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On sequences of solutions for discrete anisotropic equations

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Abstract

Taking advantage of a recent critical point theorem, the existence of infinitely many solutions for an anisotropic problem with a parameter is established. More precisely, a concrete interval of positive parameters, for which the treated problem admits infinitely many solutions, is determined without symmetry assumptions on the nonlinear data. Our goal was achieved by requiring an appropriate behavior of the nonlinear terms at zero, without any additional conditions.
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1. Introduction

For every $a, b \in \mathbb{Z}$, such that $a < b$, set $\mathbb{Z}[a, b] := \{a, a+1, \dots, b\}$. This work is concerned with the study of existence of solutions for the following anisotropic difference equation

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda f_k(u(k)), & k \in \mathbb{Z}[1, T] \\ u(0) = u(T+1) = 0, \end{cases} \qquad (A_\lambda^f)$$

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where λ is a positive parameter, $f_k : \mathbb{R} \to \mathbb{R}$ is a continuous function for every $k \in \mathbb{Z}[1, T]$ (with $T \geq 2$), and $\Delta u(k-1) := u(k) - u(k-1)$ is the forward difference operator. Further, we will assume that the map $p : \mathbb{Z}[0, T] \to \mathbb{R}$ satisfies $p^- := \min_{\mathbb{Z}[0,T]} p(k) > 1$ as well as $p^+ := \max_{\mathbb{Z}[0,T]} p(k) > 1$.

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain nonlinear problems from biological neural networks, economics, optimal control and other areas of study have led to the rapid development of the theory of difference equations; see the monograph of Agarwal [1].

Our idea here is to transfer the problem of existence of solutions for problem (A_λ^f) into the problem of existence of critical points for a suitable associated energy functional, namely J_λ.

More precisely, the main purpose of this paper is to investigate the existence of infinitely many solutions to problem (A_λ^f) by using a critical point theorem obtained in [14]; see Theorem 2.1.

Continuous analogues of problems like (A_λ^f) are known to be mathematical models of various phenomena arising in the study of elastic mechanics, electrorheological fluids or image restoration; see, for instance, Zhikov [16], Růžička [15] and Chen, Levine and Rao [4].

Variational continuous anisotropic problems have been studied by Fan and Zhang in [5] and later considered by many methods and authors, see [7] for an extensive survey of such boundary value problems.

Research concerning the discrete anisotropic problems of type (A_λ^f) was initiated by Kone and Ouaro in [9] and by Mihăilescu, Rădulescu and Tersian in [13]. In these papers known tools from the critical point theory are applied in order to get the existence of solutions.

We also recall that very recently, in [6], Galewski and Głąb considered the discrete anisotropic boundary value problem (A_λ^f) using critical point theory. Firstly they applied the direct method of the calculus of variations and the mountain pass technique in order to reach the existence of at least one nontrivial solution. Moreover, in the same paper, a discrete three critical point theorem was exploited in order to get the existence of at least two nontrivial solutions.

We note that most existence results for discrete problems assume that the nonlinearities data are odd functions. Only a few papers deal with nonlinearities for which this property does not hold; see, for instance, the interesting paper of Kristály, Mihăilescu and Rădulescu [10]; see also [11].

In analogy with the cited contributions, in our approach we do not require any symmetry hypothesis. A special case of our contributions reads as follows.

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function. Assume that*

$$\liminf_{t\to 0^+} \frac{\int_0^t g(s)ds}{t^{p^+}} = 0 \quad \textit{and} \quad \limsup_{t\to 0^+} \frac{\int_0^t g(s)ds}{t^{p^-}} = +\infty.$$

Then, for each $\lambda > 0$*, the problem*

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda g(u(k)), & k \in \mathbb{Z}[1,T] \\ u(0) = u(T+1) = 0, \end{cases} \tag{A_λ^g}$$

admits a sequence of nonzero solutions which converges to zero.

The plan of the paper is the following: Section 2 is devoted to our abstract framework, while Section 3 is dedicated to main results. A concrete example of application of the attained abstract results is presented; see Example 3.1.

Finally, we cite the recent monograph by Kristály, Rădulescu and Varga [12] as general reference on variational methods adopted here.

2. Abstract framework

Let $(X, \|\cdot\|)$ be a finite dimensional Banach space and let $J_\lambda : X \to \mathbb{R}$ be a function satisfying the following structure hypothesis:

(Λ) $J_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ *for all* $u \in X$*, where* $\Phi, \Psi : X \to \mathbb{R}$ *are functions of class* C^1 *on* X *with* Φ *coercive, i.e.* $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$*, and* λ *is a real positive parameter.*

Moreover, provided that $r > \inf_X \Phi$, put

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

and

$$\delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Clearly, one can observe that $\delta \geq 0$. Further, when $\delta = 0$, in the sequel, we agree that $1/\delta$ is $+\infty$.

Theorem 2.1. *Assume that the condition* (Λ) *is verified. If* $\delta < +\infty$ *then for each* $\lambda \in]0, 1/\delta[$*, one of the following holds:*
either

(b_1) *there is a global minimum of* Φ *which is a local minimum of* J_λ*,*

or

(b_2) *there is a sequence* $\{u_m\}$ *of pairwise distinct critical points (local minima) of* J_λ*, with* $\lim_{m\to\infty} \Phi(u_m) = \inf_X \Phi$*, which converges to a global minimum of* Φ*.*

Remark 2.1. Theorem 2.1 is a special form of the quoted variational principle of Ricceri contained in [14].

By a *strong solution* (briefly called a "solution") to (A_λ^f) we mean such a function $u : \mathbb{Z}[0, T+1] \to \mathbb{R}$ which satisfies the given equation and the associated boundary conditions. Solutions will be investigated in the space

$$H = \{u : \mathbb{Z}[0, T+1] \to \mathbb{R};\ u(0) = u(T+1) = 0\}.$$

Clearly, H is a T-dimensional Hilbert space with the inner product

$$(u, v) := \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \quad \forall\, u, v \in H.$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2},$$

see the work [2] for details.

From now on, for every $u \in H$, set

$$\Phi(u) := \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)}, \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{T} F_k(u(k)),$$

where $F_k(t) := \int_0^t f_k(s)ds$, for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}[1, T]$. Further, let us denote

$$J_\lambda(u) := \Phi(u) - \lambda \Psi(u),$$

for every $u \in H$.

Standard arguments assure that $J_\lambda \in C^1(H; \mathbb{R})$ and

$$\langle J_\lambda'(u), v\rangle = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1)\Delta v(k-1) - \lambda \sum_{k=1}^{T} f_k(u(k))v(k),$$

for all $u, v \in H$.

The next result will be crucial in the sequel.

Lemma 2.1. *The functional $\Phi : H \to \mathbb{R}$ is coercive, i.e.*

$$\lim_{\|u\|\to\infty} \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} = +\infty.$$

Proof. By [13, Lemma 1, part (a)] there exist two positive constants C_1 and C_2 such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 \|u\|^{p^-} - C_2,$$

for every $u \in H$ with $\|u\| > 1$. Hence, the conclusion simply holds taking into account that

$$\begin{aligned} \Phi(u) &:= \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \\ &\geq \frac{1}{p^+} \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)}\right) \\ &\geq \frac{C_1 \|u\|^{p^-} - C_2}{p^+} \to +\infty, \end{aligned}$$

as $\|u\| \to \infty$. $\square$

A *critical point* for J_λ, i.e. such a point $u \in H$ such that

$$\langle J_\lambda'(u), v\rangle = 0,$$

for every $v \in H$, is said to be a *weak solution* to (A_λ^f).

Summing by parts we see that any weak solution to (A_λ^f) is in fact a strong one. Hence, in order to solve (A_λ^f) we check to find critical points for J_λ and investigate their multiplicity.

Finally, let us fix a constant $p > 1$. Then

$$\|u\|_\infty := \max_{k\in\mathbb{Z}[1,T]} |u(k)| \le \frac{(T+1)^{(p-1)/p}}{2}\left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p\right)^{1/p}, \tag{1}$$

for every $u \in H$.

See Lemma 2.2 proved by Jiang and Zhou in [8].

3. Main results

Set

$$A_0 := \liminf_{t\to 0^+} \frac{\sum_{k=1}^{T} \max_{|\xi|\le t} F_k(\xi)}{t^{p^+}}, \qquad B^0 := \limsup_{t\to 0^+} \frac{\sum_{k=1}^{T} F_k(t)}{t^{p^-}},$$

and

$$\kappa := \frac{2^{p^+-1}p^-}{p^+(T+1)^{p^+-1}}.$$

Our main result is the following one.

Theorem 3.1. *Assume that the following inequality holds*

(h_0) $A_0 < \kappa B^0$.

Then, for each $\lambda \in \left]\frac{2}{p^- B^0}, \frac{2^{p^+}}{p^+(T+1)^{p^+-1}A_0}\right[$, *problem* (A_λ^f) *admits a sequence of nonzero solutions which converges to zero.*

Proof. Fix $\lambda \in \left]\frac{2}{p^- B^0}, \frac{2^{p^+}}{p^+(T+1)^{p^+-1}A_0}\right[$ and put Φ, Ψ, J_λ as in the previous section. Our aim is to apply Theorem 2.1 to function J_λ. Now, by standard arguments and bearing in mind Lemma 2.1, condition (Λ) clearly holds. Therefore, our conclusion follows provided that $\delta < +\infty$ as well as that 0_H is not a local minimum point for the functional J_λ. To this end, let $\{c_m\} \subset]0, +\infty[$ be a sequence such that $\lim_{m\to\infty} c_m = 0$ and

$$\lim_{m\to\infty} \frac{\sum_{k=1}^{n} \max_{|\xi|\le c_m} F_k(\xi)}{c_m^2} = A_0.$$

Set

$$r_m := \left(\frac{2^{p^+}}{p^+(T+1)^{p^+-1}}\right) c_m^{p^+},$$

for every $m \in \mathbb{N}$.
For $m > 0$ sufficiently large, it follows that

$$\Phi^{-1}(]-\infty, r_m[) \subset \{v \in X : |v(k)| \leq c_m, \ \forall k \in \mathbb{Z}[0, T+1]\}. \tag{2}$$

Indeed, if $v \in X$ and $\Phi(v) < r_m$, one has

$$\sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta v(k-1)|^{p(k-1)} < r_m.$$

Then,

$$|\Delta v(k-1)| < (p(k-1) r_m)^{1/p(k-1)},$$

for every $k \in \mathbb{Z}[1, T+1]$. Consequently, since $r_m < 1/p^+$ for every $m \geq \bar{m}$, one immediately one has

$$(p^+ r_m)^{1/p(k-1)} \leq (p^+ r_m)^{1/p^+},$$

hence

$$|\Delta v(k-1)| < (p^+ r_m)^{1/p^+} < 1,$$

for every $k \in \mathbb{Z}[1, T+1]$. So

$$\sum_{k=1}^{T+1} \frac{1}{p^+} |\Delta v(k-1)|^{p^+} \leq \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta v(k-1)|^{p(k-1)} < r_m,$$

for every $m \geq \bar{m}$. At this point, by (1), the following inequality

$$\max_{k \in \mathbb{Z}[1,T]} |v(k)| \leq \left(p^+ \frac{(T+1)^{p^+-1}}{2^{p^+}} r_m\right)^{1/p^+} = c_m,$$

is satisfied for every $m \geq \bar{m}$. Thus, the above computations ensure that (2) holds. The algebraic inclusion (2) implies that

$$\begin{aligned}
\varphi(r_m) &\leq \frac{\sup_{\Phi(v) < r_m} \sum_{k=1}^{T} F_k(v(k))}{r_m} \leq \frac{\sum_{k=1}^{T} \max_{|t| \leq c_m} F_k(t)}{r_m} \\
&= \frac{p^+(T+1)^{p^+-1}}{2^{p^+}} \frac{\sum_{k=1}^{T} \max_{|t| \leq c_m} F_k(t)}{c_m^{p^+}},
\end{aligned}$$

for every m sufficiently large. Hence, it follows that

$$\delta \leq \lim_{m \to \infty} \varphi(r_m) \leq \frac{p^+(T+1)^{p^+-1}}{2^{p^+}} A_0 < \frac{1}{\lambda} < +\infty.$$

In the next step we show that 0_H is not a local minimum point for the functional J_λ. First, assume that $B^0 = +\infty$. Accordingly, fix M such that $M > \frac{2}{p^-\lambda}$ and let $\{b_m\}$ be a sequence of positive numbers, with $\lim_{m\to\infty} b_m = 0$, such that

$$\sum_{k=1}^{T} F_k(b_m) > M b_m^{p^-} \quad (\forall\, m \in \mathbb{N}).$$

Thus, take in H a sequence $\{s_m\}$ such that, for each $m \in \mathbb{N}$, $s_m(k) := b_m$ for every $k \in \mathbb{Z}[1, T]$. Observe that $\|s_m\| \to 0$ as $m \to \infty$. So, it follows that

$$\sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta s_m(k-1)|^{p(k-1)} \leq \frac{2b_m^{p^-}}{p^-},$$

taking into account that

$$\sum_{k=1}^{T+1} |\Delta s_m(k-1)|^{p^-} = 2b_m^{p^-}.$$

Then, one has

$$J_\lambda(s_m) \leq \frac{2b_m^{p^-}}{p^-} - \lambda \sum_{k=1}^{T} F_k(b_m) < \left(\frac{2}{p^-} - \lambda M\right) b_m^{p^-},$$

that is, $J_\lambda(s_m) < 0$ for every sufficiently large m.
Next, assume that $B^0 < +\infty$. Since $\lambda > \frac{2}{p^- B^0}$, we can fix $\varepsilon > 0$ such that $\varepsilon < B^0 - \frac{2}{p^-\lambda}$. Therefore, also taking $\{b_m\}$ a sequence of positive numbers such that $\lim_{m\to\infty} b_m = 0$ and

$$(B^0 - \varepsilon) b_m^{p^-} < \sum_{k=1}^{T} F_k(b_m) < (B^0 + \varepsilon) b_m^{p^-}, \quad (\forall\, m \in \mathbb{N})$$

arguing as before and by choosing $\{s_m\}$ in H as above, one has

$$J_\lambda(s_m) < \left(\frac{2}{p^-} - \lambda(B^0 - \varepsilon)\right) b_m^{p^-}.$$

So, also in this case, $J_\lambda(s_m) < 0$ for every sufficiently large m. Finally, since $J_\lambda(0_H) = 0$, the above fact means that 0_H is not a local minimum of J_λ. Therefore, the unique global minimum of Φ is not a local minimum of the functional J_λ. Hence, by Theorem 2.1 we obtain a sequence $\{u_m\} \subset H$ of critical points of J_λ such that

$$\lim_{m\to\infty} \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u_m(k-1)|^{p(k-1)} = \lim_{m\to\infty} \|u_m\| = 0.$$

Thus, it follows that $\|u_m\|_\infty \to 0$ as $m \to \infty$. The proof is complete. $\square$

Remark 3.1. As pointed out earlier, Theorem 3.1 has been obtained exploiting Theorem 2.1. Via our approach we are able to determine an open subinterval of $]0, 1/\delta[$, where $\delta := \liminf_{r\to 0^+} \varphi(r)$, such that problem (A_λ^f) admits infinitely many solutions.

Indeed, the main condition (h_0) implies that the real interval of parameter

$$\left]\frac{2}{p^- B^0}, \frac{2^{p^+}}{p^+(T+1)^{p^+-1}A_0}\right[,$$

is well-defined and nonempty. Further, since $B_0 > 0$ and

$$\delta \le \lim_{m\to\infty} \varphi(r_m) \le \frac{p^+(T+1)^{p^+-1}}{2^{p^+}} A_0,$$

the following inclusion

$$\left]\frac{2}{p^- B^0}, \frac{2^{p^+}}{p^+(T+1)^{p^+-1}A_0}\right[\subseteq \left]0, \frac{1}{\delta}\right[,$$

is verified.

Remark 3.2. We note that, if f_k is a nonnegative continuous function, for every $k \in \mathbb{Z}[1,T]$, condition (h_0) assumes the form

$$\liminf_{t\to 0^+} \frac{\sum_{k=1}^{n} F_k(t)}{t^{p^+}} < \kappa \limsup_{t\to 0^+} \frac{\sum_{k=1}^{n} F_k(t)}{t^{p^-}}.$$

Consequently, Theorem 1.1 immediately follows from Theorem 3.1.

We note that there seems to be increasing interest in existence of solutions to boundary value problems for finite difference equations with p-Laplacian operator, because of their applications in many fields. In this setting, set $p > 1$ and consider the real map $\phi_p : \mathbb{R} \to \mathbb{R}$ given by $\phi_p(s) := |s|^{p-2}s$, for every $s \in \mathbb{R}$.

Further, denote

$$\widehat{A}_0 := \liminf_{t\to 0^+} \frac{\sum_{k=1}^{T} \max_{|\xi|\le t} F_k(\xi)}{t^p}, \quad \text{and} \quad \widehat{B}^0 := \limsup_{t\to 0^+} \frac{\sum_{k=1}^{T} F_k(t)}{t^p}.$$

With the previous notations, taking the map $p : \mathbb{Z}[0,T] \to \mathbb{R}$ such that $p(k) = p$, for every $k \in \mathbb{Z}[0,T]$, we have the following immediate consequence of Theorem 3.1.

Corollary 3.1. *Assume that*

($\widehat{h}_0$) $\widehat{A}_0 < \frac{2^{p-1}}{(T+1)^{p-1}} \widehat{B}^0$.

Then, for each $\lambda \in \left]\frac{2}{p\widehat{B}^0}, \frac{2^p}{p(T+1)^{p-1}\widehat{A}_0}\right[$, *the following problem*

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda f_k(u(k)), & k \in \mathbb{Z}[1,T] \\ u(0) = u(T+1) = 0, \end{cases} \qquad (D_\lambda^f)$$

admits a sequence of nonzero solutions which converges to zero.

A more technical version of Theorem 3.1 can be written as follows.

Theorem 3.2. *Assume that there exist nonnegative real sequences* $\{a_m\}$ *and* $\{b_m\}$*, with* $\lim_{m\to\infty} b_m = 0$*, such that*

(k_1) $a_m^{p^-} < \left(\frac{2^{p^+-1}p^-}{p^+(T+1)^{p^+-1}}\right) b_m^{p^+}$*, for each* $m \in \mathbb{N}$*;*

(k_2) $G_0 < \frac{B^0}{p^+(T+1)^{p^+-1}}$*, where*

$$G_0 := \lim_{m\to\infty} \frac{\sum_{k=1}^{T} \max_{|t|\le b_m} F_k(t) - \sum_{k=1}^{T} F_k(a_m)}{2^{p^+-1}p^- b_m^{p^+} - p^+(T+1)^{p^+-1}a_m^{p^-}}.$$

Then, for each $\lambda \in \left]\frac{2}{p^- B^0}, \frac{2}{p^- p^+(T+1)^{p^+-1}G_0}\right[$*, problem* (A_λ^f) *admits a sequence of nonzero solutions which converges to zero.*

Proof. Let us observe that, keeping the above notations, one has

$$\varphi(r_m) \le \inf_{w\in\Phi^{-1}(]-\infty,r_m[)} \frac{\sum_{k=1}^{T} \max_{|t|\le b_m} F_k(t) - \sum_{k=1}^{T} F_k(w(k))}{\left(\frac{2^{p^+}}{p^+(T+1)^{p^+-1}}\right) b_m^{p^+} - \Phi(u)}. \tag{3}$$

Now, for each $m \in \mathbb{N}$, let $w_m \in H$ be defined by $w_m(k) := a_m$ for every integer $k \in \mathbb{Z}[1, T]$. Clearly, since $\|w_m\| \to 0$ as $m \to \infty$, it follows that $|\Delta w_m(k-1)| < 1$ for every $k \in \mathbb{Z}[1, T+1]$ and sufficiently large m. Then, there exists $\bar{m} \in \mathbb{N}$ such that

$$\Phi(w_m) := \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta w_m(k-1)|^{p(k-1)} \le \frac{1}{p^-}\left(\sum_{k=1}^{T+1} |\Delta w_m(k-1)|^{p^-}\right),$$

for every $m \ge \bar{m}$.

Now, by condition (k_1) and taking into account that $\sum_{k=1}^{T+1} |\Delta w_m(k-1)|^{p^-} = 2a_m^{p^-}$, the above inequality implies that

$$0 < \left(\frac{2^{p^+}}{p^+(T+1)^{p^+-1}}\right) b_m^{p^+} - \frac{2a_m^{p^-}}{p^-} \le \left(\frac{2^{p^+}}{p^+(T+1)^{p^+-1}}\right) b_m^{p^+} - \Phi(w_m),$$

for every $m \ge \bar{m}$.

Hence, clearly $w_m \in \Phi^{-1}(]-\infty, r_m[)$ and inequality (3) yields

$$\varphi(r_m) \le \left(\frac{p^- p^+(T+1)^{p^+-1}}{2}\right) \frac{\sum_{k=1}^{T} \max_{|t|\le b_m} F_k(t) - \sum_{k=1}^{T} F_k(a_m)}{p^- 2^{p^+-1} b_m^{p^+} - p^+(T+1)^{p^+-1}a_m^{p^-}},$$

for every large enough m.

Further, by hypothesis (k_2), we obtain

$$\delta \le \lim_{m\to\infty} \varphi(r_m) \le \frac{p^- p^+(T+1)^{p^+-1}}{2} G_0 < \frac{1}{\lambda} < +\infty.$$

From now on, arguing exactly as in the proof of Theorem 3.1 we obtain the assertion. □

A more precise form of the intervals of parameters in our results can be obtained by using an interesting lemma contained in [3, Lemma 4]. Indeed, in the cited work, the authors improve [8, Lemma 2.2], finding that

$$\|u\|_\infty \leq \frac{1}{c_1}\left(\sum_{k=1}^{T+1}|\Delta u(k-1)|^p\right)^{1/p}, \tag{4}$$

for every $u \in H$, where

$$c_1 := \begin{cases} \left[\left(\dfrac{2}{T}\right)^{p-1} + \left(\dfrac{2}{T+2}\right)^{p-1}\right]^{1/p} & \text{if } T \text{ is even} \\ \dfrac{2}{(T+1)^{(p-1)/p}} & \text{if } T \text{ is odd.} \end{cases}$$

Note that, since the continuous function $\theta :]0, T+1[\to \mathbb{R}$ defined by

$$\theta(s) := \frac{1}{(T-s+1)^{p-1}} + \frac{1}{s^{p-1}},$$

attains its minimum $\frac{2^p}{(T+1)^{p-1}}$ at $s = \frac{T+1}{2}$, one has

$$\frac{2^p}{(T+1)^{p-1}} < \theta(T/2).$$

Hence

$$\frac{2}{(T+1)^{(p-1)/p}} < \left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]^{1/p} = \theta(T/2)^{1/p}. \tag{5}$$

Then

$$\left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]^{-1/p} < \frac{(T+1)^{(p-1)/p}}{2}.$$

For instance, an immediate consequence of the previous remarks is the following result.

Corollary 3.2. *Let $T \geq 2$ be an even number. Assume that*

($\widetilde{\mathrm{h}}_0$) $\widehat{A}_0 < 2^{p-2}\left[\frac{1}{T^{p-1}} + \frac{1}{(T+2)^{p-1}}\right]\widehat{B}^0$.

Then, for each $\lambda \in \left]\frac{2}{p\widehat{B}^0}, \left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]\frac{1}{p\widehat{A}_0}\right[$*, the following problem*

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda f_k(u(k)), & k \in \mathbb{Z}[1, T] \\ u(0) = u(T+1) = 0, \end{cases} \tag{D_λ^f}$$

admits a sequence of nonzero solutions which converges to zero.

Remark 3.3. It follows from (5) that condition $(\widehat{\mathrm{h}}_0)$ implies $(\widetilde{\mathrm{h}}_0)$ and one has

$$\left]\frac{2}{p\widehat{B}^0}, \frac{2^p}{p(T+1)^{p-1}\widehat{A}_0}\right[\subseteq \left]\frac{2}{p\widehat{B}^0}, \left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]\frac{1}{p\widehat{A}_0}\right[.$$

Moreover, if $\widehat{A}_0 > 0$, again inequality (5) ensures that the above inclusion is proper. Hence, from the previous facts, we can conclude that Corollary 3.2 is a refinement of Corollary 3.1.

An application of Theorem 1.1 is the following.

Example 3.1. Let $\gamma > 2$ be a real positive constant and $p : \mathbb{Z}[0, T] \to \mathbb{R}^+$ a map such that $p^+ := \max_{\mathbb{Z}[0,T]} p(k) = \gamma$ and $p^- := \min_{\mathbb{Z}[0,T]} p(k) = \gamma - 1$, with $T \geq 2$. Further, let $\{s_m\}$, $\{t_m\}$ and $\{\delta_m\}$ be sequences defined by

$$s_m := 2^{-\frac{m!}{2}}, \qquad t_m := 2^{-2m!}, \qquad \delta_m := 2^{-(m!)^2},$$

and consider $\nu \in \mathbb{N}$ such that

$$s_{m+1} < t_m < s_m - \delta_m, \quad \forall\, m \geq \nu.$$

Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be the nonnegative continuous function given by

$$g(s) := \begin{cases} 2^{-(\gamma-1)\nu!} & \text{if } s \in]s_\nu - \delta_\nu, +\infty[\\ y_m(s) & \text{if } s \in \bigcup\limits_{m\geq\nu}]s_{m+1} - \delta_{m+1}, s_{m+1}[\\ 2^{-(\gamma-1)m!} & \text{if } s \in \bigcup\limits_{m\geq\nu} [s_{m+1}, s_m - \delta_m] \\ 0 & \text{if } s \leq 0, \end{cases}$$

where

$$y_m(s) := \left(2^{-(\gamma-1)m!} - 2^{-(\gamma-1)(m+1)!}\right)\left(\frac{s - s_{m+1} + \delta_{m+1}}{\delta_{m+1}}\right) + 2^{-(\gamma-1)(m+1)!}.$$

Set $G(t) := \int_0^t g(s)\, ds$ for every $t \in \mathbb{R}$. Then, one has that

$$\frac{G(s_m)}{s_m^\gamma} \leq \frac{g(s_{m+1})s_m + g(s_m)\delta_m}{s_m^\gamma},$$

and

$$\frac{G(t_m)}{t_m^{\gamma-1}} \geq \frac{g(s_{m+1})(t_m - s_{m+1})}{t_m^{\gamma-1}},$$

for every large enough m.
Since

$$\lim_{m\to\infty} \frac{g(s_{m+1})s_m + g(s_m)\delta_m}{s_m^\gamma} = 0, \quad \text{and} \quad \lim_{m\to\infty} \frac{g(s_{m+1})(t_m - s_{m+1})}{t_m^{\gamma-1}} = +\infty,$$

it follows that

$$\lim_{m\to\infty} \frac{G(s_m)}{s_m^\gamma} = 0, \quad \text{and} \quad \lim_{m\to\infty} \frac{G(t_m)}{t_m^{\gamma-1}} = +\infty.$$

Thus

$$\liminf_{t\to 0^+}\frac{\int_0^t g(s)ds}{t^\gamma}=0,\quad \text{and}\quad \limsup_{t\to 0^+}\frac{\int_0^t g(s)ds}{t^{\gamma-1}}=+\infty.$$

Then, owing to Theorem 1.1, for each $\lambda > 0$, the following anisotropic discrete Dirichlet problem

$$\begin{cases}-\Delta(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1))=\lambda g(u(k)), & k\in\mathbb{Z}[1,T]\\ u(0)=u(T+1)=0,\end{cases}\qquad (A_\lambda^g)$$

admits a sequence of nonzero solutions which converges to zero.

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