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# The Groups $LS$ and Morphisms of Quadratic Extensions

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**Abstract**—For a morphism of quadratic extensions of antistructures, groups similar to the groups of obstructions to splitting along one-sided submanifolds are defined. These groups are a natural generalization of the splitting obstruction groups. The results obtained open additional possibilities for constructing groups and natural maps in $L$-theory.

Key words: *splitting obstruction group, morphism of quadratic extensions of antistructures, geometric diagram, universal spectrum square, L-groups.*

The groups $LS_{n-q}(F)$ were geometrically defined by Wall [1] (see also [2]) as the groups of obstructions to splitting a simple homotopy equivalence $f\colon M \to Y$ of $n$-dimensional manifolds along a submanifold $X \subset Y$ of codimension $q$.

Let $U$ be a tubular neighborhood of the submanifold $X$ in $Y$. The groups $LS_{n-q}(F)$ depend functorially on the universal push-out square

$$F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(Y\setminus X) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow \\ \pi_1(X) & \longrightarrow & \pi_1(Y) \end{pmatrix} \tag{1}$$

of fundamental groups with orientation and on the dimension $n-q \bmod 4$.

For the square $F$, the surgery obstruction groups $LP_{n-q}(F)$ for the pair of manifolds are also defined, and they also depend on $F$ and $n-q \bmod 4$ (see [1, 2]).

In the case of a one-sided submanifold ($q = 1$), squares of fundamental groups in which the horizontal maps are epimorphisms and the vertical ones are embeddings of index 2 (see [3–7]) naturally arise. Such squares of groups are called *geometric diagrams* [5]. The author of [5] algebraically defines splitting obstruction groups $LSA_*(F)$ under some constraints on the homotopy groups of the manifolds. The groups $LSA_*(F)$ coincide with the geometrically defined groups $LS_*(F)$ for the case in which the square $F$ is a geometric diagram of groups [7] (see also [8]).

If the horizontal maps in the square $F$ are isomorphisms, then the groups $LS_*(F)$ coincide with the Browder–Livesay groups $LN_*(\pi_1(Y\setminus X) \to \pi_1(Y))$, and the groups $LP_n(F)$ coincide with the relative groups $L_{n+1}(i^!)$ of the transfer map.

A natural algebraic generalization of a geometric diagram (1) is a geometric diagram of antistructures, where the horizontal maps are epimorphisms and the vertical ones are quadratic extensions of antistructures (see [8–10]). In this case, the groups $LS_*$ and $LP_*$ are also defined (see [8, 10]). These groups coincide with the traditional ones if the square of antistructures is obtained from a geometric diagram of groups by passing to group rings over the ring $\mathbb{Z}$ with standard involution (see [7, 8, 10]). The groups $LS_*$ and $LP_*$ are closely related to the Wall groups and serve as an effective tool for constructing $L$-groups and natural maps (see [11–15]).

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In this paper, we define the groups $LSA_*(F)$ and $LPA_*(F)$ for a square $F$ of antistructures in which the pair of horizontal maps determines a morphism of quadratic extensions of the antistructures. If the square $F$ determines a geometric diagram (i.e., the horizontal map are epimorphisms), then the formal definition of the groups $LSA_*(F)$ and $LPA_*(F)$ coincides with the definitions of the groups $LS_*(F)$ and $LP_*(F)$ given in [8] and [10], respectively. The groups introduced in this paper have the same algebraic properties as the geometrically defined groups $LS_*(F)$ and $LP_*(F)$ for the case in which $F$ is a geometric diagram of groups. Our definition of the groups $LSA_*(F)$ and $LPA_*(F)$ imposes no additional constraints on the antistructures and the morphism of quadratic extensions determining the square $F$. Thus the groups introduced here differ significantly from the geometric groups $LS_*(F)$ for the case in which the horizontal maps are not epimorphisms (see [16; 1, Sec. 12A]). We obtain new relations between the $L$-groups and natural maps for the group rings $\mathbb{Z}[\pi]$ and $\widehat{\mathbb{Z}}_2[\pi]$ (see [11, 13, 15, 17, 18]).

Let us recall the necessary definitions (see [8, 9, 13, 19–21]).

An *antistructure* [19, 20] is a triple $(R, \alpha, u)$, where $R$ is a ring with identity, $\alpha\colon R \to R$ is an antiautomorphism, and $u \in R^*$ is an invertible element such that $\alpha(u) = u^{-1}$ and $\alpha^2(x) = uxu^{-1}$ for all $x \in R$. For any subgroup $X$ of $K_i(R)$ (or of $\widetilde{K}_i(R)$) with $i = 0, 1$ invariant with respect to the involution induced by $\alpha$, the Wall groups $L_n^X(R, \alpha, u)$ are defined (see [17, 20, 21]). Let us choose one of the decorations

$$\widetilde{K}_1(R), \quad \widetilde{K}_0(R), \quad K_1(R), \quad K_0(R), \quad Y = SK_1(\widehat{\mathbb{Z}}_2[\pi]) + \{\pm\pi\}, \quad \text{etc.}$$

(see [11, 17, 21]) for all rings $R$. In what follows, all $L$-groups are considered only with this decoration.

A *morphism of antistructures* $f\colon (R, \alpha, u) \to (P, \beta, v)$ is a homomorphism of rings $f\colon R \to P$ such that $f(u) = v$ and $\beta f = f\alpha$. Let $X$ and $Z$ be decorations of one of the types under consideration for the rings $R$ and $P$, respectively. Then, for any morphism of antistructures $f\colon R \to P$, we automatically have $f_*(X) \subset Z$ (see [2; 8, Sec. 3; 13]), and the relative groups $L_*^{X,Z}(R \to P) = L_*(f)$ are defined.

A *quadratic extension of an antistructure* $(R, \alpha, u)$ with respect to a structure $(\rho, a)$ (see [9]) is the antisructure $(S, \alpha, u)$, where

$$S = R[t]/\{t^2 - a\}, \qquad tx = \rho(x)t, \qquad \alpha(t)t \in R, \qquad \alpha^2(t) = utu^{-1}.$$

In this case, a morphism of antistructures $i\colon R \to S$ arises. Let $X$ and $Y$ be decorations of one of the types under consideration for the rings $R$ and $S$, respectively. Then the transfer map $i^!\colon K_n(S) \to K_n(R)$ $(n = 0, 1)$ satisfying the condition $i^!(Y) \subset X$ is defined (see [9; 8, Sec. 3; 13]), as well as the relative transfer groups $L_*^{Y,X}(S \to R) = L_*(i^!)$.

Let $(\rho', a')$ be a structure on $P$. The morphism $f$ *preserves structure* if $f(a) = a'$ and $\rho' f = f\rho$. In this case, $f$ determines a *morphism* $\Phi$ *of quadratic extensions of antistructures* (see [13]) as follows:

$$\Phi = \begin{pmatrix} (R,\alpha,u) & \xrightarrow{f} & (P,\beta,v) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ (S,\alpha,u) & \xrightarrow{g} & (Q,\beta,v) \end{pmatrix}. \tag{2}$$

The formula $\gamma(x + yt) = (x - yt)$ $(x, y \in R)$ defines an automorphism $\gamma$ of the ring $S$ over $R$. For the similar automorphism of the ring $Q$ over $P$, we use the same symbol $\gamma$. We have the morphism of quadratic extensions of antistructures

$$\Phi_\gamma = \begin{pmatrix} (R,\alpha,u) & \xrightarrow{f} & (P,\beta,v) \\ ..\big\downarrow{\scriptstyle i_\gamma} & & \big\downarrow{\scriptstyle j_\gamma} \\ (S,\gamma\alpha,u) & \xrightarrow{g_\gamma} & (Q,\gamma\beta,v) \end{pmatrix} = \begin{pmatrix} R & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \\ S_\gamma & \longrightarrow & Q_\gamma \end{pmatrix}; \tag{3}$$

here the maps coincide with the corresponding maps of the square (2).

The automorphism $\rho$ can be extended over the ring $S$ by

$$\rho(x+yt)=t(x+yt)t^{-1} \qquad (x,y\in R).$$

The quadratic extension $i$ determines another quadratic extension of antistructures

$$\widetilde{\imath}\colon (R,\widetilde{\alpha},\widetilde{u})\to(S,\widetilde{\alpha},\widetilde{u}), \qquad \text{where} \quad \widetilde{\alpha}=\rho\gamma\alpha, \quad \widetilde{u}=-t\alpha(t^{-1})$$

(see [9]). Similarly, we can define $\widetilde{\jmath}\colon (P,\widetilde{\beta},\widetilde{v})\to(Q,\widetilde{\beta},\widetilde{v})$. We have the morphism (see [13])

$$\widetilde{\Phi}=\begin{pmatrix} (R,\widetilde{\alpha},\widetilde{u}) & \xrightarrow{\widetilde{f}} & (P,\widetilde{\beta},\widetilde{v}) \\ \big\downarrow{\scriptstyle\widetilde{\imath}} & & \big\downarrow{\scriptstyle\widetilde{\jmath}} \\ (S,\widetilde{\alpha},\widetilde{u}) & \xrightarrow{\widetilde{g}} & (Q,\widetilde{\beta},\widetilde{v}) \end{pmatrix}=\begin{pmatrix} \widetilde{R} & \longrightarrow & \widetilde{P} \\ \big\downarrow & & \big\downarrow \\ \widetilde{S} & \longrightarrow & \widetilde{Q} \end{pmatrix}, \tag{4}$$

where the maps coincide with the corresponding maps in (2) as maps of rings.

The commutative diagram (4) induces the infinite homotopy commutative diagram of $\Omega$-spectra [8] (see also [21])

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots\to & & \mathbb{L}(\widetilde{R}) & \to & \mathbb{L}(\widetilde{P}) & \to & \mathbb{L}(\widetilde{f}_*) & \to\cdots & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots\to & & \mathbb{L}(\widetilde{S}) & \to & \mathbb{L}(\widetilde{Q}) & \to & \mathbb{L}(\widetilde{g}_*) & \to\cdots & \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}, \tag{5}$$

whose rows and columns are cofibrations. Recall that $\pi_i(\mathbb{L}(\widetilde{R}))=L_i(R,\widetilde{\alpha},\widetilde{u})$. A similar result is valid for other spectra in diagram (5). Now, we can define the spectrum $\mathbb{LSA}(\Phi)$ as the homotopy cofiber of one of the following maps of diagram (5):

$$\Omega^2\mathbb{L}(\widetilde{\Phi})\to\mathbb{L}(\widetilde{R}), \qquad \Omega\mathbb{L}(\widetilde{S})\to\Omega\mathbb{L}(\widetilde{\jmath}_*), \qquad \Omega\mathbb{L}(\widetilde{P})\to\Omega\mathbb{L}(\widetilde{g}_*). \tag{6}$$

This definition formally coincides with the definition of the spectrum $\mathbb{L}S(F)$ given in [8], but in [8], the horizontal maps in the square $\Phi$ are assumed to be epimorphisms.

**Theorem 1.** *The following universal squares of spectra arise*:

$$\begin{array}{ccc} \Omega\mathbb{L}(\widetilde{f}_*) & \longrightarrow & \Omega\mathbb{L}(\widetilde{g}_*) \\ \big\downarrow & & \big\downarrow \\ \mathbb{L}(\widetilde{R}) & \longrightarrow & \mathbb{LSA}(\Phi) \end{array}, \quad \begin{array}{ccc} \Omega\mathbb{L}(\widetilde{Q}) & \longrightarrow & \Omega^2\mathbb{L}(\widetilde{\jmath}_*) \\ \big\downarrow & & \big\downarrow \\ \Omega\mathbb{L}(\widetilde{g}_*) & \longrightarrow & \mathbb{LSA}(\Phi) \end{array}, \quad \begin{array}{ccc} \Omega^2\mathbb{L}(\widetilde{\imath}_*) & \longrightarrow & \mathbb{L}(\widetilde{R}) \\ \big\downarrow & & \big\downarrow \\ \Omega^2\mathbb{L}(\widetilde{\jmath}_*) & \longrightarrow & \mathbb{LSA}(\Phi) \end{array}. \tag{7}$$

**Proof.** This theorem is a generalization of Theorem 4 from [8]. The proof given in [8] works in the case under consideration, because it only uses the homotopy properties of diagram (5). □

Recall (see [8, 10]) that each universal spectrum square generates a commutative diagram of exact sequences. To show this, it suffices to write out the homotopy long exact sequences for all maps of the square and apply the natural homotopy equivalence of the cofibers (fibers) of the parallel maps.

Similarly to diagram (5), we can construct an infinite diagram of spectra whose rows are generated by the horizontal maps (3) and columns, by the transfer maps $i^!_\gamma$ and $j^!_\gamma$ for (3) at the spectrum level [10].

We define the spectrum $\mathbb{LPA}(\Phi)$ (see [10]) as the homotopy fiber of one of the maps

$$\Omega\mathbb{L}(f_*) \to \mathbb{L}(i^!_\gamma), \qquad \mathbb{L}(S_\gamma) \to \mathbb{L}(P), \qquad \Omega\mathbb{L}(j^!_\gamma) \to \mathbb{L}(g_{\gamma*}). \tag{8}$$

**Theorem 2.** *The following universal squares of spectra arise*:

$$\begin{array}{ccc} \Omega\mathbb{L}(R) & \longrightarrow & \Omega\mathbb{L}(P) \\ \downarrow & & \downarrow \\ \mathbb{L}(i^!_\gamma) & \longrightarrow & \mathbb{LPA}(\Phi) \end{array}, \qquad \begin{array}{ccc} \Omega\mathbb{L}(Q_\gamma) & \longrightarrow & \Omega\mathbb{L}(g_{\gamma*}) \\ \downarrow & & \downarrow \\ \Omega\mathbb{L}(P) & \longrightarrow & \mathbb{LPA}(\Phi) \end{array},$$

$$\begin{array}{ccc} \Omega^2\mathbb{L}(\Phi^!_\gamma) & \longrightarrow & \Omega\mathbb{L}(i^!_\gamma) \\ \downarrow & & \downarrow \\ \Omega\mathbb{L}(g_{\gamma*}) & \longrightarrow & \mathbb{LPA}(\Phi) \end{array}, \qquad \begin{array}{ccc} \mathbb{LPA}(\Phi) & \longrightarrow & \Omega\mathbb{L}(Q) \\ \downarrow & & \downarrow \\ \mathbb{L}(S_\gamma) & \longrightarrow & \Omega\mathbb{L}(j) \end{array}. \tag{9}$$

**Proof.** This theorem is a generalization of Theorems 3.3 and 4.2 from [10] to the case in which the horizontal maps in $\Phi$ are not epimorphisms. The first three universal squares are constructed by analogy with the squares mentioned in Theorem 1. According to [9], for the quadratic extensions determined by the vertical maps of the square $\Phi$, there exist commutative diagrams of exact sequences realized at the spectrum level [13]. The existence of the universal square (9) follows if we consider the map of these diagrams induced by the horizontal maps of the square $\Phi$. □

Thus the groups

$$LSA_i(\Phi) \overset{\text{def}}{=} \pi_i(\mathbb{LSA}(\Phi))$$

are a natural algebraic generalization of the groups of obstructions to splitting along one-sided submanifolds. Similarly, the groups

$$LPA_i(\Phi) \overset{\text{def}}{=} \pi_i(\mathbb{LPA}(\Phi))$$

are a natural generalization of the groups of obstructions to the surgeries determined by pairs of manifolds to the case of geometric diagrams of groups. The results obtained in [7, 8, 10] imply that the groups constructed and the maps in the diagrams given by Theorems 1 and 2 coincide with the geometrically defined ones if the square $\Phi$ is obtained from the geometric diagram (1) by passing to group rings over $\mathbb{Z}$ with standard involution.

Consider the application of the groups introduced in this paper to the study of the natural maps in $L$-theory. The most complete results on constructing $L$-groups of finite groups and natural maps were obtained by the methods of Wall (see [17, 11, 12, 18]), where an important role is played by various localization maps. Thus, in constructing the Wall groups of finite groups, the key role is played by the map $\mathbb{Z}[\pi] \to \widehat{\mathbb{Z}}_2[\pi]$, where $\pi$ is a finite group (see [11, 12, 15, 17, 18]). Recall that all group rings are assumed to be endowed with the standard involution $\Sigma ag \to w(g)g^{-1}$ for $g \in \pi$.

An arbitrary embedding of groups $i\colon \pi \to G$ of index 2 determines the morphism of quadratic extensions

$$\Psi = \begin{pmatrix} \mathbb{Z}[\pi] & \longrightarrow & \widehat{\mathbb{Z}}_2[\pi] \\ \downarrow i & & \downarrow \hat{\imath} \\ \mathbb{Z}[G] & \longrightarrow & \widehat{\mathbb{Z}}_2[G] \end{pmatrix}.$$

Thus the spectra $\mathbb{LSA}(\Psi)$ and $\mathbb{LPA}(\Psi)$ in the universal spectrum squares mentioned in Theorems 1 and 2 are defined.

Each of these spectrum squares yields a commutative diagram of exact sequences of Wall groups in which the upper and lower rows are chain complexes with isomorphic homology groups.

For the embedding $i$, the Browder–Livesay groups $LN_n(\pi \to G)$, where $n = 0, 1, 2, 3 \bmod 4$, are defined. These groups have an algebraic analog for quadratic extensions of antistructures $\widehat{\mathbb{Z}}_2[\pi] \to \widehat{\mathbb{Z}}_2[G]$; we denote it by $LN_n(\pi \to G)_2$ (see [4, 9]). Let $G^-$ be the group $G$ with the orientation changed outside $\pi$. Theorem 1 implies the following result.

**Corollary.** *The following two-row diagrams arise*:

$$\begin{array}{ccccccc} \to & L_{n+1}(\mathbb{Z}[G^-]) & \to & L_{n+2}(\widehat{\imath}) & \to & L_{n+2}(\Psi) & \to \\ & | & & | & & | & \\ \to & L_{n+3}(\Psi) & \to & LN_n(\pi \to G) & \to & L_n(\mathbb{Z}[G^-]) & \to \end{array},$$

$$\begin{array}{ccccccc} \to & L_{n+1}(\mathbb{Z}[G^-]) & \to & L_{n+2}(\widehat{\imath}) & \to & LN_n(\pi \to G)_2 & \to \\ & | & & | & & | & \\ \to & LN_{n+1}(\pi \to G)_2 & \to & L_{n+1}(\mathbb{Z}[G] \to \widehat{\mathbb{Z}}_2[G]) & \to & L_n(\mathbb{Z}[G^-]) & \to \end{array},$$

$$\begin{array}{ccccccc} \to & LN_{n+1}(\pi \to g)_2 & \to & L_{n+1}(\mathbb{Z}[G] \to \widehat{\mathbb{Z}}_2[G]) & \to & L_{n+2}(\Psi) & \to \\ & | & & | & & | & \\ \to & L_{n+3}(\Psi) & \to & LN_n(\pi \to G) & \to & LN_n(\pi \to G)_2 & \to \end{array},$$

*whose rows are chain complexes with isomorphic homology groups.*

Theorem 2 has a similar corollary.

The results obtained in this paper give much information about the groups $L^Y$ and $L^p$ of finite groups. The relative groups $L_*(\mathbb{Z}[\pi] \to \widehat{\mathbb{Z}}_2[\pi])$, $L_*(\Psi)$, and $L_*(\mathbb{Z}[G^{\pm}] \to \widehat{\mathbb{Z}}_2[G^{\pm}])$ for finite groups are rather effectively described in [11, 17, 18].

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