Geometric properties of a spectral sequence in surgery theory

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Let X be a closed topological manifold of dimension $n \geqslant 5$ with fundamental group $G = \pi_1(X)$ having a subgroup $\pi \subset G$ of index 2. We consider a map $\chi\colon X \to \mathbb{R}P^m$ of the manifold X to an m-dimensional real projective space of high dimension. Suppose that χ induces a fundamental group epimorphism $\chi_*\colon G \to \mathbb{Z}/2$ with kernel π. We denote by Y the one-sided submanifold of X that is the transversal inverse image $\chi^{-1}(\mathbb{R}P^{m-1})$ of the one-sided submanifold $\mathbb{R}P^{m-1} \subset \mathbb{R}P^m$. The pair (X,Y) is a Browder–Livesay pair if the embedding $Y \to X$ induces an isomorphism of fundamental groups (see [1], [2]).

For a Browder–Livesay pair we have a commutative diagram of exact sequences (see [3], [4]):

$$\begin{array}{ccccccccccc}
\to & L_{n+1}(\pi) & & \longrightarrow & L_{n+1}(G^+) & & \to & & LN_{n-1}(\pi\to G) & & \to \\
& \nearrow & \searrow & & \nearrow & \searrow & & & \nearrow & \searrow & \\
& & & LP_n(F) & & & L_{n+1}(\pi\to G) & & & & \\
& \searrow & \nearrow & & \searrow & \nearrow & & & \searrow & \nearrow & \\
\to & LN_n(\pi\to G) & & \longrightarrow & L_n(G^-) & & \longrightarrow & & L_n(\pi) & & \to
\end{array} \qquad (1)$$

The diagram (1) includes the groups $L_*(\pi)$ and $L_*(G)$ of obstructions to surgery, the groups $LN_*(\pi \to G)$ of obstructions to splitting along the one-sided submanifold $Y \subset X$, and the groups $LP_*(F)$ of obstructions to surgery with respect to the pair (X,Y) of manifolds (see [5], [6]).

In [7] a spectral sequence is constructed in surgery theory. The construction uses a realization of the commutative diagram (1) at the spectral level. The basic filtration

$$\cdots \to X_{3,0} \to X_{2,0} \to X_{1,0} \to X_{0,0} \to X_{-1,0} \to \cdots \qquad (2)$$

of spectra in [7] contains the $\mathbb{L}$-spectrum $X_{0,0} = \mathbb{L}(G^+)$ and the spectrum $X_{1,0} = \Sigma\mathbb{L}P(F)$ (see [8], [9]).

The set $\mathcal{S}_n(X,Y,\xi)$ of s-triangulations of the pair (X,Y) of manifolds appears in the exact sequence

$$\cdots \to \mathcal{S}_n(X,Y,\xi) \to [X, G/TOP] \overset{v_\xi}{\to} LP_{n-1}(F) \to \cdots, \qquad (3)$$

which can be constructed at the spectral level [5].

Let $\mathbf{L}_\bullet$ denote a simply connected covering of the Ω-spectrum $\mathbf{L}_\bullet(\mathbb{Z})$ ([5], [6]). For a closed topological manifold X we have the isomorphism $[X, G/TOP] \cong H_n(X, \mathbf{L}_\bullet)$. There also exists an Ω-spectrum $\mathbb{S}(X)$ whose n-dimensional homotopy groups give the set $\mathcal{S}_n(X)$ of topological triangulations of X.

Let $(Z \subset Y \subset X)$ be a triple of manifolds such that each of the pairs $(Z \subset Y)$ and $(Y \subset X)$ of manifolds is a Browder–Livesay pair. We assume that the dimension of the submanifold Z is at least five.

We say that a normal map f *can be made by surgery into a simple homotopy equivalence of triples of manifolds* if the normal cobordism class of f contains a map g with the following properties:

1) the map $g|_X$ is a simple homotopy equivalence;
2) the map g is transversal to the submanifolds Y and Z;
3) the restrictions of g to the transversal inverse images Y, $X \setminus Y$, Z, and $Y \setminus Z$ are simple homotopy equivalences.

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We denote by α the composition of maps (see [5])

$$\mathcal{S}_n(X,Y,\xi) \to \mathcal{S}_{n-1}(Y) \to LN_{n-2}(\pi \to G^-).$$

Composition of the map α with the map $LP_n(F) \to \mathcal{S}_n(X,Y,\xi)$ in the exact sequence (3) now gives a map $\beta\colon LP_n(F) \to LN_{n-2}(\pi \to G^-)$.

Lemma 1. *There exists a map of spectra* $b\colon \Omega^2\mathbb{L}P(F) \to \mathbb{L}N(\pi \to G^-)$ *such that the induced map* b_* *of homotopy groups coincides with the map* β.

We denote by $\mathbb{L}T(X,Y,Z)$ the Ω-spectrum that is the homotopy co-fibre of the map b, and by $LT_n(X,Y,Z)$ the homotopy groups $\pi_n(\mathbb{L}T(X,Y,Z))$.

Theorem 1. *There is a map of spectra* $\psi\colon \Omega^2(X_+ \wedge \mathbf{L}_\bullet) \to \mathbb{L}T(X,Y,Z)$ *that induces a homomorphism* $\psi_*\colon H_n(X,L_\bullet) \to LT_{n-2}(X,Y,Z)$. *The normal map* $[f\colon M \to X] \in [X, G/TOP] \cong H_n(X,L_\bullet)$ *can be made by surgery into a simple homotopy equivalence of triples of manifolds if and only if* $\psi_*(f) = 0$.

Theorem 2. *There is a homotopy equivalence of spectra*

$$\Sigma^2\mathbb{L}T(X,Y,Z) \cong X_{2,0},$$

where the spectrum $X_{2,0}$ *appears in the filtration* (2) *of the spectral sequence in surgery theory.*

Corollary. *Let* F^- *denote a square of fundamental groups in the splitting problem for a Browder–Livesay pair* $Z \subset Y$. *There is a commutative diagram of exact sequences*:

$$\begin{array}{ccccccccccc}
\to & & L_{n+1}(\pi) & & \longrightarrow & & LP_n(F) & & \longrightarrow & & LN_{n-2}(\pi \to G^-) & \to \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & LT_{n-1}(F) & & & & L_n(G^-) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & LN_{n-1}(\pi \to G^-) & & \longrightarrow & & LP_{n-1}(F^-) & & \longrightarrow & & L_n(\pi) & \to
\end{array}$$

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