DE GRUYTER Forum Math. 2016; 28 (6):1095–1110

Research Article

Giovanni Molica Bisci*, Dušan Repovš and Raffaella Servadei

Nontrivial solutions of superlinear nonlocal problems

DOI: 10.1515/forum-2015-0204
Received October 13, 2015; revised January 4, 2016

Abstract: We study the question of the existence of infinitely many weak solutions for nonlocal equations of fractional Laplacian type with homogeneous Dirichlet boundary data, in presence of a superlinear term. Starting from the well-known Ambrosetti–Rabinowitz condition, we consider different growth assumptions on the nonlinearity, all of superlinear type. We obtain three different existence results in this setting by using the Fountain Theorem, which extend some classical results for semilinear Laplacian equations to the nonlocal fractional setting.

Keywords: Fractional Laplacian, nonlocal problems, variational methods, Fountain theorem, integrodifferential operators, superlinear nonlinearities

MSC 2010: Primary 49J35, 35A15, 35S15; secondary 47G20, 45G05

Communicated by: Frank Duzaar

1 Introduction and main results

Recently, nonlocal fractional problems have been appearing in the literature in many different contexts, both in the pure mathematical research and in concrete real-world applications. Indeed, fractional and nonlocal operators appear in many diverse fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves.

In this paper we are interested in the existence of infinitely many solutions of the following problem:

$$\begin{cases} -\mathcal{L}_K u - \lambda u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

Here Ω is an open bounded subset of $\mathbb{R}^n$ with continuous boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, the term f satisfies different superlinear conditions, and $\mathcal{L}_K$ is the integrodifferential operator defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is such that

$$mK \in L^1(\mathbb{R}^n), \quad \text{where } m(x) = \min\{|x|^2, 1\}, \tag{1.3}$$

***Corresponding author: Giovanni Molica Bisci:** Dipartimento PAU, Università 'Mediterranea' di Reggio Calabria, Via Melissari 24, 89124 Reggio Calabria, Italy, e-mail: gmolica@unirc.it
Dušan Repovš: Faculty of Education, and Faculty of Mathematics and Physics, University of Ljubljana, POB 2964, 1001 Ljubljana, Slovenia, e-mail: dusan.repovs@guest.arnes.si
Raffaella Servadei: Dipartimento di Scienze Pure e Applicate, Università degli Studi di Urbino 'Carlo Bo', Piazza della Repubblica 13, 61029 Urbino (Pesaro e Urbino), Italy, e-mail: affaella.servadei@uniurb.it

and

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

A model for K is given by the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$, defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n.$$

Under superlinear and subcritical conditions on f, the authors proved in [25, 28] the existence of a nontrivial solution of (1.1) for any $\lambda \in \mathbb{R}$, as an application of the Mountain Pass Theorem and the Linking Theorem (see [3, 21]). Motivated by these existence results, in this paper we shall study the existence of infinitely many solutions of (1.1), using the Fountain Theorem due to Bartsch (see [4]).

1.1 Variational formulation of the problem

In order to study problem (1.1), we shall consider its weak formulation, given by

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y)\, dx\, dy - \lambda \int_\Omega u(x)\varphi(x)\, dx = \int_\Omega f(x, u(x))\varphi(x)\, dx, & \varphi \in X_0, \\ u \in X_0, \end{cases} \tag{1.5}$$

which represents the Euler–Lagrange equation of the energy functional $\mathcal{J}_{K,\lambda} : X_0 \to \mathbb{R}$ defined as

$$\mathcal{J}_{K,\lambda}(u) := \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y)\, dx\, dy - \frac{\lambda}{2} \int_\Omega |u(x)|^2\, dx - \int_\Omega F(x, u(x))\, dx, \tag{1.6}$$

where the function F is the primitive of f with respect to the second variable, that is,

$$F(x,t) = \int_0^t f(x,\tau) d\tau. \tag{1.7}$$

Here, the space X_0 is defined as

$$X_0 := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

where the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction of any function g in X to Ω belongs to $L^2(\Omega)$ and the map

$$(x,y) \mapsto (g(x) - g(y))\sqrt{K(x-y)}$$

is in $L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy)$, with $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$.

1.2 The main results of the paper

Throughout this paper we shall assume different superlinear conditions on the term f. First of all, we suppose that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the following standard conditions:

$$f \in C(\overline{\Omega} \times \mathbb{R}), \tag{1.8}$$

$$\text{there exist } a_1, a_2 > 0 \text{ and } q \in (2, 2^*),\ 2^* = 2n/(n-2s), \text{ such that}$$
$$|f(x,t)| \leqslant a_1 + a_2 |t|^{q-1} \text{ for any } x \in \Omega,\ t \in \mathbb{R}, \tag{1.9}$$

$$\text{there exist } \mu > 2 \text{ and } r > 0 \text{ such that for any } x \in \Omega,\ t \in \mathbb{R},\ |t| \geqslant r,$$
$$0 < \mu F(x,t) \leqslant t f(x,t), \tag{1.10}$$

where F is the function from (1.7).

When looking for infinitely many solutions, it is natural to require some symmetry of the nonlinearity. Here, we assume the following condition:

$$f(x,-t) = -f(x,t) \quad \text{for any } x \in \Omega,\ t \in \mathbb{R}. \tag{1.11}$$

As a model for f we can take the function $f(x,t) = a(x)|t|^{q-2}t$, with $a \in C(\overline{\Omega})$ and $q \in (2, 2^*)$.

The first result of this paper is in the following theorem:

Theorem 1. *Let $s \in (0,1)$, $n > 2s$, and let Ω be an open bounded subset of $\mathbb{R}^n$ with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying (1.3) and (1.4) and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions (1.8)–(1.11). Then for any $\lambda \in \mathbb{R}$ problem (1.1) has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_{K,\lambda}(u_j) \to +\infty$ as $j \to +\infty$.*

Assumption (1.10) is the well-known *Ambrosetti–Rabinowitz condition*, originally introduced in [3]. This condition is often considered when dealing with superlinear elliptic boundary value problems (see, for instance, [30, 31] and the references therein). Its importance is due to the fact that it assures the boundedness of the Palais–Smale sequences for the energy functional associated with the problem under consideration.

The Ambrosetti–Rabinowitz condition is a superlinear growth assumption on the nonlinearity f. Indeed, integrating (1.10) we get that

$$\text{there exist } a_3, a_4 > 0 \text{ such that } F(x,t) \geqslant a_3|t|^{\mu} - a_4 \text{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R}; \tag{1.12}$$

see, for instance, [28, Lemma 4] for a detailed proof. As a consequence of (1.12) and the fact that $\mu > 2$, we have that

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^2} = +\infty \text{ uniformly for any } x \in \overline{\Omega}, \tag{1.13}$$

which is another superlinear assumption on f at infinity. A simple computation proves that the function

$$f(x,t) = t \log(1+|t|) \tag{1.14}$$

satisfies condition (1.13), but not (1.12) (and so, as a consequence, does not satisfy (1.10)).

Recently, many superlinear problems without the Ambrosetti–Rabinowitz condition have been considered in the literature (see, for instance, [9, 14, 16, 20, 22, 32, 33] and references therein). In particular, in [2, 13, 17–19] the local analogue of problem (1.1) (that is, problem (1.1) with $\mathcal{L}_K$ replaced by $-(-\Delta)$) has been studied. In this framework Jeanjean introduced in [16] the following assumption on f:

$$\text{there exists } \gamma \geqslant 1 \text{ such that for any } x \in \Omega,\ \mathcal{F}(x,t') \leqslant \gamma\mathcal{F}(x,t) \text{ for any } t, t' \in \mathbb{R} \text{ with } 0 < t' \leqslant t, \tag{1.15}$$

where

$$\mathcal{F}(x,t) = \frac{1}{2}tf(x,t) - F(x,t). \tag{1.16}$$

Note that (1.15) is a global condition and that the function (1.14) satisfies (1.15).

In this setting our existence result becomes:

Theorem 2. *Let $s \in (0,1)$, $n > 2s$, and let Ω be an open bounded subset of $\mathbb{R}^n$ with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying (1.3) and (1.4) and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions (1.8), (1.9), (1.11), (1.13) and (1.15). Then for any $\lambda \in \mathbb{R}$ problem (1.1) has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_{K,\lambda}(u_j) \to +\infty$ as $j \to +\infty$.*

Another interesting condition used in the classical Laplace framework is the following one introduced by Liu in [18]:

$$\begin{aligned}&\text{there exists } \bar{t} > 0 \text{ such that for any } x \in \Omega,\\ &\text{the function } t \mapsto \tfrac{f(x,t)}{t} \text{ is increasing if } t \geqslant \bar{t} \text{ and decreasing if } t \leqslant -\bar{t}.\end{aligned} \tag{1.17}$$

Under this assumption, our main result reads as follows.

Theorem 3. *Let $s \in (0,1)$, $n > 2s$, and let Ω be an open bounded subset of $\mathbb{R}^n$ with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying* (1.3) *and* (1.4) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (1.8), (1.9), (1.11), (1.13) *and* (1.17). *Then for any $\lambda \in \mathbb{R}$ problem* (1.1) *has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_{K,\lambda}(u_j) \to +\infty$ as $j \to +\infty$.*

We remark that, due to the symmetry assumption (1.11), if u is a weak solution of problem (1.1), then so is $-u$. Hence, our results give the existence of infinitely many pairs $\{u_j, -u_j\}_{j\in\mathbb{N}}$ of weak solutions of (1.1).

The proofs of Theorems 1, 2 and 3 rely on the same arguments as used in [5]. In certain steps of the proofs we only need some careful estimates of the term $\lambda\|u\|^2_{L^2(\Omega)}$. More precisely, the strategy of our proofs consists in looking for infinitely many critical points for the energy functional associated with problem (1.1), namely here we apply the Fountain Theorem proved by Bartsch in [4]. For this purpose, we have to analyze the compactness properties of the functional and its geometric features. As for the compactness, when the nonlinearity satisfies the Ambrosetti–Rabinowitz assumption (1.10), we shall prove that the Palais–Smale condition is satisfied; when f is assumed to satisfy conditions (1.13) and (1.15) or (1.17), the Cerami condition will be considered. In both cases the main difficulty is related to the proof of the boundedness of the Palais–Smale (or Cerami) sequence.

The geometry of the functional required by the Fountain Theorem consists in proving that the functional $\mathcal{J}_{K,\lambda}$ is negative in a ball in a suitable finite-dimensional subspace of X_0 and positive in a ball in an infinite-dimensional subspace.

Theorem 1, up to a normalizing constant in front of the integral definition of the operator $\mathcal{L}_K$, is the nonlocal analogue of [31, Corollary 3.9], where the limit case as $s \to 1$ (that is, the Laplace case) was considered.

Some of the results presented here could be also achieved for a larger class of nonlocal equations where the leading term is given by more some nonlinear integrodifferential operators; that is, the ones obtained by replacing the fractional Laplacian in (1.1) with the operator

$$\mathcal{F}(u)(x) = \text{P.V.} \int_{\mathbb{R}^n} K_{\text{sym}}(x,y)|u(x) - u(y)|^{p-2}(u(x) - u(y))\,dy,$$

where P.V. being a commonly used abbreviation for "in the principal value sense", K_{sym} is a symmetric kernel of differentiability order $s \in (0,1)$ and $p > 1$, with general possibly nonsmooth coefficients, as considered for instance in the very nice papers [10, 11]. However, some different technical approaches here need to be adopted in order to arrive to the analogous desired existence results for this wider class of energies. We will consider this interesting case via some further investigations.

We also point out that, in [23] the existence of infinitely many solutions of problem (1.1) was proved under assumptions on f which were different from the ones considered here and only for the case when $q \in (2, 2^* - 2s/(n-2s))$ in (1.9), but in presence of a perturbation $h \in L^2(\Omega)$.

This paper is organized as follows. In Section 2 we shall recall some preliminary notions and results. In Section 3 we shall discuss problem (1.1) under the Ambrosetti–Rabinowitz condition and we shall prove Theorem 1. Section 4 will be devoted to problem (1.1) without the Ambrosetti–Rabinowitz condition and the proof of Theorem 2 and of Theorem 3 will be provided.

2 Preliminaries

This section is devoted to some preliminary results. First of all, the functional space X_0 we shall work in is endowed with the norm

$$X_0 \ni g \mapsto \|g\|_{X_0} := \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x) - g(y)|^2 K(x-y)\,dx\,dy \right)^{1/2} \tag{2.1}$$

and $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (for this see [25, Lemma 7]), with the scalar product

$$\langle u, v \rangle_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(v(x) - v(y))\,K(x-y)\,dx\,dy. \tag{2.2}$$

The usual fractional Sobolev space $H^s(\Omega)$ is endowed with the so-called *Gagliardo norm* (see, for instance [1, 12]) given by

$$\|g\|_{H^s(\Omega)} := \|g\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}}\, dx\, dy \right)^{1/2}. \tag{2.3}$$

Note that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.3) are not the same: this makes the space X_0 not equivalent to the usual fractional Sobolev spaces and the classical fractional Sobolev space approach is not sufficient for studying our problem from a variational point of view.

We recall that by [27, Lemma 5.1] the space X_0 is nonempty, since $C_0^2(\Omega) \subseteq X_0$ and that for a general kernel K satisfying conditions (1.3) and (1.4), the following inclusion holds:

$$X_0 \subseteq \{g \in H^s(\mathbb{R}^n) : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

while in the model case $K(x) = |x|^{-(n+2s)}$, the following characterization is valid:

$$X_0 = \{g \in H^s(\mathbb{R}^n) : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

For further details on X and X_0 we refer to [25, 27–29], where various properties of these spaces were proved; for more details on the fractional Sobolev spaces H^s we refer to [12] and the references therein.

Finally, we recall that the eigenvalue problem driven by $-\mathcal{L}_K$, namely

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.4}$$

possesses a divergent sequence of positive eigenvalues

$$\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \cdots,$$

whose corresponding eigenfunctions will be denoted by e_k. By [28, Proposition 9], we know that $\{e_k\}_{k\in\mathbb{N}}$ can be chosen in such a way that this sequence provides an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in X_0. Further properties of the spectrum of the operator $-\mathcal{L}_K$ can be found in [24, Proposition 2.3], [28, Proposition 9 and Appendix A] and [26, Proposition 4]. See also the recent paper by Franzina and Palatucci [15, Theorem 4.2].

2.1 The Fountain Theorem

In order to prove our main results, stated in Theorem 1, Theorem 2 and Theorem 3, we shall apply the Fountain Theorem due to Bartsch (see [4]), which, under suitable compactness and geometric assumptions on a functional, provides the existence of an unbounded sequence of critical value for it.

The compactness condition assumed in the Fountain Theorem is the well-known *Palais–Smale condition* (see, for instance, [30, 31]), which in our framework reads as follows:

Palais–Smale Condition. The functional $\mathcal{J}_{K,\lambda}$ satisfies the *Palais–Smale compactness condition* at level $c \in \mathbb{R}$ if any sequence $\{u_j\}_{j\in\mathbb{N}}$ in X_0 such that $\mathcal{J}_{K,\lambda}(u_j) \to c$ and $\sup\{|\langle \mathcal{J}'_{K,\lambda}(u_j), \varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\} \to 0$ as $j \to +\infty$, admits a strongly convergent subsequence in X_0.

In [7, 8] Cerami introduced the so-called *Cerami condition*, as a weak version of the Palais–Smale assumption. With our notation, it can be written as follows:

Cerami Condition. The functional $\mathcal{J}_{K,\lambda}$ satisfies the *Cerami compactness condition* at level $c \in \mathbb{R}$ if any sequence $\{u_j\}_{j\in\mathbb{N}}$ in X_0 such that $\mathcal{J}_{K,\lambda}(u_j) \to c$ and $(1 + \|u_j\|) \sup\{|\langle \mathcal{J}'_{K,\lambda}(u_j), \varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\} \to 0$ as $j \to +\infty$, admits a strongly convergent subsequence in X_0.

When the right-hand side f of problem (1.1) satisfies the Ambrosetti–Rabinowitz condition, we shall prove in the following that the corresponding energy functional $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale compactness assumption, while, when we remove the Ambrosetti–Rabinowitz condition (1.10) and we replace it with assumptions (1.13) and (1.15) or (1.17), we shall show that $\mathcal{J}_{K,\lambda}$ satisfies the Cerami condition.

Following the notation in [4, Theorem 2.5] (see also [31]), for any $k \in \mathbb{N}$ we put

$$Y_k := \operatorname{span}\{e_1, \dots, e_k\} \quad \text{and} \quad Z_k := \overline{\operatorname{span}\{e_k, e_{k+1}, \dots\}}.$$

Note that, since Y_k is finite-dimensional, all norms on Y_k are equivalent and this will be used in the rest of the paper. Thanks to these notations, the geometric assumptions of the Fountain Theorem in our framework read as follows:

(i) $a_k := \max\{\mathcal{J}_{K,\lambda}(u) : u \in Y_k,\ \|u\|_{X_0} = r_k\} \leqslant 0$,

(ii) $b_k := \inf\{\mathcal{J}_{K,\lambda}(u) : u \in Z_k,\ \|u\|_{X_0} = \gamma_k\} \to \infty$ as $k \to \infty$.

3 Nonlinearities satisfying the Ambrosetti–Rabinowitz condition

This section is devoted to problem (1.1) in presence of a nonlinear term satisfying condition (1.10). In this framework we shall prove the following result about the compactness of the functional $\mathcal{J}_{K,\lambda}$:

Proposition 4. *Let $\lambda \in \mathbb{R}$ and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (1.8)–(1.10). *Then $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in X_0 such that as $j \to +\infty$,

$$\mathcal{J}_{K,\lambda}(u_j) \to c, \tag{3.1}$$

$$\sup\{|\langle \mathcal{J}'_{K,\lambda}(u_j), \varphi \rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\} \to 0. \tag{3.2}$$

We split the proof into two steps. First, we show that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0 and then that it admits a strongly convergent subsequence in X_0. In showing the boundedness of the sequence $\{u_j\}_{j\in\mathbb{N}}$ we have to treat separately the case when the parameter $\lambda \leqslant 0$ and $\lambda > 0$.

Step 1. *The sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.*

For any $j \in \mathbb{N}$ it easily follows by (3.1) and (3.2) that there exists $\kappa > 0$ such that

$$\left|\left\langle \mathcal{J}'_{K,\lambda}(u_j), \frac{u_j}{\|u_j\|_{X_0}} \right\rangle\right| \leqslant \kappa \quad \text{and} \quad |\mathcal{J}_{K,\lambda}(u_j)| \leqslant \kappa,$$

so that

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\mu}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j \rangle \leqslant \kappa(1 + \|u_j\|_{X_0}), \tag{3.3}$$

where μ is the parameter given by (1.10).

By invoking (1.9) and integrating it is easily seen that for any $x \in \overline{\Omega}$ and for any $t \in \mathbb{R}$,

$$|F(x,t)| \leqslant a_1 |t| + \frac{a_2}{q} |t|^q. \tag{3.4}$$

Hence, by (3.4) and again by (1.9), we have that for any $j \in \mathbb{N}$,

$$\left| \int_{\Omega \cap \{|u_j| \leqslant r\}} \left(F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x)) u_j(x) \right) dx \right| \leqslant \left(a_1 r + \frac{a_2}{q} r^q + \frac{a_1}{\mu} r + \frac{a_2}{\mu} r^q \right) |\Omega| =: \tilde{\kappa}. \tag{3.5}$$

Now, assume that $\lambda \leqslant 0$. Then, thanks to (1.10) and (3.5), we get that for any $j \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\mu}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j \rangle &= \left(\frac{1}{2} - \frac{1}{\mu}\right)(\|u_j\|^2_{X_0} - \lambda \|u_j\|^2_{L^2(\Omega)}) - \int_\Omega \left(F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x)) u_j(x) \right) dx \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{X_0} - \int_\Omega \left(F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x)) u_j(x) \right) dx \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{X_0} - \int_{\Omega \cap \{|u_j| \leqslant r\}} \left(F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x)) u_j(x) \right) dx \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_j\|^2_{X_0} - \tilde{\kappa}.
\end{aligned} \tag{3.6}$$

By (3.3), (3.6) and the fact that $\mu > 2$ we have that

$$\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_j\|^2_{X_0} \leqslant \kappa(1+\|u_j\|_{X_0})+\tilde{\kappa}$$

for any $j \in \mathbb{N}$, that is, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

Now, let us consider the case when $\lambda > 0$: the argument is the same as above, even though a more careful analysis is required. For reader's convenience, we prefer to give all the details.

First of all, let us fix $\sigma \in (2,\mu)$, where $\mu > 2$ is given in assumption (1.10). Arguing as above we get that for any $j \in \mathbb{N}$,

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \leqslant \kappa(1+\|u_j\|_{X_0}) \tag{3.7}$$

and

$$\Bigg|\int\limits_{\Omega\cap\{|u_j|\leqslant r\}} \Big(F(x,u_j(x)) - \frac{1}{\sigma}f(x,u_j(x))u_j(x)\Big)\,dx\Bigg| \leqslant \tilde{\kappa}, \tag{3.8}$$

for suitable positive κ and $\tilde{\kappa}$. Then, using (1.10), (1.12) and (3.8), we have that for any $j \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle &= \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}) - \int\limits_{\Omega}\Big(F(x,u_j(x)) - \frac{1}{\sigma}f(x,u_j(x))\,u_j(x)\Big)\,dx \\
&\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}) + \Big(\frac{\mu}{\sigma}-1\Big)\int\limits_{\Omega\cap\{|u_j|\geqslant r\}} F(x,u_j(x))\,dx \\
&\quad - \int\limits_{\Omega\cap\{|u_j|\leqslant r\}}\Big(F(x,u_j(x)) - \frac{1}{\sigma}f(x,u_j(x))\,u_j(x)\Big)\,dx \\
&\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}) + \Big(\frac{\mu}{\sigma}-1\Big)\int\limits_{\Omega\cap\{|u_j|\geqslant r\}} F(x,u_j(x))\,dx - \tilde{\kappa} \\
&\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}) + a_3\Big(\frac{\mu}{\sigma}-1\Big)\|u_j\|^{\mu}_{L^{\mu}(\Omega)} - a_4\Big(1-\frac{\mu}{\sigma}\Big)|\Omega| - \tilde{\kappa}.
\end{aligned} \tag{3.9}$$

Furthermore, for any $\varepsilon > 0$ the Young inequality (with conjugate exponents $\mu/2 > 1$ and $\mu/(\mu-2)$) yields

$$\|u_j\|^2_{L^2(\Omega)} \leqslant \frac{2\varepsilon}{\mu}\,\|u_j\|^{\mu}_{L^{\mu}(\Omega)} + \frac{\mu-2}{\mu}\,\varepsilon^{-2/(\mu-2)}\,|\Omega|, \tag{3.10}$$

so that, by (3.9) and (3.10), we can deduce that for any $j \in \mathbb{N}$

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle &\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\|u_j\|^2_{X_0} - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{2\varepsilon}{\mu}\|u_j\|^{\mu}_{L^{\mu}(\Omega)} - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{\mu-2}{\mu}\varepsilon^{-2/(\mu-2)}|\Omega| \\
&\quad + a_3\Big(\frac{\mu}{\sigma}-1\Big)\|u_j\|^{\mu}_{L^{\mu}(\Omega)} - a_4\Big(1-\frac{\mu}{\sigma}\Big)|\Omega| - \tilde{\kappa} \\
&= \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\|u_j\|^2_{X_0} + \Big[a_3\Big(\frac{\mu}{\sigma}-1\Big) - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{2\varepsilon}{\mu}\Big]\|u_j\|^{\mu}_{L^{\mu}(\Omega)} - C_\varepsilon,
\end{aligned} \tag{3.11}$$

where C_ε is a constant such that $C_\varepsilon \to +\infty$ as $\varepsilon \to 0$, due to $\mu > \sigma > 2$.

Now, choosing ε so small that

$$a_3\Big(\frac{\mu}{\sigma}-1\Big) - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{2\varepsilon}{\mu} > 0,$$

by (3.11), we get for any $j \in \mathbb{N}$

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\|u_j\|^2_{X_0} - C_\varepsilon. \tag{3.12}$$

Combining (3.7) and (3.12), we deduce that for any $j \in \mathbb{N}$

$$\|u_j\|^2_{X_0} \leqslant \kappa_*(1+\|u_j\|_{X_0})$$

for a suitable positive constant κ_*. This proves that the Palais–Smale sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

Hence Step 1 is proved.

Step 2. *Up to a subsequence, $\{u_j\}_{j\in\mathbb{N}}$ strongly converges in X_0.*

Since $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0 by Step 1 and X_0 is a reflexive space (being a Hilbert space, by [25, Lemma 7]), up to a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, there exists $u_\infty \in X_0$ such that

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} &(u_j(x)-u_j(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy\\ &\to \int_{\mathbb{R}^n\times\mathbb{R}^n} (u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy \quad \text{for any } \varphi \in X_0 \end{aligned}\tag{3.13}$$

as $j \to +\infty$. Moreover, by [29, Lemma 9], up to a subsequence,

$$\begin{aligned} u_j &\to u_\infty \quad \text{in } L^2(\mathbb{R}^n),\\ u_j &\to u_\infty \quad \text{in } L^q(\mathbb{R}^n),\\ u_j &\to u_\infty \quad \text{a.e. in } \mathbb{R}^n,\end{aligned}\tag{3.14}$$

as $j \to +\infty$ and there exists $\ell \in L^q(\mathbb{R}^n)$ such that

$$|u_j(x)| \leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N}\tag{3.15}$$

(see, for instance, [6, Theorem IV.9]).

By (1.9), (3.13)–(3.15), the fact that the map $t \mapsto f(\cdot\,,t)$ is continuous in $t \in \mathbb{R}$ and the Dominated Convergence Theorem we get

$$\int_\Omega f(x,u_j(x))u_j(x)\,dx \to \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx\tag{3.16}$$

and

$$\int_\Omega f(x,u_j(x))u_\infty(x)\,dx \to \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx\tag{3.17}$$

as $j \to +\infty$. Moreover, by (3.2) and Step 1 we have that

$$0 \leftarrow \langle \mathcal{J}'_{K,\lambda}(u_j),u_j\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy - \lambda\int_\Omega |u_j(x)|^2\,dx - \int_\Omega f(x,u_j(x))u_j(x)\,dx,$$

so that, by (3.14) and (3.16), we can deduce that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy \to \lambda\int_\Omega |u_\infty(x)|^2\,dx + \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx\tag{3.18}$$

as $j \to +\infty$. Furthermore, again by (3.2), we get

$$\begin{aligned}0 \leftarrow \langle \mathcal{J}'_{K,\lambda}(u_j),u_\infty\rangle = &\int_{\mathbb{R}^n\times\mathbb{R}^n} (u_j(x)-u_j(y))(u_\infty(x)-u_\infty(y))K(x-y)\,dx\,dy\\ &-\lambda\int_\Omega u_j(x)u_\infty(x)\,dx - \int_\Omega f(x,u_j(x))u_\infty(x)\,dx\end{aligned}\tag{3.19}$$

as $j \to +\infty$. By (3.13) with $\varphi = u_\infty$, (3.14), (3.17) and (3.19) we obtain

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy = \lambda\int_\Omega |u_\infty(x)|^2\,dx + \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx.\tag{3.20}$$

Thus (3.18) and (3.20) give us that

$$\|u_j\|_{X_0} \to \|u_\infty\|_{X_0}\tag{3.21}$$

as $j \to \infty$.

Finally, it is easy to see that

$$\|u_j - u_\infty\|_{X_0}^2 = \|u_j\|_{X_0}^2 + \|u_\infty\|_{X_0}^2 - 2 \int_{\mathbb{R}^n\times\mathbb{R}^n} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y))K(x-y)\,dx\,dy$$
$$\to 2\|u_\infty\|_{X_0}^2 - 2 \int_{\mathbb{R}^n\times\mathbb{R}^n} |u_\infty(x) - u_\infty(y)|^2 K(x-y)\,dx\,dy = 0$$

as $j \to +\infty$, thanks to (3.13) and (3.21). Therefore, the assertion of Step 2 is proved.

This concludes the proof of Proposition 4. □

Now, we are ready for proving Theorem 1.

3.1 Proof of Theorem 1

The idea consists in applying the Fountain Theorem. By Proposition 4 we have that $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale condition, while by (1.11), we get that $\mathcal{J}_{K,\lambda}(-u) = \mathcal{J}_{K,\lambda}(u)$ for any $u \in X_0$. Then, it remains to study the geometry of the functional $\mathcal{J}_{K,\lambda}$. For this purpose, we proceed by the following steps.

Step 1. *For any $k \in \mathbb{N}$ there exists $r_k > 0$ such that*

$$a_k := \max\{\mathcal{J}_{K,\lambda}(u) : u \in Y_k,\ \|u\|_{X_0} = r_k\} \leqslant 0.$$

By (1.12), we get that for any $u \in Y_k$,

$$\mathcal{J}_{K,\lambda}(u) \leqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{2}\|u\|_{L^2(\Omega)}^2 - a_3\|u\|_{L^\mu(\Omega)}^\mu + a_4|\Omega| \leqslant \frac{C_{k,\lambda}}{2}\|u\|_{X_0}^2 - \hat{C}_{k,\mu}\|u\|_{X_0}^\mu + a_4|\Omega| \tag{3.22}$$

for suitable positive constants $C_{k,\lambda}$, depending on k and λ, and $\hat{C}_{k,\mu}$, depending on k and μ. Here we used the fact that all the norms are equivalent in Y_k .

As a consequence of (3.22), we get that for any $u \in Y_k$ with $\|u\|_{X_0} = r_k$,

$$\mathcal{J}_{K,\lambda}(u) \leqslant 0,$$

provided $r_k > 0$ is large enough, due to the fact that $\mu > 2$. Thus Step 1 is proved.

Step 2. *Let $1 \leqslant q < 2^*$ and, for any $k \in \mathbb{N}$, let*

$$\beta_k := \sup\{\|u\|_{L^q(\Omega)} : u \in Z_k,\ \|u\|_{X_0} = 1\}.$$

Then $\beta_k \to 0$ as $k \to \infty$.

By the definition of Z_k, we have that $Z_{k+1} \subset Z_k$ and so, as a consequence, $0 < \beta_{k+1} \leqslant \beta_k$ for any $k \in \mathbb{N}$. Hence

$$\beta_k \to \beta \tag{3.23}$$

as $k \to +\infty$, for some $\beta \geqslant 0$. Moreover, by the definition of β_k, for any $k \in \mathbb{N}$ there exists $u_k \in Z_k$ such that

$$\|u_k\|_{X_0} = 1 \quad \text{and} \quad \|u_k\|_{L^q(\Omega)} > \frac{\beta_k}{2}. \tag{3.24}$$

Since X_0 is a Hilbert space, and hence a reflexive Banach space, there exist $u_\infty \in X_0$ and a subsequence of u_k (still denoted by u_k) such that $u_k \to u_\infty$ weakly converges in X_0, that is,

$$\langle u_k, \varphi\rangle_{X_0} \to \langle u_\infty, \varphi\rangle_{X_0} \quad \text{for any } \varphi \in X_0$$

as $k \to +\infty$. Since $\varphi = \sum_{j=1}^{+\infty} c_j e_j$, it follows that

$$\langle u_\infty, \varphi\rangle_{X_0} = \lim_{k\to+\infty} \langle u_k, \varphi\rangle_{X_0} = \lim_{k\to+\infty} \sum_{j=1}^{+\infty} c_j \langle u_k, e_j\rangle_{X_0} = 0,$$

thanks to the fact that the sequence $\{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions of $-\mathcal{L}_K$ is an orthogonal basis of X_0. Therefore we can deduce that $u_\infty \equiv 0$. Hence by the Sobolev Embedding Theorem (see [29, Lemma 9]), we get

$$u_k \to 0 \quad \text{in } L^q(\Omega) \tag{3.25}$$

as $k \to +\infty$. By (3.23), the fact that β is nonnegative, and by (3.24) and (3.25), we get that $\beta_k \to 0$ as $k \to +\infty$ and this concludes the proof of Step 2.

Step 3. *There exists $\gamma_k > 0$ such that*

$$b_k := \inf\{\mathcal{J}_{K,\lambda}(u) : u \in Z_k,\ \|u\|_{X_0} = \gamma_k\} \to +\infty$$

as $k \to +\infty$.

By invoking (1.9) and integrating, it is easy to see that (3.4) holds, and so, as a consequence, we get that there exists a constant $C > 0$ such that

$$|F(x,t)| \leqslant C(1 + |t|^q) \tag{3.26}$$

for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$. Then we obtain by (3.26) for any $u \in Z_k \setminus \{0\}$

$$\begin{aligned}
\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{2}\|u\|_{L^2(\Omega}^2 - C\|u\|_{L^q(\Omega)}^q - C|\Omega| \\
&\geqslant C_{k,\lambda}\|u\|_{X_0}^2 - C\left\|\frac{u}{\|u\|_{X_0}}\right\|_{L^q(\Omega)}^q \|u\|_{X_0}^q - C|\Omega| \\
&\geqslant C_{k,\lambda}\|u\|_{X_0}^2 - C\beta_k^q\|u\|_{X_0}^q - C|\Omega| \\
&= \|u\|_{X_0}^2\big(C_{k,\lambda} - C\beta_k^q\|u\|_{X_0}^{q-2}\big) - C|\Omega|,
\end{aligned} \tag{3.27}$$

where β_k is defined as in Step 2 and

$$C_{k,\lambda} = \begin{cases} \frac{1}{2} & \text{if } \lambda \leqslant 0, \\ \frac{1}{2}\big(1 - \frac{\lambda}{\lambda_1}\big) & \text{if } 0 < \lambda < \lambda_1, \\ \frac{1}{2}\big(1 - \frac{\lambda}{\lambda_k}\big) & \text{if } \lambda_k \leqslant \lambda < \lambda_{k+1}. \end{cases}$$

Defining γ_k as

$$\gamma_k = \left(\frac{2C_{k,\lambda}}{qC\beta_k^q}\right)^{1/(q-2)},$$

it is easy to see that $\gamma_k \to +\infty$ as $k \to +\infty$, thanks to Step 2, the fact that $q > 2$ and since $\{\lambda_k\}_{k\in\mathbb{N}}$ is a divergent sequence. As a consequence of this and by (3.27) we get that for any $u \in Z_k$ with $\|u\|_{X_0} = \gamma_k$,

$$\mathcal{J}_{K,\lambda}(u) \geqslant \|u\|_{X_0}^2\big(C_{k,\lambda} - C\beta_k^q\|u\|_{X_0}^{q-2}\big) - C|\Omega| = \left(1 - \frac{2}{q}\right)C_{k,\lambda}\gamma_k^2 - C|\Omega| \to +\infty$$

as $k \to +\infty$. Thus Step 3 is completed.

Hence all the geometric features of the Fountain Theorem are satisfied and the proof of Theorem 1 is complete.

We would like to emphasize that in the verification of the geometric structure of the functional $\mathcal{J}_{K,\lambda}$ the Ambrosetti–Rabinowitz condition (namely, (1.12)) was used only for proving Step 1.

4 Nonlinearities satisfying other superlinear conditions

In this section we shall deal with problem (1.1) when superlinear conditions on the term f different from the Ambrosetti–Rabinowitz are satisfied. In this framework we shall show that the functional $\mathcal{J}_{K,\lambda}$ satisfies the Cerami condition, as well as the geometric requirements of the Fountain Theorem.

4.1 Nonlinearities under the superlinear conditions (1.13) and (1.15)

First, we study the compactness properties of the functional $\mathcal{J}_{K,\lambda}$, as stated in the following result:

Proposition 5. *Let $\lambda \in \mathbb{R}$ and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (1.8), (1.9), (1.11), (1.13) *and* (1.15). *Then, $\mathcal{J}_{K,\lambda}$ satisfies the Cerami condition at any level $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a Cerami sequence in X_0, that is, let $\{u_j\}_{j\in\mathbb{N}}$ be such that

$$\mathcal{J}_{K,\lambda}(u_j) \to c \tag{4.1}$$

and

$$(1 + \|u_j\|) \sup\{|\langle \mathcal{J}'_{K,\lambda}(u_j), \varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\} \to 0 \tag{4.2}$$

as $j \to +\infty$.

First, we show that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0. For this purpose we argue as in the proof of [13, Lemma 2.2]. Suppose to the contrary that $\{u_j\}_{j\in\mathbb{N}}$ is unbounded in X_0, that is, suppose that, up to a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$,

$$\|u_j\|_{X_0} \to +\infty \tag{4.3}$$

as $j \to +\infty$.

By (4.2) and (4.3), it is easy to see that

$$\sup\{|\langle \mathcal{J}'_{K,\lambda}(u_j), \varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\} \to 0, \tag{4.4}$$

and so also

$$\sup\{|\langle \mathcal{J}'_{K,\lambda}(u_j), \varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\} \cdot \|u_j\|_{X_0} \to 0 \tag{4.5}$$

as $j \to +\infty$

Now, for any $j \in \mathbb{N}$, let

$$v_j = \frac{u_j}{\|u_j\|_{X_0}}. \tag{4.6}$$

Of course, the sequence $\{v_j\}_{j\in\mathbb{N}}$ is bounded in X_0 and so, by [29, Lemma 9], up to a subsequence, there exists $v_\infty \in X_0$ such that

$$\begin{aligned} v_j &\to v_\infty \quad \text{in } L^2(\mathbb{R}^n),\\ v_j &\to v_\infty \quad \text{in } L^q(\mathbb{R}^n),\\ v_j &\to v_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.7}$$

as $j \to +\infty$ and there exists $\ell \in L^q(\mathbb{R}^n)$ such that

$$|v_j(x)| \leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N} \tag{4.8}$$

(see [6, Theorem IV.9]). We shall separately consider the cases when $v_\infty \equiv 0$ and $v_\infty \not\equiv 0$ and we shall prove that in both cases a contradiction occurs.

Case 1. Suppose that

$$v_\infty \equiv 0. \tag{4.9}$$

As in [16], we can say that for any $j \in \mathbb{N}$ there exists $t_j \in [0, 1]$ such that

$$\mathcal{J}_{K,\lambda}(t_j u_j) = \max_{t\in[0,1]} \mathcal{J}_{K,\lambda}(t u_j). \tag{4.10}$$

Since (4.3) holds, for any $m \in \mathbb{N}$, we can choose $r_m = 2\sqrt{m}$ such that

$$r_m \|u_j\|_{X_0}^{-1} \in (0, 1), \tag{4.11}$$

provided j is large enough, say $j > \bar{\jmath}$, with $\bar{\jmath} = \bar{\jmath}(m)$.

By (4.7), (4.9) and the continuity of the function F, we get that

$$\int_\Omega |r_m v_j(x)|^2\, dx \to 0 \tag{4.12}$$

and

$$F(x, r_m v_j(x)) \to F(x, r_m v_\infty(x)) \quad \text{a.e. } x \in \Omega \tag{4.13}$$

as $j \to +\infty$ for any $m \in \mathbb{N}$. Moreover, integrating (1.9) and taking into account (4.8), we have that

$$|F(x, r_m v_j(x))| \leqslant a_1 |r_m v_j(x)| + \frac{a_2}{q} |r_m v_j(x)|^q \leqslant a_1 r_m \ell(x) + \frac{a_2}{q} (r_m \ell(x))^q \in L^1(\Omega), \tag{4.14}$$

a.e. $x \in \Omega$ and for any $m, j \in \mathbb{N}$. Hence (4.13), (4.14) and the Dominated Convergence Theorem yield that

$$F(\,\cdot\,, r_m v_j(\,\cdot\,)) \to F(\,\cdot\,, r_m v_\infty(\,\cdot\,)) \quad \text{in } L^1(\Omega) \tag{4.15}$$

as $j \to +\infty$ for any $m \in \mathbb{N}$. Since $F(x, 0) = 0$ for any $x \in \overline{\Omega}$ and (4.9) holds, (4.15) gives that

$$\int_\Omega F(x, r_m v_j(x))\, dx \to 0 \tag{4.16}$$

as $j \to +\infty$ for any $m \in \mathbb{N}$. Thus (4.10), (4.11), (4.12) and (4.16) yield

$$\begin{aligned} \mathcal{J}_{K,\lambda}(t_j u_j) \geqslant \mathcal{J}_{K,\lambda}(r_m \|u_j\|_{X_0}^{-1} u_j) = \mathcal{J}_{K,\lambda}(r_m v_j) &= \frac{1}{2}\|r_m v_j\|_{X_0}^2 - \frac{\lambda}{2}\int_\Omega |r_m v_j(x)|^2\, dx - \int_\Omega F(x, r_m v_j(x))\, dx \\ &= 2m - \frac{\lambda}{2}\int_\Omega |r_m v_j(x)|^2\, dx - \int_\Omega F(x, r_m v_j(x))\, dx \geqslant m, \end{aligned}$$

provided j is large enough and for any $m \in \mathbb{N}$. From this we deduce that

$$\mathcal{J}_{K,\lambda}(t_j u_j) \to +\infty \tag{4.17}$$

as $j \to +\infty$.

Now, note that $\mathcal{J}_{K,\lambda}(0) = 0$ and (4.1) holds. Combining these two facts and (4.17), it is easily seen that $t_j \in (0, 1)$ and so by (4.10), we get that

$$\frac{d}{dt}\Big|_{t=t_j} \mathcal{J}_{K,\lambda}(t u_j) = 0$$

for any $j \in \mathbb{N}$. As a consequence of this, we have that

$$\langle \mathcal{J}'_{K,\lambda}(t_j u_j), t_j u_j \rangle = t_j \frac{d}{dt}\Big|_{t=t_j} \mathcal{J}_{K,\lambda}(t u_j) = 0. \tag{4.18}$$

We claim that

$$\limsup_{j\to+\infty} \mathcal{J}_{K,\lambda}(t_j u_j) \leqslant \kappa, \tag{4.19}$$

for a suitable positive constant κ. Before proving this fact, we note that, as a consequence of the assumptions (1.11) and (1.15), the following condition is satisfied:

$$\text{there exists } \gamma \geqslant 1 \text{ such that for any } x \in \Omega,\ \mathcal{F}(x, t') \leqslant \gamma \mathcal{F}(x, t) \text{ for any } t, t' \in \mathbb{R} \text{ with } 0 < |t'| \leqslant |t|, \tag{4.20}$$

where $\mathcal{F}$ is the function given by (1.16).

Now, by invoking (4.18) and using (4.20), we get

$$\begin{aligned} \frac{1}{\gamma}\mathcal{J}_{K,\lambda}(t_j u_j) &= \frac{1}{\gamma}\Big(\mathcal{J}_{K,\lambda}(t_j u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(t_j u_j), t_j u_j\rangle\Big) \\ &= \frac{1}{\gamma}\Big(-\int_\Omega F(x, t_j u_j(x))\, dx + \frac{1}{2}\int_\Omega t_j u_j(x) f(x, t_j u_j(x))\, dx\Big) \\ &= \frac{1}{\gamma}\int_\Omega \mathcal{F}(x, t_j u_j(x))\, dx \\ &\leqslant \int_\Omega \mathcal{F}(x, u_j(x))\, dx \\ &= \int_\Omega \Big(\frac{1}{2} u_j(x) f(x, u_j(x)) - F(x, u_j(x))\Big)\, dx \\ &= \mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \to c \end{aligned}$$

as $j \to +\infty$, thanks to (4.1) and (4.5). This proves (4.19), which contradicts (4.17). Thus, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has to be bounded in X_0.

Case 2. Suppose that

$$v_\infty \not\equiv 0. \tag{4.21}$$

Then the set $\Omega' := \{x \in \Omega : v_\infty(x) \neq 0\}$ has positive Lebesgue measure and

$$|u_j(x)| \to +\infty \quad \text{a.e. } x \in \Omega' \tag{4.22}$$

as $j \to +\infty$, thanks to (4.3), (4.6), (4.7) and (4.21).

By (4.1) and (4.3) it is easy to see that

$$\frac{\mathcal{J}_{K,\lambda}(u_j)}{\|u_j\|_{X_0}^2} \to 0,$$

that is,

$$\frac{1}{2} - \frac{\lambda}{2}\int_\Omega \frac{|u_j(x)|^2}{\|u_j\|_{X_0}^2}\,dx - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx = o(1) \tag{4.23}$$

as $j \to +\infty$.

Now, observe that, by the variational characterization of the first eigenvalue λ_1 of $-\mathcal{L}_K$ (see [28, Proposition 9]), that is,

$$\lambda_1 = \min_{u\in X_0\setminus\{0\}} \frac{\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dx\,dy}{\int_\Omega |u(x)|^2\,dx},$$

we get that for any $u \in X_0$,

$$\|u\|_{L^2(\Omega)}^2 \leqslant \frac{1}{\lambda_1}\|u\|_{X_0}^2. \tag{4.24}$$

Hence, by (4.23) and (4.24), we can deduce that

$$\begin{aligned} o(1) &= \frac{1}{2} - \frac{\lambda}{2}\int_\Omega \frac{|u_j(x)|^2}{\|u_j\|_{X_0}^2}\,dx - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx \\ &\leqslant \frac{1}{2}\max\Big\{1, 1-\frac{\lambda}{\lambda_1}\Big\} - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx \end{aligned} \tag{4.25}$$

as $j \to +\infty$.

Let us consider separately the two integrals from formula (4.25). With respect to the first one, we have that

$$\frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2} = \frac{F(x,u_j(x))}{|u_j(x)|^2}\frac{|u_j(x)|^2}{\|u_j\|_{X_0}^2} = \frac{F(x,u_j(x))}{|u_j(x)|^2}|v_j(x)|^2 \to +\infty \quad \text{a.e. } x \in \Omega'$$

as $j \to +\infty$, thanks to (1.13), (4.7), (4.22) and the definition of Ω'. Hence, by using the Fatou Lemma, we obtain

$$\int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx \to +\infty \tag{4.26}$$

as $j \to +\infty$.

As for the second integral from (4.25), we claim that

$$\lim_{j\to+\infty} \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|_{X_0}^2}\,dx \geqslant 0 \tag{4.27}$$

(note that this limit exists thanks to (4.25) and (4.26)). Indeed by (1.13), it follows that

$$\lim_{|t|\to+\infty} F(x,t) = +\infty \quad \text{uniformly for any } x \in \overline{\Omega}. \tag{4.28}$$

Hence, by (4.28) there exist two positive constants $\tilde{t}$ and H such that

$$F(x,t) \geqslant H \tag{4.29}$$

for every $x \in \overline{\Omega}$ and $|t| > \tilde{t}$. On the other hand, since F is continuous in $\overline{\Omega} \times \mathbb{R}$, one has

$$F(x,t) \geqslant \min_{(x,t)\in\overline{\Omega}\times[-\tilde{t},\tilde{t}]} F(x,t) \tag{4.30}$$

for every $x \in \overline{\Omega}$ and $|t| \leqslant \tilde{t}$. Then it follows that by (4.29) and (4.30)

$$F(x,t) \geqslant \kappa \quad \text{for any } (x,t) \in \overline{\Omega} \times \mathbb{R} \tag{4.31}$$

for some constant κ. By (4.3) and (4.31) the claim now follows.

In conclusion, by (4.25), (4.26) and (4.27) we get a contradiction. Thus the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

In order to complete the proof of Proposition 5 we can argue from now on as in Step 2 of the proof of Proposition 4. □

We remark that in the proof of Proposition 5 assumption (1.15) was used (and was crucial) only for proving inequality (4.19).

4.2 Proof of Theorem 2

By Proposition 5 and (1.11), we have that $\mathcal{J}_{K,\lambda}$ satisfies the Cerami condition (and hence also the Palais–Smale condition) and $\mathcal{J}_{K,\lambda}(-u) = \mathcal{J}_{K,\lambda}(u)$ for any $u \in X_0$. The verification of the geometric assumption (ii) of the Fountain Theorem follows as in Step 3 in Section 3.1. It remains to verify condition (i). For this purpose we shall use the finite-dimensionality of the linear subspace Y_k and assumption (1.13).

Indeed, by (1.13) for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$F(x,t) \geqslant \varepsilon\,|t|^2 \quad \text{for any } x \in \overline{\Omega} \text{ and any } t \in \mathbb{R} \text{ with } |t| > \delta_\varepsilon, \tag{4.32}$$

while, by the Weierstrass Theorem, we have that

$$F(x,t) \geqslant m_\varepsilon := \min_{x\in\overline{\Omega},|t|\leqslant\delta_\varepsilon} F(x,t) \quad \text{for any } x \in \overline{\Omega} \text{ and any } t \in \mathbb{R} \text{ with } |t| \leqslant \delta_\varepsilon. \tag{4.33}$$

Note that $m_\varepsilon \leqslant 0$, since $F(x,0) = 0$ for any $x \in \overline{\Omega}$. By (4.32) and (4.33), it is easy to see that

$$F(x,t) \geqslant \varepsilon|t|^2 - B_\varepsilon \quad \text{for any } (x,t) \in \overline{\Omega} \times \mathbb{R}$$

for a suitable positive constant B_ε (say, $B_\varepsilon \geqslant \varepsilon\delta_\varepsilon^2 - m_\varepsilon$).

As a consequence of this and by the fact that Y_k is finite-dimensional, we have for any $u \in Y_k$

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) &= \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{2}\|u\|_{L^2(\Omega)}^2 - \int_\Omega F(x,u(x))\,dx \\ &\leqslant C_{k,\lambda}\|u\|_{X_0}^2 - \varepsilon\|u\|_{L^2(\Omega)}^2 + B_\varepsilon|\Omega| \\ &\leqslant (C_{k,\lambda} - \varepsilon C_k)\|u\|_{X_0}^2 + B_\varepsilon|\Omega|, \end{aligned} \tag{4.34}$$

where $C_{k,\lambda}$ and C_k are positive constants, the first one depending on k and λ and the second one only on k. Hence, choosing ε such that $C_{k,\lambda} - \varepsilon C_k < 0$, we get that for any $u \in Y_k$ with $\|u\|_{X_0} = r_k$,

$$\mathcal{J}_{K,\lambda}(u) \leqslant 0,$$

provided $r_k > 0$ is large enough. This proves that $\mathcal{J}_{K,\lambda}$ satisfies condition (i) of the Fountain Theorem and this completes the proof of Theorem 2.

4.3 Nonlinearities satisfying the superlinear conditions (1.13) and (1.17)

In this setting we need the following lemma, whose proof was given in [18, Lemma 2.3]: it will be crucial in the proof of Theorem 3.

Lemma 6. *If* (1.17) *holds, then for any* $x \in \Omega$, *the function* $\mathcal{F}(x, t)$ *is increasing when* $t \geqslant \bar{t}$ *and decreasing when* $t \leqslant -\bar{t}$, *where* $\mathcal{F}$ *is the function given by* (1.16). *In particular, there exists* $C_1 > 0$ *such that* $\mathcal{F}(x, s) \leqslant \mathcal{F}(x, t) + C_1$ *for any* $x \in \Omega$ *and* $0 \leqslant s \leqslant t$ *or* $t \leqslant s \leqslant 0$.

Proposition 7. *Let* $\lambda \in \mathbb{R}$ *and let* $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ *be a function satisfying conditions* (1.8), (1.9), (1.13) *and* (1.17). *Then,* $\mathcal{J}_{K,\lambda}$ *satisfies the Cerami condition at any level* $c \in \mathbb{R}$.

Proof. We can argue exactly as in the proof of Proposition 5. We only have to modify the proof of inequality (4.19): indeed, for proving it, in Proposition 5 we used condition (1.15) (actually (4.20)), which is now no more assumed.

Here we will show the validity of (4.19) by making use of assumption (1.17) and of Lemma 6. We point out that our notation is the one used in the proof of Proposition 5. In view of Lemma 6 we have that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(t_j u_j) = \mathcal{J}_{K,\lambda}(t_j u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(t_j u_j), t_j u_j\rangle &= \int_\Omega \mathcal{F}(x, t_j u_j(x))\,dx \\ &= \int_{\{u_j\geqslant 0\}} \mathcal{F}(x, t_j u_j(x))\,dx + \int_{\{u_j<0\}} \mathcal{F}(x, t_j u_j(x))\,dx \\ &\leqslant \int_{\{u_j\geqslant 0\}} [\mathcal{F}(x, u_j(x)) + C_1] + \int_{\{u_j<0\}} [\mathcal{F}(x, u_j(x)) + C_1] \\ &= \int_\Omega \mathcal{F}(x, u_j(x))\,dx + C_1|\Omega| = \mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle + C_1|\Omega| \to c + C_1|\Omega|\end{aligned}$$

as $j \to +\infty$. This proves (4.19). The proof of Proposition 7 is thus completed. □

4.4 Proof of Theorem 3

The functional $\mathcal{J}_{K,\lambda}$ satisfies the Cerami condition by Proposition 7, and so also the Palais–Smale assumption is satisfied. Moreover, $\mathcal{J}_{K,\lambda}(-u) = \mathcal{J}_{K,\lambda}(u)$ for any $u \in X_0$, thanks to (1.11).

As for the geometric features of $\mathcal{J}_{K,\lambda}$, condition (ii) of the Fountain Theorem follows as in Step 3 of the proof of Theorem 1, whereas condition (i) can be proved as in the proof of Theorem 2. Hence, the assertion of Theorem 3 is obtained.

Acknowledgment: The authors warmly thank the anonymous referee for her/his useful and nice comments on the paper.

Funding: The authors were supported by the SRA Program P1-0292-0101 *Topology, Geometry and Nonlinear Analysis*. The first and the third author were supported by the INdAM-GNAMPA Project 2015 *Modelli ed equazioni non-locali di tipo frazionario*. The third author was supported by the MIUR National Research Project *Variational and Topological Methods in the Study of Nonlinear Phenomena* and by FP7-IDEAS-ERC Starting Grant 2011 #277749 EPSILON (Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities), founded by the European Research Council (ERC).

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
[2] C. O. Alves and S. B. Liu, On superlinear $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.* **73** (2010), no. 8, 2566–2579.
[3] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
[4] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, *Nonlinear Anal.* **20** (1993), no. 1, 1205–1216.

[5] Z. Binlin, G. Molica Bisci and R. Servadei, Superlinear nonlocal fractional problems with infinitely many solutions, *Nonlinearity* **28** (2015), 2247–2264.
[6] H. Brezis, *Analyse Fonctionelle. Théorie et Applications*, Masson, Paris, 1983.
[7] G. Cerami, An existence criterion for the critical points on unbounded manifolds (in Italian), *Ist. Lombardo Accad. Sci. Lett. Rend. Sez. A* **112** (1978), no. 2, 332–336.
[8] G. Cerami, On the existence of eigenvalues for a nonlinear boundary value problem (in Italian), *Ann. Mat. Pura Appl. (4)* **124** (1980), 161–179.
[9] D. G. Costa and C. A. Magalhães, Variational elliptic problems which are nonquadratic at infinity, *Nonlinear Anal.* **23** (1994), 1401–1412.
[10] A. Di Castro, T. Kuusi and G. Palatucci, Nonlocal Harnack inequalities, *J. Funct. Anal.* **267** (2014), 1807–1836.
[11] A. Di Castro, T. Kuusi and G. Palatucci, Local behavior of fractional p-minimizers, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **267** (2015), 1807–1836.
[12] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), no. 5, 521–573.
[13] F. Fang and S. B. Liu, Nontrivial solutions of superlinear p-Laplacian equations, *J. Math. Anal. Appl.* **351** (2009), no. 1, 138–146.
[14] M. Ferrara, G. Molica Bisci and B. Zhang, Existence of weak solutions for non-local fractional problems via Morse theory, *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014), 2493–2499.
[15] G. Franzina and G. Palatucci, Fractional p-eigenvalues, *Riv. Math. Univ. Parma (N.S.)* **5** (2014), no. 2, 373–386.
[16] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), no. 4, 787–809.
[17] S. B. Liu, On ground states of superlinear p-Laplacian equations in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **361** (2010), no. 81, 48–58.
[18] S. B. Liu, On superlinear problems without Ambrosetti–Rabinowitz condition, *Nonlinear Anal.* **73** (2010), no. 3, 788–795.
[19] S. B. Liu and S. J. Liu, Infinitely many solutions for a superlinear elliptic equation, *Acta Math. Sinica (Chin. Ser.)* **46** (2003), no. 4, 625–630.
[20] O. Miyagaki and M. Souto, Superlinear problems without Ambrosetti and Rabinowitz growth condition, *J. Differential Equations* **245** (2008), no. 12, 3628–3638.
[21] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math. 65, American Mathematical Society, Providence, 1986.
[22] M. Schechter and W. Zou, Superlinear problems, *Pacific J. Math.* **214** (2004), no. 1, 145–160.
[23] R. Servadei, Infinitely many solutions for fractional Laplace equations with subcritical nonlinearity, *Contemp. Math.* **595** (2013), 317–340.
[24] R. Servadei, The Yamabe equation in a non-local setting, *Adv. Nonlinear Anal.* **2** (2013), 235–270.
[25] R. Servadei and E. Valdinoci, Mountain Pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.* **389** (2012), 887–898.
[26] R. Servadei and E. Valdinoci, A Brezis–Nirenberg result for non-local critical equations in low dimension, *Commun. Pure Appl. Anal.* **12** (2013), no. 6, 2445–2464.
[27] R. Servadei and E. Valdinoci, Lewy–Stampacchia type estimates for variational inequalities driven by (non)local operators, *Rev. Mat. Iberoam.* **29** (2013), no. 3, 1091–1126.
[28] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.* **33** (2013), no. 5, 2105–2137.
[29] R. Servadei and E. Valdinoci, The Brezis–Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.* **367** (2015), no. 1, 67–102.
[30] M. Struwe, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Ergeb. Math. Grenzgeb. (3), Springer, Berlin, 1990.
[31] M. Willem, *Minimax Theorems*, Birkhäuser, Basel, 1996.
[32] M. Willem and W. Zou, On a Schrödinger equation with periodic potential and spectrum point zero, *Indiana Univ. Math. J.* **52** (2003), 109–132.
[33] W. M. Zou, Variant fountain theorems and their applications, *Manuscripta Math.* **104** (2001), 343–358.