Applied Mathematics and Computation 242 (2014) 454–461

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Existence of solutions for p-Laplacian discrete equations

Giovanni Molica Bisci [a,*], Dušan Repovš [b]

[a] Dipartimento P.A.U., Università degli Studi Mediterranea di Reggio Calabria, Salita Melissari – Feo di Vito, 89124 Reggio Calabria, Italy
[b] Faculty of Education, and Faculty of Mathematics and Physics, University of Ljubljana, POB 2964, Ljubljana 1001, Slovenia

ARTICLE INFO

Keywords:
Discrete boundary value problem
Existence result
Discrete p-Laplacian
Critical point theory

ABSTRACT

This work is devoted to the study of the existence of at least one (non-zero) solution to a problem involving the discrete p-Laplacian. As a special case, we derive an existence theorem for a second-order discrete problem, depending on a positive real parameter α, whose prototype is given by

$$\begin{cases} -\Delta^2 u(k-1) = \alpha f(k,u(k)), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0. \end{cases}$$

Our approach is based on variational methods in finite-dimensional setting.

1. Introduction

We are interested in investigating nonlinear discrete boundary value problems by using variational methods. This approach has been recently adopted, for instance, in [2–4,14,19,29].

More precisely, for every $a, b \in \mathbb{Z}$, such that $a < b$, set $\mathbb{Z}[a,b] := \{a, a+1, \dots, b\}$ and let $T \geqslant 2$ be a positive integer.

The aim of this paper is to prove some existence results for the following discrete problem:

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = f(k,u(k)), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{D_f}$$

where $p > 1$, $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) := |s|^{p-2}s$, for every $s \in \mathbb{R}$, $f : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\Delta u(k-1) := u(k) - u(k-1)$ is the forward difference operator.

In recent years equations involving the discrete p-Laplacian operator, subject to different boundary conditions, have been widely studied by many authors and several approaches.

In particular, problem (D_f) has been previously studied, for instance, in [3,8,15] by using various methods. See the recent papers [9,23] for the discrete anisotropic case. Motivations for this interest arose in by different fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and others. Moreover, the main background in the real world for the discrete p-Laplacian operator are the problems on the boundary between different substances.

Set

$$c(p,T) := \begin{cases} \frac{1}{p}\left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right] & \text{if } T \text{ is even}, \\ \frac{2^p}{p(T+1)^{p-1}} & \text{if } T \text{ is odd}. \end{cases}$$

* Corresponding author.
E-mail addresses: gmolica@unirc.it (G. Molica Bisci), dusan.repovs@guest.arnes.si (D. Repovš).

http://dx.doi.org/10.1016/j.amc.2014.05.118

Via variational approach, we are able to prove the existence of a solution for problem $(\mathrm{D_f})$ by requiring that

$$\frac{\sum_{k=1}^{T}\max_{|\xi|\leqslant\varepsilon}\int_0^{\xi}f(k,s)ds}{\varepsilon^p}<c(p,T)$$

for some $\varepsilon>0$. See condition (2) in Theorem 3.2.

Next, by using Theorem 3.2, we study a parametric version of problem $(\mathrm{D_f})$, defined as follows

$$\begin{cases} -\Delta\big(\phi_p(\Delta u(k-1))\big)=\alpha f(k,u(k)), & \forall k\in\mathbb{Z}[1,T],\\ u(0)=u(T+1)=0, \end{cases} \qquad (\mathrm{D}^{\mathrm{f}}_{\alpha})$$

where α is a positive real parameter.

In this case, requiring a suitable behavior of the potentials at zero and at infinity, we obtain, for sufficiently large α, the existence of at least one positive solution for problem $(\mathrm{D}^{\mathrm{f}}_{\alpha})$, see Theorem 4.2. This result can be achieved exploiting Theorem 3.2 together with the well-known variational characterization of the first eigenvalue of the p-Laplacian operator in the finite-dimensional context (see [3]).

The simplest example we can deal with is a second-order boundary value problem.

Theorem 1.1. *Let $f:\mathbb{Z}[1,T]\times[0,+\infty)\to[0,+\infty)$ be a continuous function satisfying the following hypotheses:*

$$\lim_{\xi\to+\infty}\frac{\sum_{k=1}^{T}\int_0^{\xi}f(k,t)dt}{\xi^2}=0$$

and

$$\gamma_\kappa:=\liminf_{\xi\to 0^+}\frac{\int_0^{\xi}f(k,t)dt}{\xi^2}>0$$

for every $k\in\mathbb{Z}[1,T]$. Then for every

$$\alpha>\frac{2}{\min\limits_{k\in\mathbb{Z}[1,T]}\gamma_k}\sin^2\left(\frac{\pi}{2(T+1)}\right),$$

the following second-order discrete problem

$$\begin{cases} -\Delta^2u(k-1)=\alpha f(k,u(k)), & \forall k\in\mathbb{Z}[1,T],\\ u(0)=u(T+1)=0, \end{cases} \qquad (\mathrm{S}_{\alpha})$$

has at least one positive solution.

We remark that the results obtained for second-order discrete equations in [4,12] and our theorems are mutually independent. Moreover, the approach adopted here can be used studying the discrete counterpart of the following problem

$$\begin{cases} \frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+f((x,y),u(x,y))=0,\\ u(x,0)=u(x,n+1)=0, & \forall x\in(0,m+1),\\ u(0,y)=u(m+1,y)=0, & \forall y\in(0,n+1), \end{cases}$$

where $m,n\in\mathbb{N}\setminus\{0\}$ and f is a suitable continuous function. See [10,13] for details. We refer to the monograph of Cheng [7] for a geometrical interpretation of this equations.

The plan of the paper is as follows. Section 2 is devoted to our abstract framework and preliminaries. Successively, in Section 3 we prove our main result (see Theorem 3.2). The parametric case is discussed in the last section (see Theorem 4.2), where, a concrete example of an application is also presented (see Example 4.5).

2. Abstract framework

On the T-dimensional Banach space

$$H:=\{u:\mathbb{Z}[0,T+1]\to\mathbb{R}:u(0)=u(T+1)=0\}$$

endowed by the norm

$$\|u\|:=\left(\sum_{k=1}^{T+1}|\Delta u(k-1)|^p\right)^{1/p},$$

we define the functional $J:H\to\mathbb{R}$ given by

$$J(u) := \frac{1}{p}\sum_{k=1}^{T+1}|\Delta u(k-1)|^p - \sum_{k=1}^{T}\int_0^{u(k)} f(k,t)dt$$

for every $u \in H$.

We recall that a *solution* of problem $(\mathrm{D_f})$ is a function $u \in H$ such that

$$\sum_{k=1}^{T+1}\phi_p(\Delta u(k-1))\Delta v(k-1) = \sum_{k=1}^{T} f(k,u(k))v(k)$$

for every $v \in H$.

We observe that problem $(\mathrm{D_f})$ has a variational structure. Indeed, the functional J is differentiable in $u \in H$ and one has

$$\langle J'(u), v\rangle = \sum_{k=1}^{T+1}\phi_p(\Delta u(k-1))\Delta v(k-1) - \sum_{k=1}^{T} f(k,u(k))v(k)$$

for every $v \in H$.

Thus critical points of J are solutions to problem $(\mathrm{D_f})$. In order to find these critical points, we will make use of the following local minimum result due to Ricceri (see [24]) recalled here on the finite-dimensional setting.

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a finite-dimensional Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two lower semicontinuous functionals, with Ψ coercive and $\Phi(0_E) = \Psi(0_E) = 0$. Further, set*

$$J_\mu := \mu\Psi + \Phi.$$

Then for each $\sigma > \inf_{u\in X}\Psi(u)$ and each μ satisfying

$$\mu > -\frac{\inf_{u\in\Psi^{-1}((-\infty,\sigma])}\Phi(u)}{\sigma}$$

the restriction of J_μ to $\Psi^{-1}((-\infty,\sigma))$ has a global minimum.

See [27,25,26] for related abstract critical points results. We also mention the monograph [18] for some topics on variational methods adopted in this paper and [1] for general facts on finite difference equations.

3. The main result

By [6, Lemma 4] one has that

$$\|u\|_\infty := \max_{k\in\mathbb{Z}[1,T]}|u(k)| \leqslant \frac{1}{\kappa}\|u\| \tag{1}$$

for every $u \in H$, where

$$\kappa := \begin{cases} \left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]^{1/p} & \text{if } T \text{ is even,} \\ \frac{2}{(T+1)^{(p-1)/p}} & \text{if } T \text{ is odd.} \end{cases}$$

Remark 3.1. Note that

$$\left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]^{-1/p} < \frac{(T+1)^{(p-1)/p}}{2}.$$

Indeed, since the continuous function $\theta : (0, T+1) \to \mathbb{R}$ defined by

$$\theta(s) := \frac{1}{(T-s+1)^{p-1}} + \frac{1}{s^{p-1}}$$

attains its minimum $\frac{2^p}{(T+1)^{p-1}}$ at $s = \frac{T+1}{2}$, one has

$$\frac{2^p}{(T+1)^{p-1}} < \theta(T/2).$$

Then

$$\frac{2}{(T+1)^{(p-1)/p}} < \left[\left(\frac{2}{T}\right)^{p-1} + \left(\frac{2}{T+2}\right)^{p-1}\right]^{1/p} = \theta(T/2)^{1/p}$$

and the conclusion is achieved.

Set

$$F_k(\xi) := \int_0^\xi f(k,s)ds$$

for every $k \in \mathbb{Z}[1,T]$ and $\xi \in \mathbb{R}$.

With the above notations our result reads as follows.

Theorem 3.2. *Let $f : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that there exists $\varepsilon > 0$ such that*

$$\frac{\sum_{k=1}^{T} \max_{|\xi|\leqslant\varepsilon} F_k(\xi)}{\varepsilon^p} < \frac{\kappa^p}{p}. \tag{2}$$

Then problem (D_f) has at least one solution such that $\|u\|_\infty < \varepsilon$.

Proof. Let us apply Theorem 2.1 by choosing $E := H$, and

$$\Phi(u) := -\sum_{k=1}^{T} F_k(u(k)), \quad \Psi(u) := \|u\|^p$$

for every $u \in E$.

Taking $\sigma := \kappa^p \varepsilon^p$, clearly $\sigma > \inf_{u\in E}\Psi(u)$. Moreover, let us estimate from the above the following quantity

$$\varphi(\sigma) := \frac{\sup_{u\in\Psi^{-1}((-\infty,\sigma])}\Phi(u)}{\sigma}.$$

Inequality (1) yields

$$\Psi^{-1}((-\infty,\sigma]) \subseteq \{u \in E : \|u\|_\infty \leqslant \varepsilon\}.$$

Thus one has that

$$\varphi(\sigma) \leqslant \frac{\sum_{k=1}^{T} \max_{|\xi|\leqslant\varepsilon} F_k(\xi)}{\kappa^p \varepsilon^p}.$$

Hence it follows, by (2), that

$$\frac{\sup_{u\in\Psi^{-1}((-\infty,\sigma])}\sum_{k=1}^{T} F_k(u(k))}{\sigma} < \frac{1}{p},$$

that is,

$$\frac{1}{p} > -\frac{\inf_{u\in\Psi^{-1}((-\infty,\sigma])}\Phi(u)}{\sigma}.$$

Therefore, the assertion of Theorem 2.1 follows and the existence of one solution $u \in \Psi^{-1}((-\infty,\sigma))$ to our problem is established. $\square$

Remark 3.3. If in Theorem 3.2 the function f is nonnegative, hypothesis (2) assumes a simpler form

$$\frac{\sum_{k=1}^{T} F_k(\varepsilon)}{\varepsilon^p} < \frac{\kappa^p}{p}.$$

Moreover, if for some $\bar{k} \in \mathbb{Z}[1,T],\ f(\bar{k},0) \neq 0$, the obtained solution is clearly non-zero.

4. A parametric case

In this section we shall study the following discrete parametric problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \alpha f(k,u(k)), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{D_α^f}$$

where α is a real positive parameter.

For our goal, in order to obtain positive solutions to problem (D_α^f), i.e. $u(k) > 0$ for each $k \in \mathbb{Z}[1,T]$, we shall need the following consequence of the strong comparison principle, see [3, Lemma 2.3].

Lemma 4.1. *If*

$$-\Delta(\phi_p(\Delta u(k-1))) \geqslant 0, \quad \forall k \in \mathbb{Z}[1,T],$$
$$u(0) \geqslant 0, \quad u(T+1) \geqslant 0,$$

then either $u > 0$ in $\mathbb{Z}[1,T]$, or $u \equiv 0$.

Moreover, let $\lambda_{1,p},\ \varphi_1 > 0$ be the first eigenvalue and eigenfunction of the problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda\phi_p(u(k)), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0. \end{cases} \tag{$\mathrm{D}_{\lambda,p}$}$$

As observed in [3], the following variational characterization

$$\lambda_{1,p} = \min_{E\setminus\{0_H\}} \frac{\sum_{k=1}^{T+1}|\Delta u(k-1)|^p}{\sum_{k=1}^{T}|u(k)|^p} \tag{3}$$

holds. Taking into account the above facts, an important consequence of Theorem 3.2 is the following.

Theorem 4.2. *Let $f : \mathbb{Z}[1,T] \times [0,+\infty) \to [0,+\infty)$ be a continuous function satisfying the following hypotheses:*

$$\lim_{\xi\to+\infty} \frac{\sum_{k=1}^{T} F_k(\xi)}{\xi^p} = 0$$

and

$$\gamma_k := \liminf_{\xi\to 0^+} \frac{F_k(\xi)}{\xi^p} > 0$$

for every $k \in \mathbb{Z}[1,T]$. Then for every

$$\alpha > \frac{\lambda_{1,p}}{p\min_{k\in\mathbb{Z}[1,T]}\gamma_k}$$

problem (D_α^f) has at least one positive solution.

Proof. Let α be as in the conclusion, and define

$$\widetilde{f}(k,t) := \begin{cases} f(k,t) & \text{if } t \geqslant 0, \\ f(k,0) & \text{if } t < 0 \end{cases}$$

for every $k \in \mathbb{Z}[1,T]$. Consider now the following problem

$$\begin{cases} -\Delta\big(\phi_p(\Delta u(k-1))\big) = \alpha\widetilde{f}(k,u(k)), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0. \end{cases} \tag{$\mathrm{D}_\alpha^{\widetilde{f}}$}$$

By Lemma 4.1, every non-zero solution of problem $(\mathrm{D}_\alpha^{\widetilde{f}})$ is positive. Furthermore, every positive solution of $(\mathrm{D}_\alpha^{\widetilde{f}})$ also solves our initial problem (D_α^{f}). Now, since

$$\lim_{\xi\to+\infty} \frac{\sum_{k=1}^{T} F_k(\xi)}{\xi^p} = 0,$$

there exists $\varepsilon > 0$ such that

$$\frac{\sum_{k=1}^{T} F_k(\varepsilon)}{\varepsilon^p} < \frac{\kappa^p}{p}.$$

Hence, bearing in mind Remark 3.3, condition (2) of Theorem 3.2 holds.

Thus problem (D_α^{f}) admits a solution $u_\alpha \in H$ with $\|u_\alpha\| < \varepsilon$. In conclusion, we shall prove that 0_H is not a local minimum of the functional

$$J_\alpha(u) := \frac{1}{p}\sum_{k=1}^{T+1}|\Delta u(k-1)|^p - \alpha\sum_{k=1}^{T}\int_0^{u(k)} f(k,t)dt$$

i.e. the obtained solution u_α is non-zero.

For this purpose, let us observe that the first eigenfunction $\varphi_1 \in H$ is positive and it follows by (3) that

$$\|\varphi_1\|^p = \lambda_{1,p}\sum_{k=1}^{T}\varphi_1(k)^p. \tag{4}$$

Since

$$\gamma_k > \min_{k\in\mathbb{Z}[1,T]} \gamma_k > \frac{\lambda_{1,p}}{p\alpha}$$

for every $k \in \mathbb{Z}[1,T]$, there exists $\delta > 0$ such that

$$F_k(\xi) > \frac{\lambda_{1,p}}{p\alpha}\xi^p \tag{5}$$

for every $k \in \mathbb{Z}[1,T]$ and $\xi \in (0,\delta)$.

Hence, we can define $\theta_\zeta(k) := \zeta\varphi_1(k)$, for every $k \in \mathbb{Z}[0,T+1]$, where

$$\zeta \in \Lambda_\delta := \left(0, \frac{\delta}{\max\limits_{k\in\mathbb{Z}[1,T]} \varphi_1(k)}\right).$$

Taking into account (5) and (4), we easily get

$$\alpha\sum_{k=1}^{T} F_k(\theta_\zeta(k)) > \frac{\lambda_{1,p}}{p}\sum_{k=1}^{T}\theta_\zeta(k)^p = \frac{1}{p}\|\theta_\zeta\|^p,$$

that is,

$$J_\alpha(\theta_\zeta) = \frac{1}{p}\|\theta_\zeta\|^p - \alpha\sum_{k=1}^{T} F_k(\theta_\zeta(k)) < 0$$

for every $\zeta \in \Lambda_\delta$. The proof is thus complete. □

Remark 4.3. In Theorem 4.2, looking at the behavior of the function

$$h(\xi) := \frac{\sum_{k=1}^{T} F_k(\xi)}{\xi^p}, \quad (\forall \xi > 0)$$

at infinity, the existence of one positive solution has been proved. On the other hand, if the function $f(k,\cdot) : \mathbb{R} \to \mathbb{R}$ has a s-sublinear potential F_k with $s < p$, for every $k \in \mathbb{Z}[1,T]$, the behavior at zero of the map

$$\chi(\varepsilon) := \frac{\sum_{k=1}^{T} \max_{|\xi|\leqslant\varepsilon} F_k(\xi)}{\varepsilon^p}, \quad (\forall \varepsilon > 0)$$

influences the existence of multiple solutions. More precisely, requiring that

$$\lim_{\varepsilon\to 0^+} \chi(\varepsilon) = 0 \tag{6}$$

by using variational arguments, one can prove that there exists a real interval of parameters Λ such that, for every $\alpha \in \Lambda$, the problem $(\mathrm{D}^{\mathrm{f}}_{\alpha})$ admits at least three solutions. If, instead of (6), we assume that

$$\chi(c) < \frac{2^{p-1}}{(T+1)^{p-1}}\left(h(d) - \frac{c^p}{d^p}\chi(c)\right) \tag{7}$$

for some positive constants $c < d$, then for every

$$\alpha \in \left]\frac{2}{p\left(h(d) - \frac{c^p}{d^p}\chi(c)\right)}, \frac{2^p}{p\chi(c)(T+1)^{p-1}}\right[,$$

there exist at least three distinct solutions of the problem $(\mathrm{D}^{\mathrm{f}}_{\alpha})$. Clearly condition (7) is technical and quite involved. Finally, we also note that a more precise result can be obtained if T is even.

Remark 4.4. It is easy to see that Theorem 1.1 in Introduction is a consequence of Theorem 4.2 bearing in mind that the first eigenvalue of the problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda u(k), & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{D_λ}$$

is given by

$$\lambda_1 := 4\sin^2\left(\frac{\pi}{2(T+1)}\right),$$

see, for instance, [5, p. 150] and [28]. More precisely, as is well-known, the eigenvalues λ_k, for $k \in \mathbb{Z}[1,T]$, of problem (D_λ) are exactly the eigenvalues of the positive-definite matrix

$$A := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{T\times T}.$$

Thus it follows that

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2(T+1)}\right), \quad \forall k \in \mathbb{Z}[1,T].$$

A direct application of this result yields the following.

Example 4.5. For every

$$\alpha \in (\lambda_1, +\infty),$$

the following second-order discrete problem

$$\begin{cases} -\Delta^2 u(k-1) = \alpha \frac{u(k)}{1+u(k)^2}, & \forall k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{S_α}$$

has at least one positive solution.

Remark 4.6. In Example 4.5, for every α sufficiently large, our approach ensures the existence of at least one positive solution $u_\alpha \in H$ for the discrete problem (S_α). A more delicate problem is to find a concrete expression of the function u_α that one may hope to be exploited by numerical methods.

Remark 4.7. We refer to the paper of Galewski and Orpel [10] for several multiplicity results on discrete partial difference equations. See also the papers [11,16,17,20–22] for recent contributions to discrete problems.

Acknowledgments

This paper was written when the first author was visiting professor at the University of Ljubljana in 2013. He expresses his gratitude to the host institution for warm hospitality. The manuscript was realized within the auspices of the GNAMPA Project 2013 entitled *Problemi non-locali di tipo Laplaciano frazionario* and the SRA Grants P1-0292-0101 and J1-5435-0101. The authors warmly thank the anonymous referees for their useful comments on the manuscript.

References

[1] R.P. Agarwal, Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, Basel, 2000.
[2] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, Nonlinear Anal. 58 (2004) 69–73.
[3] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, Adv. Differ. Equ. 2005 (2) (2005) 93–99.
[4] D. Bai, Y. Xu, Nontrivial solutions of boundary value problems of second-order difference equations, J. Math. Anal. Appl. 326 (2007) 297–302.
[5] C. Bereanu, J. Mawhin, Existence and multiplicity results for nonlinear second order difference equations with Dirichlet boundary conditions, Math. Bohem. 131 (2006) 145–160.
[6] A. Cabada, A. Iannizzotto, S. Tersian, Multiple solutions for discrete boundary value problems, J. Math. Anal. Appl. 356 (2009) 418–428.
[7] S.S. Cheng, Partial Difference Equations, Taylor & Francis, London, 2003.
[8] J. Chu, D. Jiang, Eigenvalues and discrete boundary value problems for the one-dimensional p-Laplacian, J. Math. Anal. Appl. 305 (2005) 452–465.
[9] M. Galewski, S. Głb, On the discrete boundary value problem for anisotropic equation, J. Math. Anal. Appl. 386 (2012) 956–965.
[10] M. Galewski, A. Orpel, On the existence of solutions for discrete elliptic boundary value problems, Appl. Anal. 89 (2010) 1879–1891.
[11] M. Galewski, J. Smejda, On variational methods for nonlinear difference equations, J. Comput. Appl. Math. 233 (11) (2010) 2985–2993.
[12] J. Henderson, H.B. Thompson, Existence of multiple solutions for second order discrete boundary value problems, Comput. Math. Appl. 43 (2002) 1239–1248.
[13] M. Imbesi, G. Molica Bisci, Some existence results for partial discrete problems with Dirichlet boundary conditions, preprint.
[14] L. Jiang, Z. Zhou, Existence of nontrivial solutions for discrete nonlinear two point boundary value problems, Appl. Math. Comput. 180 (2006) 318–329.
[15] L. Jiang, Z. Zhou, Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations, Adv. Differ. Equ. 2008 (2008) 1–10.
[16] A. Kristály, M. Mihăilescu, V. Rădulescu, Discrete boundary value problems involving oscillatory nonlinearities: small and large solutions, J. Differ. Equ. Appl. 17 (2011) 1431–1440.
[17] A. Kristály, M. Mihăilescu, V. Rădulescu, S. Tersian, Spectral estimates for a nonhomogeneous difference problem, Commun. Contemp. Math. 12 (6) (2010) 1015–1029.

[18] A. Kristály, V. Rădulescu, Cs. Varga, Variational principles in mathematical physics, geometry, and economics: qualitative analysis of nonlinear equations and unilateral problems, Encyclopedia of Mathematics and its Applications, vol. 136, Cambridge University Press, Cambridge, 2010.
[19] H. Liang, P. Weng, Existence and multiple solutions for a second-order difference boundary value problem via critical point theory, J. Math. Anal. Appl. 326 (2007) 511–520.
[20] N. Marcu, G. Molica Bisci, Existence and multiplicity results for nonlinear discrete inclusions, Electron. J. Differ. Equ. (2012) 1–13.
[21] G. Molica Bisci, D. Repovš, On some variational algebraic problems, Adv. Nonlinear Anal. 2 (2013) 127–146.
[22] G. Molica Bisci, D. Repovš, Nonlinear algebraic systems with discontinuous terms, J. Math. Anal. Appl. 398 (2013) 846–856.
[23] G. Molica Bisci, D. Repovš, On sequences of solutions for discrete anisotropic equations, Expo. Math. (in press), http://dx.doi.org/10.1016/j.exmath.2013.12.001.
[24] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401–410.
[25] B. Ricceri, Well-posedness of constrained minimization problems via saddle-points, J. Global Optim. 40 (2008) 389–397.
[26] B. Ricceri, Nonlinear eigenvalue problems, in: D.Y. Gao, D. Motreanu (Eds.), Handbook of Nonconvex Analysis and Applications, International Press, 2010, pp. 543–595.
[27] B. Ricceri, A new existence and localization theorem for Dirichlet problem, Dyn. Syst. Appl. 22 (2013) 317–324.
[28] J.T. Scheick, Linear Algebra with Applications, Mathematics & Statistics Series, McGraw-Hill International Editions, 1997.
[29] G. Zhang, W. Zhang, S. Liu, Existence of 2^n nontrivial solutions for a discrete two-point boundary value problems, Nonlinear Anal. 59 (2004) 1181–1187.