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Surgery of Closed Manifolds With Dihedral Fundamental Group

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ABSTRACT. In the paper the obstruction groups to obtaining simple homotopy equivalence by surgery from normal degree 1 maps of closed manifolds with dihedral fundamental group are computed. The cases of trivial orientation for the dihedral group and nontrivial orientation for the order 2 cyclic subgroup are considered. New results concerning the Browder–Livesey groups and natural maps of L-groups arising in index 2 inclusions of the cyclic group into the dihedral group are obtained.

KEY WORDS: smooth manifold, obstruction, surgery, simple homotopy equivalence, Wall group, Browder–Livesay group, Tate cohomology, spectral sequence.

§1. Introduction

The Wall groups $L_n^s(G, w)$ are the obstruction groups for modifying a degree 1 map $f\colon (M, \partial M) \to (Y, \partial Y)$ of smooth (or piecewise linear or topological) closed manifolds with boundary by surgery into a simple homotopy equivalence. Further we shall use the notation $L_n(G, w)$ for these groups. Here n is the dimension of the manifold Y, $G = \pi_1(Y)$ and $w\colon G \to \{\pm 1\}$ is the orientation homomorphism. If the boundary of the manifold Y is nonempty, then we assume that the restriction of the map f to the boundary of the manifold M is a simple homotopy equivalence $f|_{\partial M}\colon \partial M \to \partial Y$. Each element of the group $L_n(G, w)$ is realized by a normal map of manifolds with boundary (see [1]). However, in the case of a finite fundamental group G, there usually exist very few elements of the group $L_n(G, w)$ that can be realized by normal maps of closed manifolds. The subgroup generated by such elements will be denoted by $C_n(G, w)$ (see [2]).

There exists a natural map of the Wall group $L_n(G, w)$ into the projective Novikov–Wall groups $L_n^p(G, w)$, which are much simpler. In the paper [2] a complete list of invariants describing the image of the group $C_n(G, w)$ in the group $L_n^p(G, w)$ for any finite 2-group G is obtained. In the case of trivial orientation, deep results in the realization problem were obtained in the paper [3], while in the case of nontrivial orientation complete results have been obtained only for finite Abelian 2-groups (see [3, 4]). The technique developed by Hambleton in the paper [2] for the projective case is based on results due to Browder and Livesay [5], and to Capell and Shaneson [6]. There the main role is played by the Browder–Livesay groups $LN_{n-1}(\pi \to G)$, where π is an index 2 subgroup of the group G, and by the natural maps between these groups and the Wall groups. Thus the paper [6] describes a map, known as the Browder–Livesay invariant, of the group $L_n(G)$ into the Browder–Livesay groups, which is trivial on elements of the Wall group that are realized by normal maps of closed manifolds. The Browder–Livesay groups are obstruction groups to splitting a simple homotopy equivalence $f\colon M \to Y$ along a one-sided submanifold $X \subset Y$ when the homomorphism of fundamental groups $\pi_1(X) \to \pi_1(Y)$ induced by the inclusion is an isomorphism. The natural maps (transfer, twisted transfer, induced map) between the Wall groups and the Browder–Livesay groups required for studying the realization problem by closed manifolds appear in the Levine braid (see [7, 8])

$$
\begin{array}{ccccccccc}
\longrightarrow & L_n(\pi) & & \xrightarrow{i_!} & L_n(G) & \xrightarrow{i^! t^{-1}} & LN_{n-2}(\pi \to G) & \longrightarrow & \\
\nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & & \searrow \\
 & & & L_n(i^!) & & & L_n(i_!) & & & \\
\searrow & & \nearrow & & \searrow & & \nearrow & \searrow & & \nearrow \\
\longrightarrow & LN_{n-1}(\pi \to G) & & \xrightarrow{t i_!} & L_{n-1}(G^-) & \xrightarrow{i^!} & L_{n-1}(\pi) & \longrightarrow &
\end{array}
\tag{1}
$$

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For the geometric applications, the maps appearing in the upper and lower row of diagram (1) are the most important ones (see [2, 6]). Diagrams in which we consider only the upper and lower rows will be called *two-row diagrams*. Note that the upper and lower row of diagram (1) are chain complexes with isomorphic homology in the corresponding terms. This fact yields a rather efficient method for computing groups and maps. The sign "$-$" means that the orientation homomorphism has been changed on the generator $t \in G \setminus \pi$. Here $i_!$ is the map induced by the inclusion i, while $i^!$ is the transfer map in the Wall groups. If t is an arbitrary element of $G \setminus \pi$, then the scaling isomorphism of the Wall groups for rings with antistructures (see §2) is defined [7, 8]:

$$t\colon L_n(G, w) \to L_n(\mathbb{Z}G, \beta, w(t)t^2),$$

where $\beta(x) = t\alpha(x)t^{-1}$. Note, in particular, that the map $i^!t^{-1}$ in the diagram (1) specifies the Browder–Livesay invariant.

A diagram similar to (1) beggining with the map $i_!\colon L_n(\pi) \to L_n(G^-)$ also exists. In order to distinguish them, diagram (1) will be denoted by (D), while $(\widetilde{D})$ will denote the other one.

Let $\pi = \mathbb{Z}/2^r \subset D_{r+1}$ be an inclusion of index 2 of the cyclic group with trivial orientation in the dihedral 2-group $D_{r+1} = \{x^{2^r} = y^2 = 1,\ y^{-1}xy = x^{-1}\}$. For $r = 2$ the groups $L_n(D_{r+1}, w)$ and $L_n^p(D_{r+1}, w)$ are isomorphic for any choice of the orientation homomorphism w. Therefore,the results of [2] give a complete description of the groups $C_n(D_3, w)$ for any $n = 0, 1, 2, 3 \bmod 4$. For $r \geq 3$ such a homomorphism no longer exists, so that in this case the results of [2] yield only a lower bound for the groups $C_n(D_{r+1}, w)$.

In the present paper, the groups $C_n(D_{r+1}, w)$ are computed for the cases in which the orientation homomorphism w is trivial on the generator x of the group D_{r+1}. The Wall groups of the dihedral group in these cases are computed in [8], while the Browder–Livesay groups are calculated in [9]. The main difficulty is in the calculation of the natural maps in the diagram (1). In §3, for the embeddings under consideration, the maps in diagram (1) are computed, as well as the group

$$LN_1(\pi \to D_{r+1}^{+,+}) \cong LN_3(\pi \to D_{r+1}^{+,-}),$$

obtained in [9], but only up to an extension. For the computations, the spectral sequence of surgery theory from the paper [10] is used. Note that in the case of trivial orientation on the group D_{r+1} ($w(y) = w(x) = 1$), the groups $C_n(D_{r+1}, w)$ can also be calculated (see [3]). The main results of the present paper are Theorems 3–5.

An intermediate step in the calculation of surgery obstruction groups and of the Browder–Livesay groups (see [8]) consists in computing the groups $L_*^Y(\widehat{\mathbb{Z}}_2\pi)$, where $\widehat{\mathbb{Z}}_2$ is the ring of 2-adic numbers, $Y = \ker\big(K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Q}\pi)\big) \cup \{\pm\pi\}$. These groups apppear in the Rothenberg exact sequence (see [8])

$$\longrightarrow L_n^Y(\widehat{\mathbb{Z}}_2\pi) \longrightarrow L_n^K(\widehat{\mathbb{Z}}_2\pi) \longrightarrow H^n(K/Y) \longrightarrow, \tag{2}$$

where $H^n(K/Y)$ is the Tate cohomology of the group K/Y, $K = K_1(\widehat{\mathbb{Z}}_2\pi)$, and the involution on the group K/Y is induced from the standard involution on the group ring $\widehat{\mathbb{Z}}_2\pi$ given by the formula

$$\Sigma n_g g \to \Sigma n_g w(g) g^{-1}, \qquad n_g \in \widehat{\mathbb{Z}}_2, \quad g \in \pi,$$

in which $w\colon \pi \to \{\pm 1\}$ is the orientation homomorphism. For a finite 2-group π, the group $L_*^K(\widehat{\mathbb{Z}}_2\pi)$ is isomorphic to $\mathbb{Z}/2$. The groups $L_*^Y(\widehat{\mathbb{Z}}_2\pi)$ and similar groups for the antistructures corresponding to the Browder–Livesay groups are calculated in [8, 9]. The natural maps between these groups may be calculated by using the natural maps of the Tate cohomology (see [11–13]).

For any group π we have the relative long exact sequence (see [8])

$$\longrightarrow L_n'(\mathbb{Z}\pi) \longrightarrow L_n'(\widehat{\mathbb{Z}}_2\pi) \longrightarrow L_n(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi) \longrightarrow, \tag{3}$$

in which $L'_n(\widehat{\mathbb{Z}}_2\pi)$ is the group $L^Y_n(\widehat{\mathbb{Z}}_2\pi)$ for $n = 2k$ and can be identified with the kernel of the map $L^Y_n(\widehat{\mathbb{Z}}_2\pi) \to L^K_n(\widehat{\mathbb{Z}}_2\pi)$ for $n = 2k+1$. Note that for the dihedral group or for the cyclic 2-group π the group

$$SK_1(\mathbb{Z}\pi) = \ker\big(K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Q}\pi)\big)$$

is trivial (see [8]). Therefore, in the case under consideration, we have the isomorphism $L' \cong L^s$. In the sequel all the L^s-groups will be denoted by L.

Thus, to compute the natural maps in the diagram (1) for the inclusion $i\colon \pi \longrightarrow G$ of index 2, we can use the two-row diagrams for the quadratic extension $\widehat{\mathbb{Z}}_2\pi \to \widehat{\mathbb{Z}}_2G$ (see [7]) and the two-row diagrams for the relative groups (see [11]).

§2. Natural maps for Tate cohomology

This section contains certain necessary preliminary facts about Wall groups for rings with antistructures and spectral sequences in surgery theory for quadratic extensions of antistructures (see [7–14]).

An ***antistructure*** is a triple (R, α, u), where R is a ring with unit, $u \in R^*$ is an invertible element of the ring, $\alpha\colon R \to R$ is an antiautomorphism of the ring R such that $\alpha(u) = u^{-1}$, $\alpha^2(x) = uxu^{-1}$ for any $x \in R$. A ***structure*** on a ring R is a pair (ρ, a), where $\rho\colon R \to R$ is an automorphiam of the ring R, while a is an invertible element such that $\rho(a) = a$, $\rho^2(x) = axa^{-1}$ for any $x \in R$ (see [7]). Let $S = R[t]/(t^2 - a)$ be a quadratic extension of the ring R, where t is an element, independent over R, such that $\rho(x) = txt^{-1}$ for any $x \in R$. Assume also that the automorphism α can be extended to S so that $\alpha(t)t \in R \subset S$, $\alpha^2(t) = utu^{-1} \in S$. In this case the antistructure (S, α, u) is said to be a ***quadratic extension*** of the antistructuere (R, α, u). The automorphism ρ can be extended to the ring S by means of the formula $\rho(x + yt) = t(x + yt)t^{-1}$, $x, y \in R$. One can also define the automorphism γ of the ring S by setting $\gamma(x + yt) = (x - yt)$, $x, y \in R$. From the given quadratic extension of antistructures we can construct another quadratic extension $(R, \widetilde{\alpha}, \widetilde{u}) \to (S, \widetilde{\alpha}, \widetilde{u})$, where $\widetilde{\alpha} = \rho\gamma\alpha$ and $\widetilde{u} = -t\alpha(t^{-1})u$ (see [7]). Since the squares of the antiautomorphisms $\alpha, \widetilde{\alpha}$ are inner automorphisms, they induce involutions on the groups $K_j(R), K_j(S)$ for $j = 0, 1$. Denote these involutions by T and $\widetilde{T}$ respectively.

For any subgroup $X \subset K_j(R)$ invariant with respect to the involution T, the decorated Wall groups $L^X_n(R, \alpha, u)$ are defined so that for any two invariant subgroups $X \subset X' \subset K_j(R)$ we have the Rothenberg exact sequence (see, for example [8, 14])

$$\longrightarrow L^X_n(R) \longrightarrow L^{X'}_n(R) \longrightarrow H^n(X'/X) \longrightarrow ,$$

where H^* is the Tate cohomology. Let $i\colon (R, \alpha, u) \to (S, \alpha, u)$ be a quadratic extension of antistructures. Then for $j = 0, 1$ the induced homomorphism $i_*\colon K_j(R) \to K_j(S)$ and the transfer homomorphism $i^*\colon K_j(S) \to K_j(R)$ are defined. The maps i^*, i_* commute with the involutions $T, \widetilde{T}$ and therefore induce maps of the Tate cohomology, which we shall denote by $i^!$ and $i_!$ respectively.

Similarly, let $X \subset X' \subset K_j(R)$ and $Y \subset Y' \subset K_j(S)$ be T-, $\widetilde{T}$-invariant subgroups for $j = 0, 1$. Assume that $i_*(X) \subset Y$, $i^*(Y) \subset X$, $i_*(X') \subset Y'$, $i^*(Y') \subset X'$, and denote $A = X'/X$, $B = Y'/Y$. In this case we also obtain maps of the Tate cohomology groups A and B induced by i^*, i_*, which we still denote by $i^!$ and $i_!$ respectively.

Under these assumptions, there exists a Levine braid consisting of relative exact sequencces of Tate cohomology (see [11, 12]):

$$\begin{array}{ccccccccccccc}
\longrightarrow & H^n(B,\widetilde{T}) & & \longrightarrow & & H^n(A,T) & & \longrightarrow & & H^n(B,T) & & \longrightarrow & \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
& & & H^{n-1}(i^+_*) & & & & H^n(i^*_-) & & & & & \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
\longrightarrow & H^{n-1}(B,T) & & \longrightarrow & & H^{n-1}(A,\widetilde{T}) & & \longrightarrow & & H^{n-1}(B,\widetilde{T}) & & \longrightarrow &
\end{array} \qquad (4)$$

This diagram contains relative exact sequences for the induced maps

$$i_*\colon (A, T) \to (B, T), \qquad i_*\colon (A, \widetilde{T}) \to (B, \widetilde{T})$$

and for the transfer maps

$$i^*: (B, T) \to (A, \widetilde{T}), \qquad i^*: (B, \widetilde{T}) \to (A, T).$$

Denote this diagram by (D). The other diagram, $(\widetilde{D})$, is obtained for the induced maps

$$i_*: (A, \widetilde{T}) \to (B, T), \qquad i_*: (A, T) \to (B, \widetilde{T})$$

and the transfer maps

$$i^*: (B, T) \to (A, T), \qquad i^*: (B, \widetilde{T}) \to (A, \widetilde{T}).$$

The diagram (1) can be realized on the spectral level (see [10]). Using this fact, the spectral sequence of surgery theory is constructed in the paper [10]; in it $E_1^{p,q} = LN_{q+2}(\pi \to G)$, and the first differential coincides with the composition

$$LN_n(\pi \to G^{\pm}) \longrightarrow L_n(G^{\mp}) \longrightarrow LN_{n-2}(G^{\mp}),$$

where one map comes from diagram (D), and the other from $(\widetilde{D})$. Here the first differential coincides with the map $1 \mp \Phi$, where Φ is the involution on the Browder–Livesay groups (see [2, 10]). The diagram (4) can be realized on the spectrum level (see [11, 13]) similarly to diagram (1). Thus the method of Hambleton and Kharshiladze [10], who constructed the spectral sequence of surgery theory from diagram (1), yields the spectral sequence of Tate cohomology in the case under consideration. A detailed construction of this spectral sequence for this case appears in the paper [13]. The first differential $d_1^{p,q}$ for even p and $q = 0, 1$ coincides with the composition

$$H^q(A, \widetilde{T}) \xrightarrow{i_!} H^q(B, \widetilde{T}) \xrightarrow{i_!} H^q(A, \widetilde{T}),$$

where the map $i_!$ comes from diagram (D), while the map $i^!$ comes from the diagram $(\widetilde{D})$. For odd p and $q = 0, 1$ the first differential $d_1^{p,q}$ coincides with the composition

$$H^q(A, \widetilde{T}) \xrightarrow{i_!} H^q(B, T) \xrightarrow{i_!} H^q(A, \widetilde{T}),$$

where the map $i_!$ comes from diagram $(\widetilde{D})$, while the map $i^!$ comes from (D). Denote by ρ the involution on the group $H^q(A, \widetilde{T})$, induced by the involution ρ on the ring R.

Theorem 1 [13]. *The differential $d_1^{p,q}: E_1^{p,q} \to E_1^{p+1,q}$ does not depend on p and q and coincides with the homomorphism $1 + \rho$.*

Let $\pi = \mathbb{Z}/2^r \subset D_{r+1} = G$ be the index 2 inclusion of the cyclic group with trivial orientation into the dihedral group also supplied with the trivial orientation, $w = 1$. By i^- we shall denote the inclusion $\pi \to G^-$ of the oriented groups. If the orientation is clear from the context, we shall not indicate it in our notation for groups and maps. The inclusion i induces a quadratic extension of rings with the standard involutions $\widehat{\mathbb{Z}}_2\pi \to \widehat{\mathbb{Z}}_2 G$. In this casee the involution ρ for any element $x \in \pi$ is given by the formula $\rho(x) = x^{-1}$. Let us denote $A = K_1(\widehat{\mathbb{Z}}_2\pi/Y)$, $B = K_1(\widehat{\mathbb{Z}}_2 G/Y)$. Then $H^1(A, T) = H^1(A, \widetilde{T}) = H^1(B, T) = H^1(B, \widetilde{T}) = 0$, $H^0(A, T) = (\mathbb{Z}/2)^2$, $H^0(A, \widetilde{T}) = (\mathbb{Z}/2)^{2^r}$, $H^0(B, T) = (\mathbb{Z}/2)^{2^{r-1}+3}$, $H^0(B, \widetilde{T}) = (\mathbb{Z}/2)^{2^{r-1}-1}$ (see [8, 9]).

Theorem 2 [11, 13]. 1) *In the two-row diagram for Tate cohomology for the inclusion i, only the even-dimensional row is nontrivial and this row is the exact sequence*

$$\longrightarrow H^0(A, T) \xrightarrow{i_!} H^0(B, T) \xrightarrow{i^!} H^0(A, \widetilde{T}) \xrightarrow{i_!} H^0(B, \widetilde{T}) \xrightarrow{i^!},$$

in which the left-most map is a monomorphism.

2) *In the two-row diagram for the inclusion i^- only the even-dimensional row is nontrivial and this row is the exact sequence*

$$\longrightarrow H^0(A, T) \xrightarrow{i_!} H^0(B, \widetilde{T}) \xrightarrow{i^!} H^0(A, \widetilde{T}) \xrightarrow{i_!} H^0(B, T) \xrightarrow{i^!},$$

in which the left-most map is trivial.

§3. Natural maps of L-groups

In this section we compute the natural maps in the diagram (1) for the inclusion $i\colon \pi = \mathbb{Z}/2^r \subset D_{r+1} = \{x^{2^r} = y^2 = 1, y^{-1}xy = x^{-1}\} = G$ and calculate certain Browder–Livesay groups.

Recall that we only consider the case of a trivial orientation for the cyclic subgroup. Let us introduce the following notation: $R = \mathbb{Z}\pi$, $T^{\pm} = \mathbb{Z}G^{\pm}$, $\widehat{R}_2 = \widehat{\mathbb{Z}}_2\pi$, $\widehat{T}_2^{\pm} = \widehat{\mathbb{Z}}_2G^{\pm}$, $A = K_1(\widehat{\mathbb{Z}}_2\pi)/Y$, $B = K_1(\widehat{\mathbb{Z}}_2G)/Y$. The sign "+" means that the orientation homomorphism w is trivial on the generator y, while the symbol "−" means that $w(y) = -1$. Whenever we consider the Tate cohomology with respect to the involution induced by the standard involution in the group ring (it corresponds to the involution T from §2), we shall write $H^n(A)$ and $H^n(B)$. The Tate cohomology with respect to the identity involution corresponding to the antistructure of the Browder–Livesay groups will be denoted by $H^n(A,*)$ and $H^n(B,*)$ (it corresponds to the involution $\widetilde{T}$ from §2). Recall that the antistructure $L_n(\widehat{\mathbb{Z}}_2\pi, \mathrm{Id}, 1)$ correponds to the antistructure of the Browder–Livesay groups in the case under consideration via the isomorphism $LN_{n+2}(\pi \to D_{r+1}^{+}) \cong LN_n(\pi \to D_{r+1}^{-}) \cong L_n(\mathbb{Z}\pi, \mathrm{Id}, 1)$.

All the groups appearing in the diagrams (1) for the quadratic extensions $\widehat{\mathbb{Z}}_2\pi \to \widehat{\mathbb{Z}}_2G^{+}$ and $\widehat{\mathbb{Z}}_2\pi \to \widehat{\mathbb{Z}}_2G^{-}$, are known from the papers [8, 9]. We have the following isomorphisms:

$$L_n(\widehat{R}_2) \cong \begin{cases} 0 & \text{for } n = 0 \bmod 4, \\ (\mathbb{Z}/2)^2 & \text{for } n = 1 \bmod 4, \\ \mathbb{Z}/2 & \text{for } n = 2,3 \bmod 4, \end{cases}$$

$$L_n(\widehat{T}_2^{-}) \cong \begin{cases} \mathbb{Z}/2 & \text{for } n = 0,2 \bmod 4, \\ (\mathbb{Z}/2)^{2^{r-1}-1} & \text{for } n = 1,3 \bmod 4, \end{cases}$$

$$L_n(\widehat{T}_2^{+}) \cong \begin{cases} 0 & \text{for } n = 0 \bmod 4, \\ (\mathbb{Z}/2)^{2^{r-1}+3} & \text{for } n = 1 \bmod 4, \\ \mathbb{Z}/2 & \text{for } n = 2 \bmod 4, \\ (\mathbb{Z}/2)^{2^{r-1}+2} & \text{for } n = 3 \bmod 4, \end{cases}$$

$$L_n(\widehat{\mathbb{Z}}_2\pi, \mathrm{Id}, 1) \cong \begin{cases} 0 & \text{for } n = 0 \bmod 4, \\ (\mathbb{Z}/2)^{2^r} & \text{for } n = 1 \bmod 4, \\ \mathbb{Z}/2 & \text{for } n = 2 \bmod 4, \\ (\mathbb{Z}/2)^{2^r-1} & \text{for } n = 3 \bmod 4. \end{cases}$$

Lemma 1. *1) The map $i_!\colon L_n(\widehat{R}_2) \to L_n(\widehat{T}_2^{+})$, induced by i, is an isomorphism for any $n = 0,1,2,3 \bmod 4$.*

2) The map $i_!\colon L_n(\widehat{R}_2) \to L_n(\widehat{T}_2^{-})$ is an isomorphism for even n and is trivial for odd n, while the map $i^!\colon L_{2n+1}(\widehat{T}_2^{+}) \to L_{2n+1}(\widehat{R}_2)$ is an epimorphism for $n = 0,1 \bmod 2$.

Proof. In even dimensions, the statements of the lemma are either trivial or follow from the preservation of the Arf-invariant by the map $i_!$. The latter, in its turn, follows from the obvious isomorphisms $L_n^K(\widehat{R}_2) \xrightarrow{\cong} L_n^K(\widehat{T}_2^{\pm}) \cong \mathbb{Z}/2$.

Consider the commutative diagram

$$\begin{array}{ccccccc} & & & & L_2^K(\widehat{T}_2^{+}) & \cong & \mathbb{Z}/2 \\ & & & & \downarrow & & \\ (\mathbb{Z}/2)^2 & \cong & H^0(A) & \xrightarrow{i_!} & H^0(B) & \cong & (\mathbb{Z}/2)^{2^{r-1}+3} \\ & & \downarrow & & \downarrow & & \\ (\mathbb{Z}/2)^2 & \cong & L_1(\widehat{R}_2) & \xrightarrow{i_!} & L_1(\widehat{T}_2^{+}) & \cong & (\mathbb{Z}/2)^{2^{r-1}+3} \end{array}.$$

The right column is the Rothenberg exact sequence (2). The upper map in it is trivial, since the Arf-invariant is realized in the group $L_2(\widehat{T}_2^+)$. The upper horizontal map is a monomorphism by Theorem 2, hence so is the lower horizontal one. In dimension 3 in case 1) the argument is similar. In case 2), we consider the commutative diagram

$$\begin{array}{ccccccc} (\mathbb{Z}/2)^{2^{r-1}+3} & \cong & H^0(B) & \xrightarrow{i^!} & H^0(A) & \cong & (\mathbb{Z}/2)^2 \\ & & \downarrow & & \downarrow & & \\ (\mathbb{Z}/2)^{2^{r-1}+3} & \cong & L_1(\widehat{T}_2^+) & \xrightarrow{i^!} & L_1(\widehat{R}_2) & \cong & (\mathbb{Z}/2)^2 \end{array}$$

The upper horizontal map is an epimorphism by Theorem 2. It follows from the previous diagram that the right-most vertical map is an isomorphism. This implies the statement of the lemma in dimension 1. The case of dimension 3 is studied similarly. Note also that the map $L_2(\widehat{\mathbb{Z}}_2\pi, \mathrm{Id}, 1) \to L_2(\widehat{T}_2^+)$, appearing in one of the diagrams considered, is an isomorphism. It preserves the Arf-invariant. Now the remaining statements of the Lemma are obtained by diagram chasing in two-row diagrams. The lemma is proved. □

It should be noted that the results established in Lemma 1 specify all the maps in diagrams of type (1) for quadratic extensions $\widehat{R}_2 \to \widehat{T}_2^{\pm}$.

The relative Wall groups appearing in the sequence (3) are also known (see [8, 9]). We have the following isomorphisms:

$$L_n(R \to \widehat{R}_2) \cong \begin{cases} 0 & \text{for } n = 0, 2 \bmod 4, \\ \mathbb{Z}^{2^{r-1}+1} \oplus (\mathbb{Z}/2)^2 & \text{for } n = 1 \bmod 4, \\ \mathbb{Z}^{2^{r-1}-1} & \text{for } n = 3 \bmod 4, \end{cases}$$

$$L_n(T^+ \to \widehat{T}_2^+) \cong \begin{cases} 0 & \text{for } n = 0, 2, 3 \bmod 4, \\ \mathbb{Z}^{2^{r-1}+3} \oplus (\mathbb{Z}/2)^{r+3} & \text{for } n = 1 \bmod 4, \end{cases}$$

$$L_n(T^- \to \widehat{T}_2^-) \cong \begin{cases} 0 & \text{for } n = 0, 1, 2 \bmod 4, \\ (\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}^{2^{r-1}-1} & \text{for } n = 3 \bmod 4, \end{cases}$$

$$L_n(R \to \widehat{R}_2, \mathrm{Id}, 1) \cong \begin{cases} (\mathbb{Z}/2)^{r-1} & \text{for } n = 0 \bmod 4, \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{2^{r-1}+r} & \text{for } n = 1 \bmod 4, \\ 0 & \text{for } n = 2, 3 \bmod 4. \end{cases}$$

In the diagrams of the next lemma, we write out only the nontrivial relative groups in the upper and lower row.

Lemma 2. 1) *In the two-row diagram* (D) *for the relative groups*

$$\begin{array}{lcl} L_1(R \to \widehat{R}_2) \xrightarrow{i_!} L_1(T^+ \to \widehat{T}_2^+) \xrightarrow{i^! t^{-1}} L_1(R \to \widehat{R}_2, \mathrm{Id}, 1) & \xrightarrow{t i_!} & L_3(T^- \to \widehat{T}_2^-) \\ & \xrightarrow{i^!} & L_3(R \to \widehat{R}_2) \\ & & \quad | \\ & & L_0(R \to \widehat{R}_2, \mathrm{Id}, 1) \end{array}$$

the map $i_!$ *is a monomorphism with the cokernel* $\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{2^{r-1}+1}$; *the cokernel of the map* $i^! t^{-1}$ *is* $(\mathbb{Z}/2)^{r-1}$ *and lies in the torsion subgroup; the map* $t i_!$ *is an epimorphism onto the torsion subgroup; the cokernel of the map* $i^!$ *is* $(\mathbb{Z}/2)^{r-1}$.

2) *In the two-row diagram* $(\widetilde{D})$ *for the relative groups*

$$\begin{array}{lcl} L_3(R \to \widehat{R}_2) \xrightarrow{i_!} L_1(T^- \to \widehat{T}_2^-) \xrightarrow{i^! t^{-1}} L_1(R \to \widehat{R}_2, \mathrm{Id}, 1) & \xrightarrow{t i_!} & L_1(T^+ \to \widehat{T}_2^+) \\ & \xrightarrow{i^!} & L_1(R \to \widehat{R}_2) \\ & & \quad | \\ & & L_0(R \to \widehat{R}_2, \mathrm{Id}, 1) \end{array}$$

the map $i_!$ has the cokernel $(\mathbb{Z}/2)^{2^{r-1}-1}$; the image of the map $i^!t^{-1}$ is $(\mathbb{Z}/2)^{2^{r-1}-1}$, and is contained in the torsion subgroup; the image of the map $ti_!$ is $(\mathbb{Z}/2)^{r+1}\oplus\mathbb{Z}^2$, and is a direct summand; the map $i^!$ is an epimorphism onto the torsion group and has the cokernel $(\mathbb{Z}/2)^{r-1}$.

Proof. Consider the case 1). According to [15] (also see [11]), for any finite 2-group π the group $L_*(\mathbb{Z}\pi\to\widehat{\mathbb{Z}}_2\pi)$ is isomorphic to $L^S_*(\mathbb{Z}[1/2]\pi\to\widehat{\mathbb{Q}}_2\pi)$, where $S=0\subset K_1$. The group L^S splits into the direct sum in accordance with the decomposition of the ring $\mathbb{Z}[1/2]\pi$, and the the map of group rings induced by i can be decomposed into a direct sum similarily (see [11, 15]). In the case under consideration, the two-row diagram for the relative groups splits into a direct sum of

a) two two-row diagrams corresponding to the diagonal inclusion

$$\mathbb{Z}[1/2]\to\mathbb{Z}[1/2]\oplus\mathbb{Z}[1/2],$$

b) $r-1$ two-row diagrams corresponding to the quadratic extensions

$$(\Gamma_k,c,1)\to M_2(R_{k-1},\mathrm{Id},1),\qquad k=1,2,\dots,r-1$$

(see [11]), where $\Gamma_k=\mathbb{Z}[1/2][\theta_{k+1}]$, $R_{k+1}=\mathbb{Z}[1/2][\theta_{k+2}+\overline{\theta}_{k+2}]$, c denotes complex conjugation, θ_m is a primitive root of 1 of degree 2^m.

The relative group diagrams for the maps in case b) are isomorphic to the two-row diagrams for the inclusions $(R_{k-1},\mathrm{Id},1)\to(\Gamma_k,c,1)$ up to a change of notation for the maps (see [15]). Thus all the required diagrams from a) and b) are described in [11] (they are the diagram $(D4)$ for $N=1$ and diagram $(D3)$ for $N=k$). A straightforward calculation establishes statement 1) of Lemma 2. Case 2) is considered in a similar way. The lemma is proved. □

Lemma 3. *Let A be a finite Abelian 2-group and let Φ be an involution on the group A such that the images of the homomorphisms $d^{\pm}\colon A\to A$ specified by the formulas $d^+(x)=x\Phi(x)$, $d^-(x)=x(\Phi(x))^{-1}$, are elemenrary 2-groups and direct summands in the group A. Then A is an elementary 2-group.*

Proof. For any $x\in A$ consider the subgroup $\rho\subset A$, generated by the elements x and $\Phi(x)$. If ρ is a cyclic subgroup, then x and $\Phi(x)$ are its generators since Φ is an involution. Here if ρ is of order 2, then x is of order 2. If ρ is of order greater than 2, then $x,\Phi(x)\in\mathrm{Im}(\Phi|_\rho)$. Therefore, $x\Phi(x)=x(\Phi(x))^{-1}=1$ since the only elementary 2-group that can be a direct summand in the group ρ is the trivial group. Thus $(\Phi(x))^2=1$, which contradicts the assumption that the order of ρ is greater than 2. If ρ is not a cyclic group, then it can be represented in the form $\rho_1\oplus\rho_2$, where ρ_1 and ρ_2 are cyclic 2-groups interchanged by the involution. These groups are generated by x and $\Phi(x)$ respectively. Hence the order of the element $x\Phi(x)$ is equal to the order of the group ρ_1, which coincides with that of ρ_2. By assumption, the order of $x\Phi(x)$ is 2, i.e., the groups ρ_1,ρ_2 and the element x are of order 2. The lemma is proved. □

Theorem 3. *We have the isomorphism*

$$LN_3(\mathbb{Z}/2^r\to D^{+;-}_{r+1})\cong LN_1(\mathbb{Z}/2^r\to D^{+;+}_{r+1})\cong(\mathbb{Z}/2)^{2^r+r-2}.$$

Proof. The group $LN_3=LN_3(\mathbb{Z}/2^r\to D^{+;-}_{r+1})$ is described up to extension in the paper [9]. Hence it suffices to prove that this group is elementary. Consider the commutative diagram

$$\begin{array}{ccccc}
LN_3 & \xrightarrow{ti_!} & L_3(T^-) & \xrightarrow{i^!t^{-1}} & LN_3\\
\Big\downarrow{\scriptstyle\mathrm{epi}} & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\mathrm{epi}}\\
L_3(\widehat{R}_2,\mathrm{Id},1) & \xrightarrow{ti_!} & L_3(\widehat{T}_2^-) & \xrightarrow{i^!t^{-1}} & L_3(\widehat{R}_2,\mathrm{Id},1)
\end{array},$$

in which the extreme vertical maps come from the relative exact sequence (3) and so are epimorphisms according to [9]. The lower left horizontal map is an epimorphism, while the lower right horizontal map is a monomorphism by virtue of Lemma 1. The middle vertical map is an isomorphism according to [8]. The composition of the upper horizontal maps is the first differential of the surgery theory spectral sequence $d_1 = 1 + \Phi$, where Φ is the involution on the group LN_3 (see [10]). Since $L_3(T^-)$ is an elementary 2-group, the image of the map d_1 in the upper row of the diagram is also an elemenatry 2-group. The group $L_3(\widehat{R}_2, \mathrm{Id}, 1)$ is an elementary 2-group. Diagram chasing in the right square of the diagram shows that the image of d_1 is a direct summand.

Similarly, let us consider the commutative diagram

$$\begin{array}{ccccc} LN_3 & \xrightarrow{ti_!} & L_3(T^+) & \xrightarrow{i^!t^{-1}} & LN_3 \\ \downarrow{\scriptstyle \mathrm{epi}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{epi}} \\ L_3(\widehat{R}_2, \mathrm{Id}, 1) & \xrightarrow{ti_!} & L_3(\widehat{T}_2^+) & \xrightarrow{i^!t^{-1}} & L_3(\widehat{R}_2, \mathrm{Id}, 1) \end{array};$$

in it the right-most horizontal maps have isomorphic images $(\mathbb{Z}/2)^{2^{r-1}}$ by virtue of Lemma 1. The composition of the upper horizontal maps is the first differential $1-\Phi$ in the spectral sequence of surgery theory. By diagram chasing in the right square, we see that the image of the map $1-\Phi$ is an elementary 2-group and a direct summand in the group LN_3. An application of Lemma 3 concludes the proof of Theorem 3. □

Now we can compute the natural maps of the Wall groups in the two-row diagram (1) for the inclusions $i\colon \pi = \mathbb{Z}/2 \to D^{\pm}_{r+1} = G^{\pm}$. Further we shall use these results to compute the groups $C_n(D_{r+1}, w)$.

Let us recall some results concerning Wall groups and Browder–Livesay groups needed for these computations (see [8, 9]). Let Σ, Σ' be infinite abelian groups of ranks $2^{r-1}-1$, $2^{r-1}+3$, respectively. We have the following isomorphisms:

$$L_n(\pi) \cong \begin{cases} \Sigma \oplus \mathbb{Z}^2 & \text{for } n = 0 \bmod 4, \\ 0 & \text{for } n = 1 \bmod 4, \\ \Sigma \oplus \mathbb{Z}/2 & \text{for } n = 2 \bmod 4, \\ \mathbb{Z}/2 & \text{for } n = 3 \bmod 4, \end{cases}$$
$$L_n(G^+) \cong \begin{cases} \Sigma' & \text{for } n = 0 \bmod 4, \\ (\mathbb{Z}/2)^{2^{r-1}-r} & \text{for } n = 1 \bmod 4, \\ \mathbb{Z}/2 & \text{for } n = 2 \bmod 4, \\ (\mathbb{Z}/2)^{2^{r-1}+2} & \text{for } n = 3 \bmod 4, \end{cases}$$
$$L_n(G^-) \cong \begin{cases} \mathbb{Z}/2 & \text{for } n = 0 \bmod 4, \\ (\mathbb{Z}/2)^{2^{r-1}-1} & \text{for } n = 1 \bmod 4, \\ \Sigma \oplus \mathbb{Z}/2 & \text{for } n = 2 \bmod 4, \\ (\mathbb{Z}/2)^{2^{r-1}-r} & \text{for } n = 3 \bmod 4. \end{cases}$$

The groups $L_n(\mathbb{Z}\pi, \mathrm{Id}, 1) \cong LN_n(\pi \to G^-) \cong LN_{n+2}(\pi \to G^+)$ will be denoted by LN_n. Then $LN_3 \cong (\mathbb{Z}/2)^{2^r+r-2}$ by Theorem 3. According to [9], the group LN_2 is isomorphic to $\mathbb{Z}/2$ (Arf-invariant), and we have the exact sequence

$$0 \longrightarrow LN_1 \longrightarrow (\mathbb{Z}/2)^{2^{r-1}+r} \oplus \mathbb{Z}^2 \longrightarrow LN_0 \longrightarrow 0.$$

Consider the following part of the two-row diagram (1)

$$\begin{array}{ccccc} L_n(\pi) & \xrightarrow{i_!} & L_n(G^+) & \xrightarrow{i^!t^{-1}} & LN_n \\ \downarrow & & \downarrow{\scriptstyle \Gamma} & & \downarrow \\ LN_{n+1} & \xrightarrow{ti_!} & L_{n-1}(G^-) & \xrightarrow{i^!} & L_{n-1}(\pi) \end{array} \tag{5}$$

for the inclusion $i\colon \pi \to G^+$ and a similar part

$$\begin{array}{ccccc} L_n(\pi) & \xrightarrow{i_!} & L_n(G^-) & \xrightarrow{i^!t^{-1}} & LN_{n-2} \\ \downarrow & & \downarrow{\scriptstyle \Gamma} & & \downarrow \\ LN_{n-1} & \xrightarrow{ti_!} & L_{n-1}(G^+) & \xrightarrow{i^!} & L_{n-1}(\pi) \end{array} \qquad (6)$$

for the inclusion $i\colon \pi \to G^-$.

Theorem 4. *1) In diagram (5) the map $i^!t^{-1}\colon L_n(G^+) \to LN_n$ is a monomorphism for $n = 1 \bmod 4$, is trivial for $n = 2 \bmod 4$, has the kernel $(\mathbb{Z}/2)^2$ for $n = 3 \bmod 4$, and is epimorphic for $n = 0 \bmod 4$. The map $i_!\colon L_n(\pi) \to L_n(G^+)$ is trivial for $n = 1 \bmod 4$, monomorphic for $n = 3 \bmod 4$, epimorphic for $n = 2 \bmod 4$, and monomorphic with cokernel $\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{2^{r-1}-r}$ for $n = 0 \bmod 4$.*

2) In diagram (6) the map $i^!t^{-1}\colon L_n(G^-) \to LN_{n-2}$ is monomorphic for $n = 1, 3 \bmod 4$, trivial for $n = 0 \bmod 4$, and epimorphic for $n = 2 \bmod 4$. The map $i_!\colon L_n(\pi) \to L_n(G^-)$ is trivial for $n = 0, 1, 3 \bmod 4$ and monomorphic with cokernel $\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{2^{r-1}-r}$ for $n = 2 \bmod 4$.

Proof. We only consider case 1). The map $i_!$ is epimorphic in dimension 2 since it preserves the Arf-invariant. In dimension 1 it is trivial since $L_1(\pi) = 0$. In this case let us consider the following part of diagram (1):

$$\mathbb{Z}/2 \;\cong\; L_3(\pi) \xrightarrow{i_!} L_3(G^+) \xrightarrow{i^!t^{-1}} LN_3 \xrightarrow{ti_!} L_1(G^-) \longrightarrow 0\cdot \qquad (7)$$

The homology in the term $L_1(G^-)$ is trivial as we can see by inspecting the lower row. The homology in the term LN_3 is isomorphic to that in the term $\xrightarrow{i^!} L_2(\pi) \xrightarrow{i_!}$ of the lower row of the diagram. Consider the commutative diagram

$$\begin{array}{ccccc} L_3(T^- \to \widehat{T}_2^-) & \xrightarrow{i^!} & L_3(R \to \widehat{R}_2) & \xrightarrow{i_!} & L_3(T^+ \to \widehat{T}_2^+) \\ \downarrow & & \downarrow{\scriptstyle \Gamma} & & \downarrow \\ L_2(G^-) & \xrightarrow{i^!} & L_2(\pi) & \xrightarrow{i_!} & L_2(G^+) \end{array},$$

in which the vertical maps come from the corresponding relative exact sequences (3). The homology in the term $L_3(R \to \widehat{R}_2)$ equals $(\mathbb{Z}/2)^{r-1}$ and is isomorphically mapped into that of the term $L_2(\pi)$ of the lower row, since the middle vertical map is the inclusion of a direct summand, while the right vertical map is trivial. The homology in the term $L_3(G^+)$ of diagram (7) is isomorphic to the direct summand $\mathbb{Z}/2$ of the group $L_2(G^-)$ from the corresponding lower row since the Arf-invariant is not realized in the group LN_0, while the map $L_2(G^-) \xrightarrow{i^!} L_2(\pi)$, from the lower row is a monomorphism on the free part by virtue of Lemma 3. Therefore, in diagram (7) we have

$$\operatorname{rang}(\operatorname{Im} i^!t^{-1}) = \operatorname{rang} LN_3 - \operatorname{rang} L_1(G^-) - (r-1) = 2^r + r - 2 - (2^{r-1} - 1) - (r-1) = 2^{r-1}.$$

This immediately implies both statements from 1) in dimension 3. The description of the maps $i^!t^{-1}$ in dimension 1 and 2 presents no difficulties, since in this case the map $\mathbb{Z}/2 \cong LN_2 \xrightarrow{ti_!} L_0(G^-) \cong \mathbb{Z}/2$ from the diagram (D) is an isomorphism. To prove the claims in dimension 0, let us consider the commutative diagram

$$\begin{array}{ccc} L_1(R \to \widehat{R}_2) & \xrightarrow{i_!} & L_1(T^+ \to \widehat{T}_2^+) \\ \downarrow{\scriptstyle \text{epi}} & & \downarrow{\scriptstyle \text{epi}} \\ L_0(\pi) & \xrightarrow{i_!} & L_0(G^+) \end{array}$$

in which the vertical maps belong to the relative exact sequences. Lemma 2 implies that the upper horizontal map is a monomorphism. Here the free part is taken to the free part with cokernel $\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{2^{r-1}-r}$. Since the vertical maps are epimorphisms and the lower groups are torsion-free, the lower horizontal map has the same cokernel. Thus we have obtained a description of the map $i_!$ in dimension zero. The map $i^!t^{-1}$ is an epimorphism since the homology in the term LN_0 is trivial. This follows from the fact that the map $i_!\colon L_3(\pi) \to L_3(G^+)$ from the other row is monomorphic. Case 2) is similar. The theorem is proved. □

Remark. The results of Theorem 4 with the help of diagram chasing allow to compute almost all maps and homology groups in the two-row diagrams (D) and $(\widetilde{D})$ for the inclusions $\mathbb{Z}/2^r \to D_{r+1}$ studied in this paper. However, at this point the authors do not know the homology in the term $L_0(G^+)$ of diagram (D) and the homology in the term $L_2(G^-)$ of diagram $(\widetilde{D})$. To answer this question, it would suffice to know the maps in the exact sequence relating the groups LN_0 and LN_1.

§4. Realizability by closed manifolds

According to the results of [6] (also see [2, 16]), elements of the group $L_n(G^+)$ not lying in the kernel of the map $i^!t^{-1}$ from diagram (5) cannot be realized by maps of closed manifolds. Thus $i^!t^{-1}(x)$ for $x \in L_n(G)$ is the first obstruction to realizing the element x by normal maps of closed manifolds. Suppose that for an element x the first obstruction is trivial, i.e., $i^!t^{-1}(x) = 0$. Using the homology homomorphism Γ from diagram (5), we can specify the coset $\Gamma(x) \subset L_{n-1}(G^-)$, now containing the map $i^!t^{-1}$ from diagram (6). According to [2], this coset is the second obstruction to realizability; if $0 \notin \Gamma(x)$, then the element x cannot be realized by a normal map of closed manifolds. These two invariants suffice for the study of the realizability problem in its projective version (see [2]). The process may be continued further, yielding iterated Browder–Livesay invariants (see [16]). But for dihedral 2-groups in our case this is not necessary. Recall that $D_{r+1} = G$ is a group with trivial orientation homomorphism, while $D_{r+1}^- = G^-$ is a group with the orientation homomorphism w for which $w(x) = 1$, $w(y) = -1$.

Theorem 5. *We have the following isomorphisms:*

$$C_n(D_{r+1}) = \begin{cases} 0 & \text{for } n = 1 \bmod 4, \\ \mathbb{Z}/2 & \text{for } n = 2 \bmod 4, \\ (\mathbb{Z}/2)^2 & \text{for } n = 3 \bmod 4, \end{cases} \quad C_n(D_{r+1}^-) = \begin{cases} \mathbb{Z}/2 & \text{for } n = 0 \bmod 4, \\ 0 & \text{for } n = 1, 3 \bmod 4. \end{cases}$$

Proof. All the required maps were computed in Theorem 4. First consider the case of nontrivial orientation. The maps $i^!t^{-1}$ in the diagrams (6) are monomorphisms in odd dimensions. Therefore, no elements of the group $L_{2n+1}(G^-)$ can be realized by normal maps of closed manifolds. The group $L_0(G^-)$ is isomorphic to $\mathbb{Z}/2$ and, according to [2], the image of $C_0(D_{r+1}^-)$ in the projective group L^p is $\mathbb{Z}/2$. This implies the assertion of the theorem in the case of nontrivial orientation.

In the trivial orientation case, the map $i^!t^{-1}$ in diagram (5) is monomorphic in dimension 1, and in dimension 3 has the kernel $(\mathbb{Z}/2)^2$. According to [2] the image of $C_3(D_{r+1})$ in the projective group is $(\mathbb{Z}/2)^2$. Thus in dimension 3 the upper and lower estimates of the group $C_3(D_{r+1})$ coincide, and hence $C_3(D_{r+1}) \cong (\mathbb{Z}/2)^2$. In dimension 2 we can repeat the arguments used in dimension 0 for the nonoriented case. The theorem is proved. □

In dimension zero in the oriented case diagrams (5) and (6) also yield the upper estimate $\mathbb{Z}^{r+1}$ for the group $C_0(D_{r+1})$, but this estimate considerably exceeds the lower estimate $\mathbb{Z}$ from [2]. For dimension 2, the situation in the nonorientable case is similar.

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