1\mathbb{L}(i_-^*) & \longrightarrow & \Sigma^1\mathbb{L}(B) \end{array} . \tag{41}$$

Then the obstruction groups $LT_*(A \to B^-) = \pi_*(\mathbb{L}T(A \to B^-))$ are defined. The homotopy long exact sequences of the square (41) give one the following commutative diagram:

$$\begin{array}{ccccccccc}
\to & L_{n+1}(A) & \to & L_{n+1}(i_+^*) & \to & LN_{n-2}(A \to B) & \to \\
& \nearrow \quad \searrow & & \nearrow \quad \searrow & & \nearrow \quad \searrow & \\
& & LT_{n-1}^- & & L_n(B) & & \\
& \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\
\to & LN_{n-1}(A \to B) & \to & L_n(i_-^*) & \to & L_n(A) & \to
\end{array}, \quad (42)$$

where $LT_{n-1}^- = LT_{n-1}(A \to B^-)$. The groups $LT_*(A \to B^-)$ are distinct from the groups $LT_*(A \to B^+)$ in the general case. In fact one can transfer all results of this section to the case of a quadratic extension of antistructures and decorated L-groups (see [6] and [15]).

Proposition 3. *The spectrum $\mathbb{X}_{3,0}$ of the fibration* (38) *fits in the following pull-back square of spectra*:

$$\begin{array}{ccc}
\mathbb{X}_{3,0} & \longrightarrow & \Sigma^2\mathbb{L}T(A \to B) \\
\downarrow & & \downarrow \\
\Sigma^3\mathbb{L}T(A \to B^-) & \longrightarrow & \Sigma^1\mathbb{L}(i_+^*)
\end{array}.$$

Proof. This immediately follows by the consideration of the pull-back squares (36) and (41) and the definition of the filtration (38) in [8].

Example 1. Consider the triple of real projective spaces $\mathbb{RP}^{2l-1} \subset \mathbb{RP}^{2l} \subset \mathbb{RP}^{2l+1}$, $2l-1 \geqslant 3$. We have $\pi_1(\mathbb{RP}^k) = \mathbb{Z}/2$ for each $k \geqslant 2$. The orientation homomorphism $w\colon \mathbb{Z}/2 \to \{\pm 1\}$ is trivial for odd k and an isomorphism for even k. All the groups $L_n(1)$, $L_n(\mathbb{Z}/2^+)$, and $L_n(\mathbb{Z}/2^-)$ are well known [3], and we have the isomorphisms $LN_*(1 \to \mathbb{Z}/2^+) \cong L_{*+2}(1)$ and $LN_*(1 \to \mathbb{Z}/2^-) \cong L_*(1)$. Using diagram chasing in the diagram (6) for the inclusions $1 \to \mathbb{Z}/2^+$ and $1 \to \mathbb{Z}/2^-$ we can compute the corresponding surgery obstruction groups for manifold pairs. Then we obtain

$$LP_n(F) = L_{n+1}(i_-^*) = \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}$$

and

$$LP_n(F^-) = L_{n+1}(i_+^*) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively.

Proposition 4. *Let M^m be a topological manifold of dimension m with trivial fundamental group. Let $n = m + 2l - 1 \geqslant 5$. Then for the triple of manifolds*

$$\mathbb{RP}^{2l-1} \times M \subset \mathbb{RP}^{2l} \times M \subset \mathbb{RP}^{2l+1} \times M$$

with $l \geqslant 2$ the groups LT_n are isomorphic to

$$\mathbb{Z} \oplus \mathbb{Z}/2, \quad \mathbb{Z}/2, \quad \mathbb{Z} \oplus \mathbb{Z}/2, \quad \mathbb{Z}/2$$

for $n = 0, 1, 2, 3 \pmod 4$, respectively. Furthermore, the natural forgetful maps $LT_n(F) \to LP_n(F^-)$ and $LT_n(F) \to LP_{n+1}(F)$ are epimorphisms for all n.

Proof. The result follows by diagram chasing in the diagram (37).

Example 2. Consider the triple of real projective spaces $\mathbb{RP}^{2l} \subset \mathbb{RP}^{2l+1} \subset \mathbb{RP}^{2l+2}$ with $l \geqslant 1$. Similarly to Example 1 we obtain the following result.

Proposition 5. *Let M^m be a topological manifold of dimension m with trivial fundamental group. Let $n = m + 2l \geqslant 5$. For the triple of manifolds*

$$\mathbb{RP}^{2l} \times M \subset \mathbb{RP}^{2l+1} \times M \subset \mathbb{RP}^{2l+2} \times M$$

one has the isomorphisms $LT_0^- \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $LT_1^- \cong \mathbb{Z}/2$. The groups LT_3^- and LT_2^- fit in the following exact sequence:

$$0 \to LT_3^- \to \mathbb{Z} \to \mathbb{Z} \to LT_2^- \to \mathbb{Z}/2 \to 0.$$

Proof. The result follows by diagram chasing in the diagram (42).

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