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ON 1-CYCLES AND THE FINITE DIMENSIONALITY OF HOMOLOGY 4-MANIFOLDS

W. J. R. MITCHELL, D. REPOVŠ and E. V. ŠČEPIN

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1. INTRODUCTION

IN HIS remarkable paper [12] A. N. Dranišnikov gave a method for constructing for every $n \geq 3$ examples of infinite-dimensional compacta (i.e. compact metric spaces) X_n with integral cohomological dimension $\text{c-dim}_{\mathbb{Z}} X_n = n$. It follows by a well-known result of R. D. Edwards [34] that there are therefore n-dimensional compacta Y_n and cell-like surjections $f_n: Y_n \to X_n$, i.e. for every $x \in X_n$, the pre-image $f_n^{-1}(x)$ has trivial shape. By the Nöbelung-Pontrjagin embedding theorem [29] there exist embeddings $\phi_n: Y_n \to \mathbb{R}^{2n+1}$ which in turn yield upper semicontinuous decompositions G_n of $\mathbb{R}^{2n+1}$, with non-degeneracy set given by $\{\phi_n f_n^{-1}(x) | x \in X_n\}$, whose quotient spaces $\mathbb{R}^{2n+1}/G_n$ contain X_n and so are infinite dimensional. Thus cell-like maps can raise dimension on manifolds of dimensions 7 and above.

Such phenomena are impossible in $\mathbb{R}^q$ for $q \leq 3$. Cell-like images of topological q-manifolds are always $\mathbb{Z}$-homology q-manifolds [36], and for these cohomological and covering dimensions agree if $q \leq 3$; for $q \leq 2$ it is classical that homology q-manifolds are topological manifolds [37], for $q = 3$ see [35]. Recently J. Dydak and J. J. Walsh [16] have shown that there exists an infinite-dimensional compactum X with $\text{c-dim}_{\mathbb{Z}} X = 2$. The preceding argument shows that cell-like maps can also raise dimension on $\mathbb{R}^5$ and $\mathbb{R}^6$. For more on the cell-like mapping problem and its history, see the survey [25].

We are interested in dimension four, the remaining unsettled case of the cell-like mapping problem. A. N. Dranišnikov and E. V. Ščepin conjectured [15] that cell-like maps cannot raise dimension on 4-manifolds. As evidence for this conjecture we prove:

THEOREM 1.1. *Let X be a $\mathbb{Z}$-homology 4-manifold. Then* $\dim X < \infty$ *(equivalently* $\dim X = 4$*) if and only if for some* $n \geq 3$, *X has the disjoint Pontrjagin n-tuples property.*

COROLLARY 1.2. *Let M^4 be a topological 4-manifold and $f: M \to X$ be a proper cell-like onto map. Then* $\dim X < \infty$ *if and only if for some* $n \geq 3$, *X has the disjoint Pontrjagin n-tuples property.*

Remarks. In general a $\mathbb{Z}$-homology manifold need not be finite-dimensional. For example the spaces $\mathbb{R}^{2n+1}/G_n$ mentioned above are infinite-dimensional $\mathbb{Z}$-homology $(2n+1)$-manifolds.

In the first version of this paper, written in 1988 [26], Theorem 1.1 was obtained with the additional restriction that X should be locally 1-connected with respect to singular homology (lc_s^1). This property is automatic for X as in 1.2 [9]. As we remark in Section 6, it is possible to state the theorem without mentioning the Pontrjagin n-tuples property or the

Pontrjagin disc. However the proof makes heavy use of these concepts and so it is natural to state the theorem using them†.

A metric space X is said to have the *disjoint Pontrjagin n-tuples property*, denoted dd_n, if for every $\varepsilon > 0$ and every collection of maps $f_1, f_2, \ldots, f_n: \mathbb{D}^2 \to X$ of the Pontrjagin disc $\mathbb{D}^2$ into X, there exist maps $g_1, g_2, \ldots, g_n: \mathbb{D}^2 \to X$ such that (i) for every i, $d(f_i, g_i) < \varepsilon$, and (ii) $\bigcap_{i=1}^n g_i(\mathbb{D}^2) = \varnothing$. The definition of the Pontrjagin disc is given in section 3. In brief, $\mathbb{D}^2$ is obtained from the standard 2-cell by repeatedly subdividing and replacing the interior of each 2-simplex by a small punctured torus. As will be shown later, Daverman's disjoint triples property DD_3 [9] (which has the same definition as dd_3 except that $\mathbb{D}^2$ is replaced by the standard 2-cell) implies dd_3. Hence we obtain the following corollary, which was originally obtained by D. J. Garity [18; Proposition 1]. Note that our results also show that the hypothesis of finite-dimensionality in Theorem 1 of [18] is unnecessary.

COROLLARY 1.3. [18] *Let G be an upper semi-continuous cell-like decomposition of a topological 4-manifold M and suppose that the quotient space M/G has the property DD_3. Then M/G has (covering) dimension 4.*

On the other hand the ghastly 4-dimensional examples of R. J. Daverman and J. J. Walsh [10] do not have DD_3, whereas Theorem 5.3 below implies that they do possess dd_3. The conjecture of Dranišnikov and Ščepin is, by 5.3, clearly equivalent to the following question.

QUESTION 1.4. *Does every cell-like quotient of a topological 4-manifold possess property dd_3?*

The idea of our proof of finite-dimensionality of an X satisfying dd_n is roughly as follows. By duality, a subset A of a homology 4-manifold has cohomological dimension at most 1 if all "small" 1-cycles in its complement $X - A$ are nullhomologous in (a small subset of) $X - A$. The homology theory here is Borel–Moore, in which the definition of cycles has no clear geometric meaning. We prove that in a locally connected space, Borel–Moore 1-cycles are represented by maps in of circles. We also show that if a map of a circle represents a trivial 1-cycle in a space which is locally 1-connected with respect to Borel–Moore homology, then it extends to a map of the Pontrjagin disc. (These two results, and their proofs, are of independent interest, involving new applications of the Menger universal curve and of an argument due to W. Hurewicz.) Now under dd_n it is easy to show that there is a dense *finite-dimensional* set A of images of the Pontrjagin disc. By the above results about 1-cycles, it follows easily from the special form of A that the complement $X - A$ has cohomological dimension—and so covering dimension—at most 1. Hence X is the union of two finite-dimensional pieces, and so finite-dimensional by the sum theorem of dimension theory.

2. PRELIMINARIES

All spaces will be separable metric spaces, with metric denoted by d, and, except for function spaces, they will also be locally compact. By $N_\varepsilon(K)$ we denote the open ε-

† The disjoint Pontrjagin n-tuples property can be viewed as specifying a degree of homological general position. J. J. Walsh has recently informed us that, adopting the latter point of view, there is a comparable analysis in each dimension, leading to the result that homology n-manifolds which satisfy homological general position are finite dimensional. Details will appear in *Homological General Position* and the *Finite Dimensionality of Homology Manifolds* by J. J. Walsh.

neighborhood of $K \subset X$, i.e. $N_\varepsilon(K) = \{x \in X \mid d(x, K) < \varepsilon\}$. By I we denote the unit interval $[0, 1]$, while S^k and B^k denote the k-dimensional sphere and closed ball respectively. The Hilbert cube $\Pi_1^\infty[0, 1]$ with usual metric is denoted by I^∞. We will use a variety of homology theories (explained in the next paragraph), while cohomology will be Aleksandrov-Čech-sheaf, denoted by H^*, or by H_c^* if compact supports are used. A tilde will denote reduced homology and cohomology. Coefficients will be usually the integers $\mathbb{Z}$, in which case they will be omitted from notation.

We shall consider four homology theories, Čech (denoted by ${}_CH_i$), Steenrod–Sitnikov with compact supports (denoted by ${}_EH_i$), Borel–Moore (for which we reserve no special symbol), and singular (denoted by ${}_sH_i$). The subscript E in ${}_EH_i$ stands for "exact", and we follow the treatment of E. G. Skljarenko [31], in which a version of Steenrod–Sitnikov homology called *exact homology* is developed. A natural transformation between two homology theories ${}_AH_i$ and ${}_BH_i$ will be denoted by $T_{A,B}$. There are well-known natural transformations

$$T_{s,A}\colon {}_sH_i \to {}_AH_i \quad \text{and} \quad T_{A,C}\colon {}_AH_i \to {}_CH_i$$

where ${}_AH_i$ is any homology theory. If $X = \overleftarrow{\lim}\, K_j$ is an inverse limit of polyhedra K_j, then for every i, there is a short exact sequence

$$0 \to \overleftarrow{\lim}{}^1\, H_{i+1}(K_j) \to {}_EH_i(X) \xrightarrow{T_{E,C}} {}_CH_i(X) \to 0. \tag{2.1}$$

When the coefficients in the above homology theories are the integers, Steenrod–Sitnikov homology coincides with Borel–Moore homology, but in general, for example with rational coefficients, this is not so [31]. Borel–Moore theory is an appropriate one to use in the definition of homology manifold, since it is the theory which allows the most natural formulation and proof of the various duality theorems; for a development of Borel–Moore theory, see G. E. Bredon's book [7]. We shall use without further comment the coincidence in our circumstances of Borel–Moore and Steenrod–Sitnikov "exact" homology. We recall that for general spaces, singular theory behaves badly, while Čech theory fails to be exact.

We say that a locally compact metric space X is *locally connected up to dimension n with respect to a homology theory* ${}_AH_i$, denoted by lc_A^n, if for all open sets U and points x in U, there exists an open set V such that $x \in V \subset U$ and for all $j \leq n$, $i_*\colon {}_A\tilde{H}_j(V) \to {}_A\tilde{H}_j(U)$ is zero. (${}_A\tilde{H}_j$ denotes reduced homology.) Similarly we say X is *locally homotopy n-connected*, denoted by LC^n, if a similar conclusion holds with ${}_A\tilde{H}_j$ replaced by the homotopy groups π_j.

A set $Z \subset X$ is *locally homologically k-co-connected, with respect to Steenrod–Sitnikov* (= *Borel–Moore*) *homology*, denoted lcc_E^k, if for all open neighborhoods U of an arbitrary point $x \in X$, there exists a smaller open neighborhood V of x such that the map i_*: ${}_E\tilde{H}_*(V - Z) \to {}_E\tilde{H}_*(U - Z)$ is zero for $* \leq k$, and $V - Z \neq \varnothing$.

A space has *finite* (*integral*) *cohomological dimension* N if for all open sets U, we have $H_c^{N+1}(U) = 0$, but for no smaller N is this true (here cohomology is with compact supports). We write $c\text{-}\dim_{\mathbb{Z}} X = N$.

A space X is said to be a *$\mathbb{Z}$-homology n-manifold*, denoted n-hm, if

(i) X has finite (integral) cohomological dimension, and
(ii) for every $x \in X$, ${}_EH_*(X, X - x) \cong {}_EH_*(\mathbb{R}^n, \mathbb{R}^n - 0)$.

Here ${}_EH$ denotes Borel–Moore homology—see previous paragraph for further details. Note that (i) and (ii) imply that

(iii) X is cohomologically locally connected (clc^∞) [19, 24] and that for each neighborhood U of an arbitrary point $x \in X$ there exists a smaller neighborhood V such that $j^*\colon H_c^*(V) \to H_c^*(U)$ has a finitely generated image, and also that

(iv) the orientation sheaf of X is locally constant [8],

(v) the cohomological dimension of X is *exactly* n [24, 32],

and finally, and crucially for our purposes,

(vi) X is lc_E^∞ [5].

A compact set C is said to be *cell-like* if it has the shape of a point. A proper map $f: X \to Y$ is said to be *cell-like* if for every $y \in Y, f^{-1}(y)$ is cell-like. [A map $f: X \to Y$ is said to be *proper* if for any compact subset K of $Y, f^{-1}(K)$ is compact.] A map $f: X \to Y$ is said to be *one-to-one over* $A \subset Y$ if $f|f^{-1}(A): f^{-1}(A) \to A$ is a bijection. For metric spaces X and Y we shall use the following notation: $\mathscr{C}(X, Y) = \{f: X \to Y | f \text{ is continuous}\}$ and $\mathscr{C}_k(X, Y) = \{f \in \mathscr{C}(X, Y) | \text{ for every } y \in Y, \operatorname{card} f^{-1}(y) \leq k\}$. All spaces of functions will carry the usual sup-norm metric d given by $d(f, g) = \sup\{d(f(x), g(x)) | x \in X\}$.

We work with locally compact spaces, but in practice arguments reduce to the case of compact spaces, because the spaces mapped in are all compact. The following scholium explains this. The reader may find it helpful to consider the case when $P(V, U)$ is the property "any two points in V may be joined by an arc in U".

SCHOLIUM 2.2. *Let X be a locally compact space, and $P = P(K, L)$ a property of pairs of subsets of X with $K \subset L$. Suppose further that for all open sets U and all $x \in U$, there exists an open set V such that $x \in V \subset U$ and $P(V, U)$ holds. Then given any relatively compact open sets V, U with $\bar{V} \subset U$, and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, V, U) > 0$ such that for any $x \in \bar{V}$, the property $P(N_\delta(x), N_\varepsilon(x) \cap U)$ holds.*

Demonstration. For each $x \in V$, let W_x be an open set such that $x \in W_x \subset N_\varepsilon(x) \cap U$ and $P(W_x, N_\varepsilon(x) \cap U)$ holds. Then take δ to be a Lebesgue number for the cover $\{W_x\}_{x \in \bar{V}}$ of $\bar{V}$.

3. THE PONTRJAGIN DISC

A key instrument in our work will be a homogeneous (modulo the boundary), 2-dimensional compactum $\mathbb{D}^2$ called the *Pontrjagin disc*. The construction is a variant of the classical procedure of L. S. Pontrjagin [28]. It has been axiomatized by R. F. Williams [38], and has found applications in several more recent papers [21], [22], [1]. The space $\mathbb{D}^2$ is constructed as the inverse limit of an inverse sequence of discs with orientable handles.

We begin the construction by fixing notation. Let F denote a torus, with some fixed triangulation $\tau(F)$, from which the interior of a 2-simplex σ has been removed. Denote the boundary of F by K. Let $\phi = \phi(F, K): F \to \sigma$ be a fixed PL map which is one-to-one over $\partial\sigma$ and such that $\phi^{-1}(\partial\sigma) = K$, and which sends all simplexes of F not meeting K to the barycentre of σ. Finally whenever we glue together two oriented manifolds along the boundary of a 2-simplex, we shall use a fixed orientation reversing PL homeomorphism ψ as the glueing map, so that the result will be an orientable manifold.

Inductively we shall construct an inverse sequence of discs with orientable handles $\{Q_k, p_{k,k+1}\}$, where each Q_k is embedded in $\mathbb{R}^3$, along with triangulations $\tau(Q_k)$ of Q_k with mesh $\tau < \dfrac{1}{k+1}$. To begin let Q_0 be some tame 2-cell in $\mathbb{R}^3$ and $\tau(Q_0)$ some triangulation of it with mesh less than 1 (measured in the standard metric on $\mathbb{R}^3$). To construct Q_{k+1} from Q_k, take the second barycentric subdivision $[\tau(Q_k)]''$ of Q_k and for each 2-simplex $v \in \tau(Q_k)$, let $\lambda_v = \operatorname{St}(\hat{v}, [\tau(Q_k)]'')$ so that $\lambda_v \subset \operatorname{Int} v$. Then define

$$Q_{k+1} = \left(Q_k - \bigcup_v \operatorname{Int} \lambda_v\right) \cup_{\{\psi_v\}} \left(\bigcup_v F_v\right)$$

where ψ_v: $K_v \to \partial\lambda_v$ is a copy of the fixed orientation-reversing homeomorphism chosen above. We may clearly assume that (for some suitable embedding in $\mathbb{R}^3$) diam $F_v < 1/(k+1)$ for every v. Finally choose $\tau(Q_{k+1})$ to be some triangulation of mesh less than $1/(k+1)$ in the PL structure induced on Q_{k+1} by the given triangulations of $(Q_k - \bigcup_v \text{Int}\,\lambda_v)$ and $\bigcup_v F_v$. To define the bonding maps $p_{k,k+1}$: $Q_{k+1} \to Q_k$, take the map induced by the identity on $(Q_k - \bigcup_v \text{Int}\,\lambda_v)$ and by an appropriate copy ϕ_v of ϕ on F_v, i.e. such that $\phi_v|K_v = \psi_v$.

We define the Pontrjagin disc to be

$$\mathbb{D}^2 = \overleftarrow{\lim}_k\{Q_k, p_{k,k+1}\}.$$

We define the *boundary* $\partial\mathbb{D}^2$ of $\mathbb{D}^2$ to be $p_0^{-1}(\partial Q_0)$ and the *interior* of $\mathbb{D}^2$ to be $\mathbb{D}^2 - \partial\mathbb{D}^2$. It is easy to see that the definitions are independent of choices made in the construction, and it follows as in [21] that $\mathbb{D}^2 - \partial\mathbb{D}^2$ is indeed homogeneous (although we shall not need this latter fact). We shall use several times the fact that $\mathbb{D}^2$ is independent of choice of $\tau(Q_0)$. This may be verified, for example by direct construction of maps between inverse sequences using 3.1 below. For every $k > 0$, let $Z_k = p_{k-1,k}^{-1}(([\tau(Q_{k-1})]'')^{(1)})$, i.e. the inverse image under the bonding map $p_{k-1,k}$ of the 1-skeleton of the second derived of Q_{k-1}, and $Z_k^* = p_k^{-1}(Z_k)$, where p_k: $\mathbb{D}^2 \to Q_k$ is the canonical projection. Note that $p_k|Z_k^*$ is injective. Moreover each of the components of $\mathbb{D}^2 - Z_k^*$ is homeomorphic to the interior of $\mathbb{D}^2$.

PROPOSITION 3.1. *Let E be a disc with orientable handles. Then there exists an onto map* f: $(\mathbb{D}^2, \partial\mathbb{D}^2) \to (E, \partial E)$ *such that* $f|\partial\mathbb{D}^2$ *is a homeomorphism.*

Proof. Let E have $p \geq 0$ handles. Choose k large enough that Q_k has at least p handles. Clearly there is an onto map g: $(Q_k, \partial Q_k) \to (E, \partial E)$ such that $g|\partial Q_k$ is a homeomorphism. Then $f = gp_k$ is the required map.

LEMMA 3.2. *Let X be a compact metric space and* f: $\mathbb{D}^2 \to X$ *any map. Then for every* $\varepsilon > 0$ *there exists a finite cover* $\mathscr{C} = \{\mathbb{D}_i^2\}_{i=1}^m$ *of* $\mathbb{D}^2$ *with Pontrjagin discs* $\mathbb{D}_i^2$ *such that*

(i) *for every* $i \neq j$, $\mathbb{D}_i^2 \cap \mathbb{D}_j^2 = \partial\mathbb{D}_i^2 \cap \partial\mathbb{D}_j^2$ *is an arc, a point, or empty,*
(ii) *for every* i, *diam* $f(\mathbb{D}_i^2) < \varepsilon$,
(iii) *The nerve of the covering* $\mathscr{C}$ *(as a simplicial complex) is some 2-dimensional simplicial complex.*

Proof. Given $\varepsilon > 0$, it follows by uniform continuity of f that for sufficiently large k, each component C_i of $\mathbb{D}^2 - Z_k^*$ will map under f to a set of diameter less than ε. The desired cover consists of the closures (in $\mathbb{D}^2$) of each component C_i. Its nerve is Q_k.

We shall refer to such a cover, $\mathscr{C}$ say, of $\mathbb{D}^2$ as a *Pontrjagin cellulation* (*for f of mesh less than* ε). Note that the mesh is measured in X, not in $\mathbb{D}^2$. We remark that there is evidently a theory of Pontrjagin surfaces just like the usual one, in which $\mathbb{D}^2$ replaces the usual 2-cell. For example, just as in the usual proof of the Jordan Curve theorem, for a Pontrjagin subdisc D of $\mathbb{D}^2$ which is a component of $\mathbb{D}^2 - Z_k^*$ for some k, the discs in some sufficiently fine cellulation of $\mathbb{D}^2$ which meet D form a neighborhood of D which is itself homeomorphic to $\mathbb{D}^2$. One may summarize this by saying that $\mathbb{D}^2$ is a self-similar fractal; arbitrarily small pieces are equivalent to the whole space, just as for the familiar 2-cell. We shall exploit this in Section 4.

We shall adopt an obvious terminology for cellulations. By the 0-*skeleton* or *vertices* we mean all the singletons arising as $\mathbb{D}_i^2 \cap \mathbb{D}_j^2$ as in 3.2(i). Similarly by the 1-*skeleton* or *edges* we mean the arcs so arising.

PROPOSITION 3.3. *Let X be a lc^1_s space and $f\colon \mathbb{D}^2 \to X$ an arbitrary map. Then for every $\varepsilon > 0$ there can be found an integer N and a map $f'\colon Q_N \to X$ such that f and $f'p_N$ are ε-close.*

Proof. Given $\varepsilon > 0$ let δ be such that

(i) every subset of $\mathbb{D}^2$ of diameter less than δ maps under f to a subset of diameter less than $\varepsilon/3$.
(ii) for every loop $\alpha\colon S^1 \to f(\mathbb{D}^2)$ of diameter less than δ there exists an extension $\tilde{\alpha}\colon \Sigma \to X$, where Σ is a disc with orientable handles with $\partial\Sigma = S^1$ and diam $\tilde{\alpha}(\Sigma) < \varepsilon/3$.

The existence of such a δ follows easily from compactness and the lc^1_s hypothesis. By lemma 3.2 choose a Pontrjagin cellulation for f of mesh less than δ. For each i, let $\tilde{\alpha}_i\colon \Sigma_i \to X$ be the map guaranteed by (ii) for the loop $f|\partial\mathbb{D}^2_i$. As in the proof of Lemma 3.1 there exist onto maps $\phi_i\colon Q_{n(i)} \to \Sigma_i$. Let $N = \max\{n(i)|1 \le i \le m\}$ and define the required map by $f'|p_N(\mathbb{D}^2_i) = \tilde{\alpha}_i\phi_i p_{N,n(i)}$. By the definition of δ it is clear that $f'p_N$ and f are indeed ε-close.

PROPOSITION 3.4. *Let X be LC^1 and satisfy the disjoint triples property DD_3 [9]. Then X satisfies dd_3.*

Proof (sketch): Given maps $f_i\colon \mathbb{D}^2_i \to X$, $(i = 1, 2, 3)$ and $\varepsilon > 0$, by 3.3 there are maps $f'_i\colon Q_N \to X$ (for some N) such that f_i and $f'_i p_N$ are $\varepsilon/2$-close. By DD_3 and an argument as in Proposition 24.1 of [9], there exist maps $g_i\colon Q_N \to X$ which are $\varepsilon/2$-close to f'_i and satisfy $g_1(Q_N) \cap g_2(Q_N) \cap g_3(Q_N) = \emptyset$. Then the maps $g_i p_N$ verify dd_3.

In developing the properties of $\mathbb{D}^2$, we next recall two well-known facts about singular homology—see [30]. Firstly the Hurewicz homomorpism is onto, so any element $\lambda \in {}_sH_1(X)$ is represented by a map $f\colon S^1 \to X$. Secondly if $\lambda = 0$, then there exists a compact orientable surface N with one boundary component B and a map $F\colon N \to X$ such that $F|B = f$ under some identification of $B = \partial N$ and S^1.

PROPOSITION 3.5. *With the above notation, if $\lambda = 0 \in {}_sH_1(X)$, there exists a map $G\colon \mathbb{D}^2 \to X$ such that $G|\partial\mathbb{D}^2 = f|S^1$ (under some identification of $\partial\mathbb{D}^2$ and S^1). Moreover given any map $F\colon N \to X$ representing a singular nullhomology of f as above, G may be chosen so that there exists a surjection $K\colon \mathbb{D}^2 \to N$ with $G = FK$.*

Proof. Given some singular nullhomology $F\colon N \to X$, by 3.1 there is a surjection $K\colon \mathbb{D}^2 \to N$ which is a homeomorphism over ∂N. Then $G = FK$ yields the desired map.

LEMMA 3.6. *If D_1 and D_2 are Pontrjagin discs such that $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ is an arc, then $D_1 \cup D_2 \cong \mathbb{D}^2$.*

Proof. It is easy to see that if D_1 and D_2 are inverse limits of $\{Q^{(1)}_i\}$ and $\{Q^{(2)}_i\}$ respectively, then $D_1 \cup D_2$ may be constructed inductively as above starting from $Q^{(1)}_1 \cup Q^{(2)}_1$. Since p_1 is $1-1$ on Z^*_1, which contains $\partial\mathbb{D}^2$, $Q^{(1)}_1 \cap Q^{(2)}_1$ is an arc homeomorphic to $\partial D_1 \cap \partial D_2$, and so $Q^{(1)}_1 \cup Q^{(2)}_1$ is a 2-cell. The result follows by the independence of triangulation noted in the construction of $\mathbb{D}^2$.

Definition 3.7. By the Pontrjagin annulus $\mathbb{A}$ we mean the space obtained from $S^1 \times [1, 2]$ by replacing each of the discs $\{\theta \in S^1|0 \le \theta \le \pi\} \times [1, 2]$ and $\{\theta \in S^1|\pi \le \theta \le 2\pi\}$

$\times [1, 2]$ by Pontrjagin discs. We define $\partial\mathbb{A}$ to be the set corresponding to $(S^1 \times 1) \cup (S^1 \times 2)$ in $S^1 \times [1, 2]$, and (when convenient) regard both $S^1 \times 1$ and $S^1 \times 2$ as identified to S^1 via $(\theta, 1) \equiv (\theta, 2) \equiv \theta$. It follows from 3.6 that $\mathbb{A}$ is homeomorphic to the space obtained from $S^1 \times [1, 2]$ by replacing each of the discs $E_j = \{\theta \in S^1 | \theta_{j-1} \leq \theta \leq \theta_j\} \times [1, 2]$, $(0 = \theta_0 < \theta_1 < \ldots < \theta_k = 2\pi)$, by a Pontrjagin disc, for any choice of such θ_j.

Definition 3.8. A locally compact metric space is said to be *locally Pontrjagin 1-connected*, denoted by $1 - pc$, if it is locally connected and if for all open sets U and points $x \in U$, there exists an open set V such that $x \in V \subset U$ and if $f: \partial\mathbb{D}^2 \to V$ is any map, then there exists an extension $F: \mathbb{D}^2 \to U$. If X is compact this implies as usual that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $f: \partial\mathbb{D}^2 \to X$ is any map with diam $f(\partial\mathbb{D}^2) < \delta$, then there exists an extension $F: \mathbb{D}^2 \to X$ with diam $(F(\mathbb{D}^2)) < \varepsilon$.

THEOREM 3.9. *Let X be a locally compact metric space which is locally Pontrjagin 1-connected. Then for all $\kappa > 0$ and $g: S^1 \equiv S^1 \times 1 \to X$ there exists $\eta = \eta(\kappa, g) > 0$ such that given $h: S^1 \equiv S^1 \times 2 \to X$ with $d(g, h) < \eta$, there exists $G: \mathbb{A} \to N_\kappa(g(S^1))$ such that $G|\partial\mathbb{A} = g \cup h$.*

Proof. By Scholium 2.2, we know that given $\kappa > 0$, there exists $\delta(\kappa) > 0$ such that if $f: \partial\mathbb{D}^2 \to X$ is a map with diam $f(\partial\mathbb{D}^2) < \delta$ which takes values in some fixed compact neighborhood of $g(S^1)$, there exists an extension $F: \mathbb{D}^2 \to X$ with diam $F(\mathbb{D}^2) < \kappa$. Suppose given $g: S^1 \equiv S^1 \times 1 \to X$. Pick a sequence $0 = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_k = 2\pi$ such that diam $g([\theta_{j-1}, \theta_j]) < \delta/2$ for $1 \leq j \leq k$. Since X is locally connected, there exists $\mu > 0$ such that any two μ-close points in the chosen compact neighborhood of $g(S^1)$ can be joined by a path of diameter at most $\delta/2$. Let $\eta = \min(\kappa, \mu, \delta/2)$ and suppose also given any $h: S^1 \equiv S^1 \times 2 \to X$ such that $d(h, g) < \eta$. As $g(\theta_j)$ and $h(\theta_j)$ are μ-close for $1 \leq j \leq k$, they may be joined by paths of diameter at most $\delta/2$. Such paths together with h and g define maps of the boundaries ∂E_j of the discs $E_j = \{\theta \in S^1 | \theta_{j-1} \leq \theta \leq \theta_j\} \times [1, 2]$ for $j = 1, \ldots, k$. By construction, each ∂E_j is mapped to a set of diameter no more than $\delta/2 + \delta/2$. Hence replacing (the interior of) each E_j by a Pontrjagin disc, and using $1 - pc$, we can extend these maps to a map G of the new space to X. By the remark at the end of Definition 3.7, this new space is homeomorphic to $\mathbb{A}$, while it is clear that $G|\partial\mathbb{A} = g \cup h$ and $G(\mathbb{A}) \subset N_\kappa(g(S^1))$, as required.

4. REPRESENTATION OF 1-DIMENSIONAL HOMOLOGY CLASSES

In this section we show that under local connectedness hypotheses for the appropriate homology theory, 1-dimensional homology classes and relations between them have singular representations. There are two main ideas. Firstly we use a "resolution" involving the Menger universal curve to show that 1-dimensional homology classes in locally connected spaces have a singular representation. Secondly we revive an old argument of Hurewicz in our "fractal" setting to show that an approximate mapping property implies an exact one. It follows that in Borel–Moore (= Steenrod–Sitnikov) homology, trivial 1-cycles in a homologically locally 1-connected space are represented by maps of Pontrjagin discs.

THEOREM 4.1. *Let X be a connected, locally connected and locally compact metric space. Then the natural transformation $T_{s,E}: {}_sH_1(X) \to {}_EH_1(X)$ is surjective (and hence so is the natural transformation $T_{s,C}: {}_sH_1(X) \to {}_CH_1(X)$).*

Proof. Consider any $z' \in {}_EH_1(X)$. Since we are taking homology with compact supports, we can find compact subsets K and L of X such that $K \subset \text{Int } L$ and $z' \in \text{Im } i_*$: ${}_EH_1(K) \to {}_EH_1(X)$, say $z' = i_*(z)$. There is no harm in assuming L lies in the Hilbert cube I^∞. By a theorem first obtained by R. D. Anderson [2] (see [39] or [11] for a proof) there exists a continuous surjection $\pi: \mu^1 \to I^\infty$ such that for all points $x \in I^\infty$, $\pi^{-1}(x) \cong \mu^1$. Here μ^1 denotes the Menger universal curve. The map $\pi|\pi^{-1}(K): \pi^{-1}(K) \to K$ is a Vietoris map in exact homology in degree zero, i.e. ${}_E\tilde{H}_m(\pi^{-1}(y)) \cong 0$ for $m \leq 0$ and all $y \in K$, by Lemma 4.2 below. Hence by L. A. Nguen's extension of the Vietoris–Begle mapping theorem [27], $\pi_*: {}_EH_1(\pi^{-1}(K)) \to {}_EH_1(K)$ is onto. Choose $\bar{z} \in {}_EH_1(\pi^{-1}(K))$ such that $\pi_*(\bar{z}) = z$. Since $\text{Int } L$ is locally connected (LC^0) and μ^1 1-dimensional, $\pi|\pi^{-1}(K): \pi^{-1}(K) \to K \subset \text{Int } L$ extends by the Kuratowski–Dugundji extension theorem [6] to a map $P: N \to \text{Int } L$, where N is some neighborhood of $\pi^{-1}(K)$ in μ^1. There is a commutative diagram

$$\begin{array}{ccccccc} \pi^{-1}(K) & \xrightarrow{i_1} & N & & & & \\ \downarrow \pi & & \downarrow P & & & & \\ K & \xrightarrow{i_2} & \text{Int } L & \xrightarrow{i_3} & L & \xrightarrow{i_4} & X \end{array}$$

in which the horizontal maps are all inclusions. By Proposition 4.3 below $i_{1*}(\bar{z})$ is singularly induced, i.e. can be written as $T_{s,E}(w)$ where $w \in {}_sH_1(N)$. Then we have

$$\begin{aligned} z' = (i_4 i_3 i_2)_*(z) = (i_4 i_3)_*(i_2)_* \pi_*(\bar{z}) &= (i_4 i_3 P)_*(i_1)_*(\bar{z}) = (i_4 i_3 P)_*(T_{s,E}(w)) \\ &\in (i_4 i_3 P)_*(T_{s,E}({}_sH_1(N)) \\ &\subset (i_4 i_3)_* T_{s,E}({}_sH_1(\text{Int } L)) \subset i_{4*}(T_{s,E}({}_sH_1(L)), \end{aligned}$$

as required, using the naturality of $T_{s,E}$.

LEMMA 4.2. *If μ^k is the Menger universal k-dimensional space, then for all i we have*

$${}_EH_i(\mu^k) \cong {}_CH_i(\mu^k)$$

and

$${}_CH_i(\mu^k) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \prod_1^\infty \mathbb{Z} & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The space μ^k may be written as the inverse limit of a sequence of maps of $(2k+1)$-dimensional PL manifolds M_j such that each bonding map is onto and has a single non-degenerate inverse image, which is homotopy equivalent to a finite wedge of k-spheres [3]. Such spaces are constructed by taking neighborhoods of the dual k-skeleton of finer and finer triangulations starting from an initial I^{2k+1}. It is easy to see that the inverse sequence $\{{}_sH_i(M_j)\}_{j \in \mathbb{N}}$ vanishes except when $i = 0$, where it is constant and isomorphic to $\mathbb{Z}$, and when $i = k$, where it is isomorphic to the inverse sequence

$$\mathbb{Z} \xleftarrow{p_1} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{p_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{p_3} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{p_4} \ldots,$$

where p_n denotes projection on the first n summands. The conclusion follows at once from the sequence 2.1 relating exact and Čech homology, since $\overleftarrow{\lim}^1$ vanishes on sequences with surjective bonding maps.

PROPOSITION 4.3. *If N is an open subset of μ^k, any exact or Čech 1-cycle is singularly induced, i.e. $T_{s,E}({}_sH_1(N)) = {}_EH_1(N)$ and $T_{s,C}({}_sH_1(N)) = {}_CH_1(N)$.*

Proof. If $k > 1$, there is nothing to prove since μ^k is LC^{k-1} and so all groups agree [23], [7; V.11.9].

Next suppose $N = \mu^1$. By 4.2, ${}_EH_1(\mu^1) \cong {}_CH_1(\mu^1)$ and so it suffices to consider ${}_CH_1$. Now in the notation of 4.2, ${}_CH_1(N) \cong \overleftarrow{\lim}^1 {}_sH_1(M_j)$ where $N = \mu^1 = \bigcap_1^\infty M_j$, the M_j being the stages in the usual Menger construction [17; 1.11.5, p. 122]. Here we regard all M_j as subsets of $M_1 = I^3$, with the inherited metric. To produce a singular 1-cycle realizing a given Čech class, it clearly suffices to prove the following: given a PL map f_j: $S^1 \to M_j$ representing $\lambda \in {}_sH_1(M_j)$, and such that $f_j(S^1)$ meets all the cubes of the jth stage of the construction, then there exists a PL map f_{j+1}: $S^1 \to M_{j+1}$ such that $f_{j+1}(S^1)$ meets all the cubes of the $(j + 1)$st stage of the construction, and such that f_{j+1} represents an arbitrary element of $((p_{j+1,j})_*)^{-1}(\lambda) \in {}_sH_1(M_{j+1})$ and $d(f_j, f_{j+1}) < 3^{-j}\sqrt{3}$, the diameter of the cubes used in the jth stage of the construction. Given this statement, it is easy to construct inductively maps f_j converging to a (surjective) map f: $S^1 \to N$ whose singular homology class represents any given element of $\overleftarrow{\lim}\, {}_sH_1(M_j)$.

The above conditions are easy to arrange. The second condition follows if in altering f_j to f_{j+1} we move no point out of the cube C_α of the jth stage of the construction in which it lies. To achieve this simply ensure that each arc of $f_j(S^1) \cap C_\alpha$ is moved (keeping endpoints fixed) within C_α so as to miss the new holes drilled in C_α, at the same time ensuring that we avoid the places on the faces of this and all subsequent cubes where holes are to be drilled. Moreover since $\ker (p_{j+1,j})_*$ is a sum of copies of $\mathbb{Z}$, one corresponding to each hole drilled, and is a direct summand in ${}_sH_1(M_{j+1})$, we can wind $f_i(S^1) \cap C_\alpha$ around the new holes to produce any required element as in the second condition. [We must also ensure inductively that f_{j+1} does meet every cube of the $(j + 1)$st stage of the construction. This poses no problems, one simply runs feelers into any uninvolved cubes.]

Finally if N is open in μ^1, it follows by M. Bestvina's triangulation theorem [3] that N has a cover by closed sets each of which is homeomorphic to μ^1 and which overlap, if at all, in sets homeomorphic to μ^1. By a routine Mayer–Vietoris and induction argument, plus the taking of direct limits, it follows from the above case that for arbitrary unions of such sets, and in particular for N itself, the desired conclusion is true.

THEOREM 4.4. *Let X be a connected, locally compact and lc_E^1 metric space. Suppose f: $S^1 \to X$ is a map such that the corresponding homology class $\lambda \in {}_EH_1(X)$ is trivial. Then there exists an extension of f to $\mathbb{D}^2$, i.e. a map F: $\mathbb{D}^2 \to X$ such that (under some identification of $\partial\mathbb{D}^2$ and S^1), $F|\partial\mathbb{D}^2 = f$.*

Remark. Here and in 4.9, one can replace lc_E^1 by lc_C^1 (local 1-connectedness with respect to Čech homology), without change to the proof.

Proof of 4.4. The proof will employ two subsidiary lemmas. The strategy of the proof is as follows; λ is nullhomologous in some compact set K, which we can regard as being embedded in the Hilbert cube. Then λ is also nullhomologous in arbitrarily close neighborhoods of K. Since these neighborhoods are ANRs, we may take singular homology, and so by 2.5 regard the nullhomology as represented by a map of $\mathbb{D}^2$. The aim is then to use the lc_E^1 condition to push this map into K. To do this, the following definition is required.

Definition 4.5. We say that a locally compact metric space X is *weakly Pontrjagin* 1-*connected*, denoted by $1 - wpc$, if it is locally connected, and if for all relatively compact open sets U and points $x \in U$, there exists an open set V such that $x \in V \subset U$, and such that if $f: \partial \mathbb{D}^2 \to V$ is any map, then there exists an embedding of $\bar{U}$ in the Hilbert cube I^∞ such that for any neighborhood P of $\bar{U}$ in I^∞, there exists an extension of f to $F: \mathbb{D}^2 \to P$.

A routine uniform continuity argument shows that the definition of $1 - wpc$ is independent of the choice of embedding of $\bar{U}$ in the Hilbert cube. Of course such an embedding always exists. Equally it is clear that $1 - pc$ implies $1 - wpc$, since the former condition guarantees extensions to U itself (rather than merely arbitrarily close neighborhoods thereof).

LEMMA 4.6. *Let X be a connected, locally compact and lc_E^1 metric space. Then X is $1 - wpc$.*

Proof. It is well-known that an lc_E^1 space is locally connected. Without loss of generality, suppose $\bar{U}$ lies in the Hilbert cube (where x and U are as in Definition 4.5). Pick V for U as in the definition of lc_E^1. Given $f: \partial \mathbb{D}^2 \to V$, by the Hurewicz homomorphism f determines a class $\mu \in {}_sH_1(V)$. Let P be an arbitrary open neighborhood of $\bar{U}$ in I^∞. The composition

$${}_sH_1(V) \xrightarrow{T_{s,E}} {}_EH_1(V) \xrightarrow{i_*} {}_EH_1(U) \to {}_EH_1(P) \xrightarrow{\cong} {}_sH_1(P)$$

clearly carries μ to the class μ' obtained by applying the Hurewicz homomorphism to f regarded as a map to P. By lc_E^1, we know that i_* is zero, while the last map is an isomorphism (the inverse of $T_{s,E}$) since P is locally contractible and so HLC [7]. Hence $\mu' = 0$ and by Proposition 3.5, there exists a map $F: \mathbb{D}^2 \to P$ such that $F|\partial\mathbb{D}^2 = f$. Clearly F satisfies the requirements for $1 - wpc$.

We now show that with an "extra" $1 - wpc$ hypothesis (in fact not extra, by 4.6), the statement of 4.4 holds.

LEMMA 4.7. *Let X be a connected, locally compact and lc_E^1 metric space which is $1 - wpc$. Suppose $f: S^1 \to X$ is a map such that the corresponding homology class $\lambda \in {}_EH_1(X)$ is trivial. Then there exists an extension of f to $\mathbb{D}^2$, i.e. a map $F: \mathbb{D}^2 \to X$ such that (under some identification of $\partial\mathbb{D}^2$ and S^1), $F|\partial\mathbb{D}^2 = f$.*

Remark. This is essentially due to Hurewicz [20], who applied the self-similar "fractal" structure of the standard cell to prove that a weak LC^n property implies the usual LC^n property. The idea behind the proof is simple and beautiful—repeatedly subdivide the cell into smaller and smaller similar pieces, applying weak local connectedness with tighter and tighter controls to push the map closer and closer to X.

Proof. Since ${}_EH_1$ has compact supports, we may suppose that λ is nullhomologous in some compact subset $L \subset X$. Choose compact subsets M, N such that $L \subset \operatorname{Int} M \subset M \subset \operatorname{Int} N \subset N$. There is no loss of generality in assuming N is a subset of the Hilbert cube I^∞, with inherited metric. By 2.2, we may assume that given $\eta > 0$,

(i) there exists $\theta = \theta(\eta) > 0$ such that if $x, x' \in \operatorname{Int} M$ and $d(x, x') < \theta(\eta)$, then x and x' may be joined by a path in N of diameter less than η.

(ii) there exists $\delta = \delta(\eta) > 0$ such that if $f: \partial\mathbb{D}^2 \to M$ satisfies $\operatorname{diam} f(\partial\mathbb{D}^2) < \delta$, then for any neighborhood P of N in I^∞ there is an extension $F: \mathbb{D}^2 \to P$ such that $\operatorname{diam} F(\mathbb{D}^2) < \eta$.

We require a sublemma.

SUBLEMMA. *Suppose given open sets* $V \subset U$ *of* I^∞ *such that* $U \cap N \subset \text{Int } M$, *and a map* $g\colon \mathbb{D}^2 \to V$ *such that* $g(\partial\mathbb{D}^2) \subset V \cap \text{Int } M$ *and, for any* $\alpha > 0, g|\partial\mathbb{D}^2\colon \partial\mathbb{D}^2 \to V \cap \text{Int } M$ *extends to a map* $G\colon \mathbb{D}^2 \to N_\alpha(U \cap M)$. *Then given* $\beta > 0$, $\gamma > 0$, *there can be found a cellulation* $\mathscr{C} = \mathscr{C}(\beta, \gamma)$ *of* $\mathbb{D}^2$ *and a map* $h = h(\beta, \gamma)\colon \mathbb{D}^2 \to N_\beta(U \cap M)$ *such that*:

(a) $h|\partial\mathbb{D}^2 = g|\partial\mathbb{D}^2$;
(b) *for any cell* $C \in \mathscr{C}$, $h(\partial C) \subset M$;
(c) *for any cell* $C \in \mathscr{C}$, diam $h(C) < \gamma$.

Remark. The hypotheses of the sublemma hold if $V \cap M$ and $U \cap M$ are as in the definition of $1 - wpc$, or if $g|\partial\mathbb{D}^2$ represents a nullhomologous class in $V \cap M$.

Proof of Sublemma. We first define a new map G whose image lies much closer to N. Let $\mu = \frac{1}{3}\theta(\delta(\gamma)/2)$, where the functions δ and θ are as in (i) and (ii) above. Without loss of generality, we may also assume that $\mu < \frac{1}{3}$ dist $(U, N - \text{Int}\, M)$; we remark for future reference that as a result G and the map g defined below both meet N only in a subset of $U \cap \text{Int}\, M$. Choose a map $G\colon \mathbb{D}^2 \to N_\mu(U \cap M)$ as guaranteed by the hypotheses of the Sublemma. Pick a cellulation $\mathscr{C} = \mathscr{C}(\beta, \gamma)$ of $\mathbb{D}^2$ such that for each cell $D \in \mathscr{C}$, diam $G(D) < \mu$ (possible by uniform continuity of G). We will define g inductively on the skeleta of $\mathscr{C}$.

For each vertex v of $\mathscr{C}$, define $g(v) = G(v)$ if $G(v) \in U \cap \text{Int}\, M$; otherwise let $g(v)$ be any point of $U \cap \text{Int}\, M$ within μ of $G(v)$. This defines g on the 0-skeleton of $\mathscr{C}$.

Suppose v and v' are vertices of the same edge e in the 1-skeleton of $\mathscr{C}$, e being a face of $D \in \mathscr{C}$ say. If $G(e) \subset U \cap \text{Int}\, M$, define $g|e = G|e$; otherwise note that

$$d(g(v), g(v')) \le d(g(v), G(v)) + \text{diam } G(D) + d(G(v'), g(v')) < \mu + \mu + \mu \le \theta(\delta(\gamma)/2).$$

Hence we may join $g(v)$ and $g(v')$ in N by a path of diameter $\delta(\gamma)/2$, and we use this path to define g on the edge e. Note that by the above remark, $g(e) \subset U \cap \text{Int}\, M$. By doing this process for all edges, we define g on the 1-skeleton of $\mathscr{C}$.

Now for any cell D of $\mathscr{C}$, diam $g(\partial D)$ is at most twice the diameter of (the image of) an edge, i.e. at most $2.\delta(\gamma)/2 = \delta(\gamma)$. Thus we may complete the definition of g by extending over each cell D of $\mathscr{C}$ by a map equalling the already defined g on ∂D and whose image has diameter less than γ and lies in $N_\beta(N)$, using the $1 - wpc$ property (ii) stated at the start of the proof. By the remark, the extension in fact takes values in $N_\beta(M)$ as required. The properties (a), (b) and (c) claimed in the Sublemma are clear. This proves the Sublemma.

To complete the proof of the main result, we shall apply the sublemma inductively, to produce a sequence of maps converging to the desired map F.

Let $\varepsilon_0 = \frac{1}{10}\text{dist}\,(L, N - \text{Int}\, M)$ and $\varepsilon_k = \frac{1}{4}\varepsilon_{k-1}$ for $k \ge 1$. Let $\delta_k = \delta(\varepsilon_k)$ where δ is the above function. Without loss of generality, $\delta_{k+1} < \delta_k$ for $k \ge 0$.

Inductively we shall construct maps $f_0, f_1, \ldots, f_k\colon \mathbb{D}^2 \to I^\infty$ and cellulations $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_k$ of $\mathbb{D}^2$ such that, for $0 \le i \le k$,

1_i $f_i|\partial\mathbb{D}^2 = f|\partial\mathbb{D}^2$,
2_i f_i sends the 1-skeleton of $\mathscr{C}_i$ to M,
3_i for any cell D of $\mathscr{C}_i$, diam $f_i(D) < \varepsilon_i$,
4_i for any cell D of $\mathscr{C}_i$, diam $f_i(\partial D) < \delta_{i+1}$,

and, for $0 \le i \le k-1$,

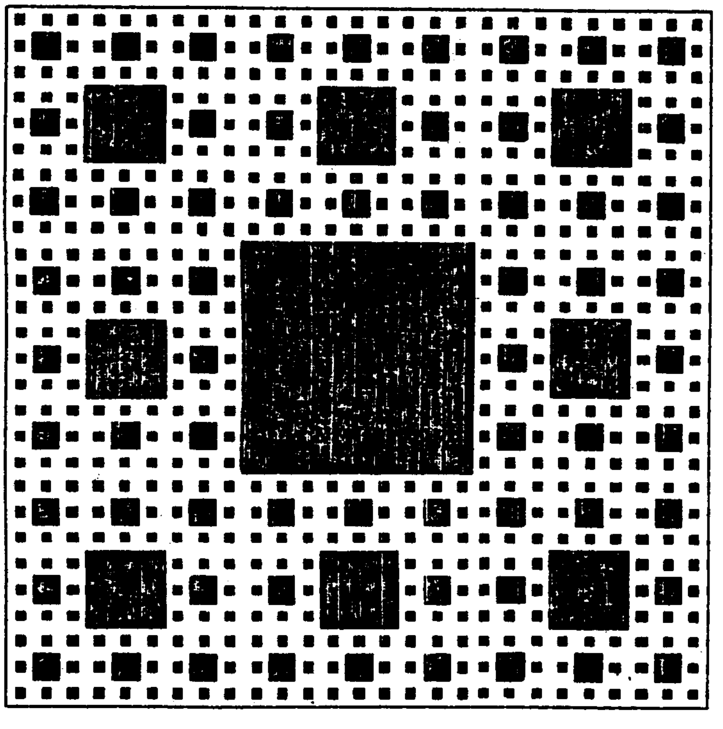
5_i $\mathscr{C}_{i+1}$ refines $\mathscr{C}_i$,
6_i $f_i = f_{i+1}$ on the 1-skeleton of $\mathscr{C}_i$,
7_i $d(f_i, f_{i+1}) < \varepsilon_i$.

To obtain f_0, apply the Sublemma with $U = V = N_{\varepsilon_0}(L)$ and $\beta = \gamma = \min(\varepsilon_0, \delta_1)$. Note that by the definition of ε_0 and the hypothesis on $f: S^1 \to L$, the hypotheses of the Sublemma hold. Inductively given f_i for $i \le k$, we construct f_{k+1} by applying the Sublemma to each cell of $\mathscr{C}_k$. More precisely, given $C \in \mathscr{C}_k$, by 4_k, diam $f_k(\partial C) < \delta_{k+1}$. Thus choosing (for some convenient point $y \in M$) $V = N_{\delta_{k+1}}(y)$ and $U = N_{\varepsilon_k}(y)$, the hypotheses of the Sublemma hold for $f_k|C: C \to V$. Choosing $\beta = \gamma = \min(\varepsilon_{k+1}, \delta_{k+2})$, the Sublemma produces a map on C and a cellulation of C; amalgamating the cellulations, and taking the unions of the maps, for each cell, produces $\mathscr{C}_{k+1}$ and f_{k+1} as required. Properties 1_{k+1} to 6_{k+1} are clear from the construction. Finally it is clear that on any cell C as above, $f_k(C)$ and $f_{k+1}(C)$ both lie in $U = N_{\varepsilon_k}(y)$, from which 7_{k+1} is clear. This completes the inductive construction of the maps f_k.

Clearly (f_n) is a Cauchy sequence in the complete metric space of maps from $\mathbb{D}^2$ to I^∞, and so converges to a map $F: \mathbb{D}^2 \to I^\infty$. By 2 and 3, sup $\{d(f_n(t), M) \mid t \in \mathbb{D}^2\}$ tends to zero as $n \to \infty$, so $F(\mathbb{D}^2) \subset M \subset X$. Clearly $F|\partial\mathbb{D}^2 = f|\partial\mathbb{D}^2$, thanks to the above construction of F from f.

This completes the proof of 4.7, and hence thanks to 4.6 the proof of 4.4 is complete.

Remark. In fact we can work with an apparently slightly weaker condition than $1 - wpc$, in which the "extensions" F of $f: \partial\mathbb{D}^2 \to X$ are not precise extensions, but instead can be chosen so that $f|\partial\mathbb{D}^2$ and $F|\partial\mathbb{D}^2$ are arbitrarily close as maps to X, (the rest of F taking values in $P \subset I^\infty$). The proof that this weaker condition in fact implies $1 - wpc$ uses a similar fractal trick. Since we do not require the result, we merely draw a picture, leaving the interested reader to figure out the necessary epsilonics.



Exactly the same fractal ideas can be used to prove the following.

PROPOSITION 4.8. *Suppose X is a compact locally connected metric space such that for any map* $f: S^1 \to X$ *and any* $\lambda > 0$ *there exists a map* $g: B^2 \to X$ *such that* $d(f, g) < \lambda$ *(where we identify* S^1 *and* ∂B^2*). Then* $\pi_1(X) \cong 1$.

We again leave the proof of this result to the interested reader.
Putting this together, we have

THEOREM 4.9. *Let X be a locally compact,* lc_E^1 *metric space. Then every class in* ${}_EH_1(X)$ *is represented by a map* $f: S^1 \equiv \partial \mathbb{D}^2 \to X$. *Such a map represents the trivial class if and only if f extends to a map* $F: \mathbb{D}^2 \to X$.

Proof. The representation and backward implication for triviality are simply restatements of 4.1 and 4.4. For the forward implication, note that by naturality and the diagram

$$\partial \mathbb{D}^2 \xrightarrow{i} \mathbb{D}^2 \xrightarrow{F} X,$$

it suffices to show that $i_*: {}_EH_1(\partial \mathbb{D}^2) \to {}_EH_1(\mathbb{D}^2)$ is zero. In the inverse sequence $\{Q_n, p_{n,n+1}\}$ defining $\mathbb{D}^2$, we note that $i_*: {}_sH_1(\partial Q_n) \to {}_sH_1(Q_n)$ is zero, and that the two inverse sequences $\{{}_sH_j(Q_n)\}_{n\in\mathbb{N}}$ and $\{{}_sH_j(\partial Q_n)\}_{n\in\mathbb{N}}$ satisfy the Mittag–Leffler condition in all degrees j. Hence by continuity of Čech homology and the $\varprojlim^1$ sequence 2.1 for exact homology, we find $i_* = 0$, as required.

THEOREM 4.10. *Let X be a locally compact metric space. Then the following are equivalent:*

(i) *X is* lc_E^1,
(ii) *X is* lc_C^1 *(locally* 1*-connected with respect to Čech homology),*
(iii) *X is* $1 - pc$ *(locally Pontrjagin* 1*-connected).*

Proof. (*i*) *implies* (*ii*): This is immediate from the sequence 2.1, since the natural map ${}_EH_1(U) \to {}_CH_1(U)$ is onto.

(*ii*) *implies* (*iii*): Local connectivity is immediate. Given x and U as in 3.8, pick V as guaranteed by lc_E^1. Then a map $f: S^1 \to V$ can be regarded as a Borel–Moore 1-cycle in V, and so is nullhomologous in U. Thus by 4.9, f extends to $F: \mathbb{D}^2 \to U$, as required.

(*iii*) *implies* (*i*): Given x and U as in the definition of lc_E^1, pick V as guaranteed by $1 - pc$. By 4.9, any 1-cycle in V is represented by a map $f: S^1 \to V$. By $1 - pc$, this extends to F: $\mathbb{D}^2 \to U$ and by 4.9 again this reveals the original 1-cycle to have been trivial.

5. THE PROOF OF THEOREM 1.1

Firstly we show that dd_n guarantees finite dimensionality. To begin, we split X into two pieces, one of which is always 1-dimensional. Let $\{f_i | i \in \mathbb{N}\}$ be any dense set of maps in $\mathscr{C}(\mathbb{D}^2, X)$; such a set exists since $\mathscr{C}(\mathbb{D}^2, X)$ is separable. Let $A = \bigcup_{i=1}^{\infty} f_i(\mathbb{D}^2)$. Our plan is (roughly) to show that $X - A$ is always 1-dimensional, and that A can be chosen to be finite-dimensional in the presence of dd_n.

Assertion 1. For any closed set $K \subset X - A$, $\dim K \leq 1$.

Proof. Since having covering dimension 1 is the same as having integral cohomological dimension 1 (see [34, Corollary 3.3]), it suffices to show $c\text{-}\dim_Z K \le 1$. We require a lemma.

LEMMA 5.1. *Let K be a closed subset of an n-hm X. Suppose $c\text{-}\dim_Z K \le k$. Then $c\text{-}\dim_Z K < k$ if and only if K is lcc^{n-k-1}*

Proof: *Sufficiency.* By [7: II.15.13] it is enough to show that for all open sets $Q \subset X$ and points $x \in Q \cap K$, there is an open neighborhood $U \cap K$ of x in K (where U is open in X) such that $j^*\colon H_c^k(U \cap K) \to H_c^k(Q \cap K)$ is zero. By local orientability choose an open neighborhood $W \subset Q$ of $x \in X$ such that on W the homology sheaf of X is constant [8]. Since X is clc^∞, by [7: V, Exercise 31] we may find open sets $U \subset V \subset W$ such that $i_*\colon {}_E\tilde{H}_*(U) \to {}_E\tilde{H}_*(V)$ and $i_*\colon {}_E\tilde{H}_*(V) \to {}_E\tilde{H}_*(W)$ are zero. Then for $k < n$ there is a commutative diagram

$$\begin{array}{ccccccc}
{}_E\tilde{H}_{n-k}(W) & \to & {}_EH_{n-k}(W, W-K) & \to & {}_E\tilde{H}_{n-k-1}(W-K) & \to & {}_E\tilde{H}_{n-k-1}(W) \\
\uparrow i_*=0 & & \uparrow \beta & & \uparrow \gamma & & \uparrow \\
{}_E\tilde{H}_{n-k}(V) & \to & {}_EH_{n-k}(V, V-K) & \to & {}_E\tilde{H}_{n-k-1}(V-K) & \to & {}_E\tilde{H}_{n-k-1}(V) \\
\uparrow & & \uparrow \alpha & & \uparrow \delta & & \uparrow i_*=0 \\
{}_E\tilde{H}_{n-k}(U) & \to & {}_EH_{n-k}(U, U-K) & \to & {}_E\tilde{H}_{n-k-1}(U-K) & \to & {}_E\tilde{H}_{n-k-1}(U)
\end{array}$$

arising from the exact sequence of a pair. Further by the collapse of the Poincaré duality spectral sequence [7; V.8], the vertical maps of the relative groups may be identified with the maps

$$H_c^k(U \cap K) \xrightarrow{j^*} H_c^k(V \cap K) \xrightarrow{j^*} H_c^k(W \cap K).$$

Then if K is lcc^{n-k-1}, we may suppose $\delta = 0$ and a diagram chase verifies that $\beta\alpha$ is zero. Hence so is $j^*\colon H_c^k(U \cap K) \to H_c^k(Q \cap K)$.

If $k > n$ the result is trivial since for any subset K of X, $c\text{-}\dim_Z K \le c\text{-}\dim_Z X = n$. Finally if $k = n$, there is no loss of generality in assuming U, V and W are connected. Then ${}_EH_0(U, U-K) \cong \operatorname{Coker}(i_*\colon {}_EH_0(U-K) \to {}_EH_0(U))$ is zero if and only if $U - K \ne \varnothing$. Now by definition of lcc_E^k, we know $U - K \ne \varnothing$.

Necessity. Suppose $c\text{-}\dim_Z K < k$, and that $k < n$. Then the groups in the second column of the above diagram are all zero, and by a similar diagram chase it can be verified that $\gamma\delta = 0$. If $k > n$, there is again nothing to prove, as ${}_EH_j(U, U-K) = 0$ if $j < 0$ [31]. Finally if $k = n$, the result is immediate by the above necessary and sufficient condition for the vanishing of ${}_EH_0(U, U-K)$.

COROLLARY 5.2. *Let K be a closed subset of an n-hm X. Then*

(i) *$c\text{-}\dim_Z K = n$ if and only if* $\operatorname{Int} K \ne \varnothing$.
(ii) *$c\text{-}\dim_Z K = n-1$ if and only if* $\operatorname{Int} K = \varnothing$ *and K separates X locally at some point.*
(iii) *$c\text{-}\dim_Z K \le n-3$ if and only if* $\operatorname{Int} K = \varnothing$, *$K$ does not separate X locally anywhere, and given any open set U, there exists an open set $V \subset X$ with $V \subset U$ and i_*: ${}_EH_1(V-K) \to {}_EH_1(U-K)$ zero.*

This is immediate from the proof of 5.1. To complete the proof of Assertion 1, we use Corollary 5.2 (iii) with $n = 4$. Note that by choice of A, $\operatorname{Int} K = \varnothing$ and K does not separate X locally anywhere, since any image of $\mathbb{D}^2$ is path connected. We claim that for open sets U as in the above diagram, $\alpha = i_*\colon {}_EH_1(U-K) \to {}_EH_1(U)$ is injective. From this the required

$lcc_E^{\frac{1}{2}}$ condition follows. Indeed in the diagram

$$\begin{array}{ccc} {}_EH_1(U-K) & \xrightarrow{\alpha} & {}_EH_1(U) \\ \uparrow \beta & & \uparrow \gamma \\ {}_EH_1(V-K) & \rightarrow & {}_EH_1(V), \end{array}$$

where all maps are induced by inclusion, we may use the $lc_E^{\frac{1}{2}}$ condition to ensure that $\gamma = 0$. Hence $\alpha\beta = 0$ and so $\beta = 0$ if α is injective.

Suppose then that $\lambda \in \operatorname{Ker} \alpha$. By 4.9, $\lambda \in \operatorname{Ker} \alpha$ gives rise to a map $F\colon \mathbb{D}^2 \to U$ such that $F(\partial\mathbb{D}^2) \subset U - K$ represents $\lambda \in {}_EH_1(U-K)$. Let $\varepsilon = \min(\operatorname{dist}(F(\mathbb{D}^2), \operatorname{Fr} U), \operatorname{dist}(F(\partial\mathbb{D}^2), K))$. By definition of A, there exists $h\colon \mathbb{D}^2 \to A \subset X - K$ such that $d(h, F) < \varepsilon$ (so $h(\mathbb{D}^2) \subset U$), and $d(h|\partial\mathbb{D}^2, F|\partial\mathbb{D}^2) < \eta(\varepsilon, F|\partial\mathbb{D}^2)$, where η is as in 3.9. Let $D = \mathbb{D}^2 \cup \mathbb{A}$, where the component $S^1 \times 1$ of $\partial\mathbb{A}$ is attached to $\partial\mathbb{D}^2$. By applying 3.5 twice, it is easy to see that $D \cong \mathbb{D}^2$. By taking h on $S^1 \times 1$ and the restriction of F on $S^1 \times 2$ and applying 3.9 (with the roles of $S^1 \times 1$ and $S^1 \times 2$ interchanged) and 4.10, we produce a map $k\colon \mathbb{A} \to N_\varepsilon(h(S^1))$ extending the given map on the boundary $\partial\mathbb{A}$. Then k and h determine a map $l\colon \mathbb{D}^2 \cong D \to X$. Now $l|\partial\mathbb{D}^2$ represents λ and by choice of ε, $l(\mathbb{D}^2) \subset U - K$. Hence by 4.9, λ represents the zero class in ${}_EH_1(U-K)$, i.e. α is injective. This proves Assertion 1.

We now use the special dd_n hypothesis to produce a finite dimensional A. Let $\mathscr{C} = \mathscr{C}(\mathbb{D}^2, X)$. We claim that if X satisfies dd_n, then $\mathscr{C}_{n-1} = \mathscr{C}_{n-1}(\mathbb{D}^2, X)$ is dense in $\mathscr{C}$. To this end take a countable collection $\{(D_k^{(1)}, D_k^{(2)}, \ldots, D_k^{(n)}) | k \in \mathbb{N}\}$ of n-tuples of pairwise disjoint Pontrjagin subdiscs $D_k^{(i)} \subset \mathbb{D}^2$ such that for every i and k, $\partial D_k^{(i)} \subset Z_l^*$ for some $l = l(i)$, and such that for any n distinct points $x_1, x_2, \ldots, x_n \in \mathbb{D}^2$ there is an integer $k \geq 1$ such that for every $j \in \{1, 2, \ldots, n\}$, $x_j \in D_k^{(j)}$. Without loss of generality, we may assume that if $A_k^{(j)} = D_k^{(j)} \cap \partial\mathbb{D}^2$ is non-empty, then it is an arc—see the remark after Lemma 3.2. Informally the collection separates n-tuples of points in $\mathbb{D}^2$. Define $\mathscr{E}_k = \{f \in \mathscr{C} | \bigcap_{j=1}^n f(D_k^{(j)}) = \varnothing\}$.

Assertion 2. $\mathscr{E}_k$ is dense in $\mathscr{C}$.

Proof. Suppose given $f \in \mathscr{C}$ and $\varepsilon > 0$. For simplicity write D^j and A^j for $D_k^{(j)}$ and $A_k^{(j)}$ respectively. First we consider the case where all the A^j are empty. By the remark after Lemma 3.2, we may choose slightly larger pairwise disjoint Pontrjagin discs E^j such that $D^j \subset \operatorname{Int} E^j$ and (as in 3.7) E^j—$\operatorname{Int} D^j$ is a Pontrjagin annulus $\mathbb{A}^j$. By 4.10 X is $1 - pc$ and 3.9 applies. Thus provided E^j is only slightly larger than D^j, the maps induced on the two components of $\partial\mathbb{A}^j$ by f will be $\frac{1}{2}\eta(\varepsilon/2, f|\partial E^j)$ close, where η is as in Proposition 3.9. By dd_n pick maps $g_j\colon D^j \to X$ such that $\bigcap_{j=1}^n g_j(D_j) = \varnothing$ and $\mathrm{d}(f|D^j, g_j) < \min(\varepsilon, \frac{1}{2}\eta(\varepsilon/2, f|\partial E^j))$. Then by 3.9, $f|\partial E^j$ and $g_j|\partial D^j$ extend to a map $F_j\colon \mathbb{A}^j \to N_{\varepsilon/2}(f(\partial E^j))$. Define $f'\colon \mathbb{D}^2 \to X$ using f on $\mathbb{D}^2 - \bigcup_{j=1}^n \operatorname{Int} E^j$, F_j on $\mathbb{A}^j$, and g_j on D^j. Clearly $f' \in \mathscr{E}_k$, while by choice of g_j and F_j, it follows that f and f' are ε-close.

The general case is proved in a similar way. Again we may choose slightly larger Pontrjagin subdiscs E^j of $\mathbb{D}^2$ such that if $A^j = \varnothing$ then as before $D^j \subset \operatorname{Int} E^j$, whereas if $A^j \neq \varnothing$, then $D^j - A^j \subset \operatorname{Int} E^j$, $D^j \cap \partial E^j = A^j$ and $f|\overline{\partial D^j - A^j}$ and $f|\overline{\partial E^j - \partial\mathbb{D}^2}$ are sufficiently close. More precisely they should be so close as to satisfy the condition in a version of 3.9 based on a Pontrjagin strip instead of a Pontrjagin annulus, where a Pontrjagin strip is a copy of $\mathbb{D}^2$ with $\partial\mathbb{D}^2$ identified with $\partial([0,1] \times [1,2])$, and with $\overline{\partial D^j - A^j}$ and $\overline{\partial E^j - \partial\mathbb{D}^2}$ corresponding to $[0,1] \times 1$ and $[0,1] \times 2$ respectively. As in 3.9, the strip is divided into a chain of "small" Pontrjagin discs, and local Pontrjagin 1-connectivity is used on each. Further details are left to the reader.

Assertion 3. $\mathscr{E}_k$ is open in $\mathscr{C}$.

Proof. We show the complement is closed. Let $f_p \to f$ as $p \to \infty$, where $f_p \notin \mathscr{E}_k$. Then there are points $x_p^i \in D_k^{(i)}$ such that $f_p(x_p^1) = f_p(x_p^2) = \ldots = f_p(x_p^n)$. As $D_k^{(1)}$ is compact, (x_p^1) has a subsequence converging to (say) $x^1 \in D_k^1$. A subsequence of this subsequence may then be chosen so that the corresponding subsequence of (x_p^2) converges to (say) $x^2 \in D_k^{(2)}$. Proceeding in this way, we obtain simultaneous subsequences $(x_{p'}^1), (x_{p'}^2), \ldots, (x_{p'}^n)$ such that for each j, $x_{p'}^j \to x^j \in D_k^{(j)}$ as $p' \to \infty$. Since the $f(x_{p'}^j)$ are all equal, we have $f(x^1) = f(x^2) = \ldots = f(x^n)$ and so $f \notin \mathscr{E}_k$.

The density of $\mathscr{C}_{n-1}$ follows by applying the Baire category theorem to $\bigcap_1^\infty \mathscr{E}_k$, which is clearly a subset of $\mathscr{C}_{n-1}$. Note that if f is in $\mathscr{C}_{n-1}$, then f is at most $(n-1)$ to 1, and so by [17; 1.12.2], we have $\dim f(\mathbb{D}^2) \leq 2 + (n-1) - 1 = n$. Since $\mathscr{C}(\mathbb{D}^2, X)$ is separable, we may choose a countable dense sequence $f_1, f_2, \ldots$ with each f_i in $\mathscr{C}_{n-1}$. By the sum theorem [17; 1.5.3] the set $A = \bigcup_{i=1}^\infty f_i(\mathbb{D}^2)$ is at most n-dimensional.

The proof of the finite-dimensionality is now completed as in [35]. There exists by Tumarkin's theorem [17; 1.5.11] a G_δ set $A' \supset A$ with $\dim A' = \dim A$. By the sum theorem and assertion 1, $\dim(X - A') = 1$, since it is a countable union of 1-dimensional sets. By the non-closed sum theorem [17; 1.5.10],

$$\dim X \leq 1 + \dim A' + \dim(X - A') \leq 1 + n + 1 < \infty.$$

The proof of the converse follows at once on setting $N = 4$ in the following general result.

THEOREM 5.3. *Let X be a finite-dimensional N-hm ($N \geq 4$) (e.g. X could be a finite-dimensional cell-like quotient of a manifold). Then X satisfies dd_3, and hence dd_n for every $n \geq 3$.*

Remarks. Conversely if X is a homology N-manifold (not necessarily finite-dimensional), then provided $n \geq 3$ and $N \geq 4$, it is true that dd_{n+1} implies dd_n. It is also true that a $\mathbb{Z}$-homology 3-manifold satisfies dd_n for all $n \geq 4$. These results, generalized to homology with coefficients other than $\mathbb{Z}$, will appear in another paper.

The proof of 5.3 proceeds in several stages.

Assertion 1. Any map of $\mathbb{D}^2$ to X can be approximated arbitrarily closely by a map under which $\mathbb{D}^2$ has 2-dimensional image.

To prove this, let $F = \bigcup_{i=1}^\infty F_i$ be a union of closed sets in X such that c-dim $F_i = \dim F_i \leq \dim F = N - 3$, and $\dim(X - F) = 2$ [17; 1.5.8]. Let $\mathscr{B}_i = \{f\colon \mathbb{D}^2 \to X \mid f(\mathbb{D}^2) \cap F_i = \varnothing\}$. Clearly $\mathscr{B}_i$ is open in $\mathscr{C} = \mathscr{C}(\mathbb{D}^2, X)$. We claim it is also dense. Suppose $f \in \mathscr{C}$ and $\varepsilon > 0$ are given. Replacing F_i by its intersection with a compact neighbourhood of $f(\mathbb{D}^2)$ if necessary, we may suppose F_i is compact. Then applying Corollary 5.2(iii) and compactness, we may find a $\delta > 0$ such that any loop in $X - F_i$ of diameter less than δ is nullhomologous in a set of diameter less than $\varepsilon/2$ and lying in $X - F_i$. To achieve this, cover F_i by pairs of open sets $V \subset U$ such that (by $lcc_{\mathbb{E}}^1$) $i_*\colon {}_{\mathbb{E}}H_1(V - F_i) \to {}_{\mathbb{E}}H_1(U - F_i)$ is zero, and each set U has diameter at most ε. Then let δ be a Lebesgue number for the cover $\{V\}$. Clearly there is no harm in supposing that $\delta < \varepsilon/4$. By 3.2 pick a Pontrjagin cellulation $\{\mathbb{D}_j^2 \mid 1 \leq j \leq m\}$, of mesh less than $\delta/2$ (measured in X) for f. Let $Z = \bigcup_{j=1}^m \partial\mathbb{D}_j^2$. By ordering the cells $\mathbb{D}_1^2, \mathbb{D}_2^2, \ldots, \mathbb{D}_m^2$, and using 5.2(iii) to alter f on any arcs $\partial\mathbb{D}_k^2 \cap \partial\mathbb{D}_j^2$ whose images under f intersect F_i, we construct $g\colon Z \to X - F_i$ such that g and $f|Z$ are $\delta/2$-close, and $g = f$ on $\partial\mathbb{D}_j^2$ if $f(\mathbb{D}_j^2) \cap F_i = \varnothing$. Then $g(\partial\mathbb{D}_j^2)$ still has diameter at most δ, and so $g(\partial\mathbb{D}_j^2)$ is nullhomologous in a set of diameter less than $\varepsilon/2$ lying in the complement of F_i. Thus by 4.9 there is an extension of $g|\partial\mathbb{D}_j^2$ to h_j: $\mathbb{D}_j^2 \to X$ such that diam $h_j(\mathbb{D}_j^2) < \varepsilon/2$. Define f': $\mathbb{D}^2 \to X - F_i$ by using f on $\mathbb{D}_k^2$ if $f(\mathbb{D}_k^2) \cap F_i = \emptyset$, and using h_k otherwise. Clearly $f' \in \mathscr{B}_i$ and by construction $d(f, f') < \varepsilon$.

Thus by the Baire category theorem, $\bigcap_{i=1}^{\infty} \mathscr{B}_i$ is dense in $\mathscr{C} = \mathscr{C}(\mathbb{D}^2, X)$. Clearly if $g \in \bigcap_{i=1}^{\infty} \mathscr{B}_i$, then $g(\mathbb{D}^2) \cap F = \emptyset$, and so the image of g lies in the 2-dimensional set $X - F$. This proves Assertion 1.

Assertion 2. Any two maps f_1, f_2: $\mathbb{D}^2 \to X$ can be approximated arbitrarily closely by maps f'_1, f'_2: $\mathbb{D}^2 \to X$ such that dim $(f'_1(\mathbb{D}^2) \cap f'_2(\mathbb{D}^2)) \leq 0$ and dim $(f'_i(\mathbb{D}^2)) \leq 2$ $(i = 1, 2)$.

Approximate f_1 as in Assertion 1. Let $H = \bigcup_{i=1}^{\infty} H_i$ be a 1-dimensional F_σ set in the 2-dimensional set $f'_1(\mathbb{D}^2)$ such that dim $(f'_1(\mathbb{D}^2) - H) \leq 0$. Since $n \geq 4$, dim $H \leq n - 3$ and by constructing an approximation to f_2 as in Assertion 1, but with each F_i replaced by $F_i \cup H_i$, an arbitrary close approximation f'_2 to f_2 may be found. The required dimensional conditions follow since the image of f'_2 misses H and F. This proves Assertion 2.

Given three maps f_1, f_2, f_3: $\mathbb{D}^2 \to X$, use Assertion 2 to obtain approximations f'_1, f'_2 to f_1, f_2. Then by the method of proof of Assertion 1, it is easy to find an approximation f'_3 to f_3 whose image avoids the closed 0-dimensional set $f'_1(\mathbb{D}^2) \cap f'_2(\mathbb{D}^2)$. It is clear that $f'_1(\mathbb{D}^2) \cap f'_2(\mathbb{D}^2) \cap f'_3(\mathbb{D}^2) = \emptyset$, verifying dd_3. This completes the proof of 5.3 and hence of 5.1. Clearly we have the following Corollary.

COROLLARY 5.4. *For $n \geq 4$, the ghastly generalized n-manifolds of Daverman and Walsh* [10] *satisfy dd_3, but do not satisfy DD_q for any $q \geq 2$.*

Remark. The proof of 4.7 shows that there are indeed many non-constant maps of $\mathbb{D}^2$ to a generalized manifold, however ghastly.

6. PROOFS OF COROLLARIES AND CONCLUDING REMARKS

Corollary 1.2 follows from 1.1. Corollary 1.3 follows from 1.2 and 3.4, since by [9, Theorem 16.11] M/G is LC^1.

Remarks. In Corollary 1.2, it is sufficient to assume only that f is acyclic, rather than cell-like. By constructing a dense 1-dimensional set as in [35], it is easy to see that cell-like maps on 4-manifolds cannot raise dimension *if and only if* cell-like maps cannot raise dimension on the class of all 2-dimensional compacta embeddable in some 4-manifold. (This class includes all Boltjanskij compacta, see the questions below).

As remarked in the introduction, the theorem can be stated without mention of the Pontrjagin disc. More precisely one may replace the disjoint Pontrjagin n-tuples property by the following: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that given any map f: $S^1 \to X$ with diam $f(S^1) < \delta$, then $f(S^1)$ is nullhomologous in a *finite-dimensional* subset of X of diameter at most ε. (Here for simplicity we assume that X is compact.) Indeed if X is finite-dimensional, this condition clearly holds. Conversely, by choosing a suitable cellulation, the condition allows us to approximate arbitrarily closely any map f: $\mathbb{D}^2 \to X$ by one with finite-dimensional image. Then the proof of Theorem 5.3 above shows X satisfies dd_3, and so it is finite-dimensional by Theorem 1.1.

Questions 6.1. A compactum X is said to be a *Boltjanskij compactum* if $\dim(X \times X) = 3$. The first example of such a space was constructed by V. Boltjanskij [4]. Note that such an X must always be 2-dimensional [13].

(i) Can a cell-like map defined on a Boltjanskij compactum raise dimension?

(ii) Does every $\mathbb{Z}$-homology 4-manifold contain a copy of some (or perhaps every) Boltjanskij compactum? (Note that by [14] every topological 4-manifold contains a copy of every Boltjanskij compactum (see also [33])).

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Aston University
Aston Triangle
Birmingham, B4 7ET
U.K.
Present address:
Magdalene College
Cambridge, CB3 0AG
U.K.

University of Ljubljana
P.O. Box 64
Ljubljana, 61111
Yugoslavia

Steklov Mathematical Institute
42 Vavilova Street
Moscow, 117966
U.S.S.R.