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THE TOPOLOGY OF CELL-LIKE MAPPINGS

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ABSTRACT: This is a survey of some recent developments in the topology of cell-like mappings, a central branch of modern geometric topology. The main subject is the so-called cell-like mapping problem. We describe its history and explain its importance. Then we describe in detail the work on it since the late 1970s by various people, notably J.J. Walsh and E.V. Ščepin, culminating in the solution of the problem by A.N. Dranišnikov in 1988. The survey includes a list of unsolved problems and an extensive bibliography.

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CONTENTS

1. INTRODUCTION

This paper is concerned with cell–like maps. Very roughly the notion of cell–like map is a generalization of the idea of homeomorphism. Under a homeomorphism, the inverse image of a point is a single point. Under a cell–like map, the inverse image of a point may consist of more than one point, but has homotopical and geometrical properties very like that of a point. It turns out that cell–like maps arise everywhere in geometric topology — for example a limit of homeomorphisms between manifolds is necessarily cell-like. Moreover cell-like maps enjoy pleasant homotopy properties, and enjoy easier-to-use categorical properties than homeomorphisms. Finally they are a tool which features strongly in the solution of key problems, such as the four–dimensional Poincaré conjecture, the characterization of topological manifolds, and certain profound problems in dimension theory.

We introduce the notion of cell–likeness by describing a special case, namely that of *cellularity*. The concept of cellularity originated in the work of M. Brown [**31**]: a subset $K \subset M$ of an n-dimensional manifold M is said to be *cellular in* M if $K = \bigcap_{i=1}^{\infty} B_i$ for some sequence $\{B_i\}$ of n-dimensional cells $B_i \subset M$, such that, for every i, $K \subset \text{int } B_i$ and $B_{i+1} \subset \text{int } B_i$. (Here and elsewhere in this paper, manifolds are assumed to have empty boundary, but are not assumed to have any PL or smooth structure.) Clearly every point, and more generally every PL embedded cell in $\mathbf{R}^n$ (Euclidean n-space) is cellular in $\mathbf{R}^n$. The so–called topologist's sine curve, i.e. the set

$$K = \{(x, y) \in \mathbf{R}^2 \mid y = \sin(1/x), 0 < x \leq 1\} \cup (\{0\} \times [-1, 1]),$$

provides a less trivial example of a cellular subset of the plane $\mathbf{R}^2$ — in particular it shows that cellular sets need not be locally connected.

Some clue as to the relevance of cellular sets lies in the fact that (with the above notation) $M - K$ is homeomorphic to M – point. Brown used cellularity to prove that a nicely embedded $(n-1)$-sphere S^{n-1} in S^n is equivalent to the usual embedding as the equator. The proof is of stunning elegance, and any reader unfamiliar with it is urged to refer to the original immediately [**31**].

As further evidence of the connection of this idea with homeomorphisms, suppose $f : M \to N$ is a limit of a sequence of homeomorphisms f_n. If M and N are n-manifolds, it is a simple exercise to verify that if $\{C_n\}$ is a base of neighbourhoods of some point $z \in N$, with each C_n an n-cell, then the sequence of n-cells $\{f_n^{-1}(C_n)\}$ is a neighbourhood base of $f^{-1}(z)$, which is thus cellular in M. Thus study of the closure of homeomorphism groups leads inevitably to the idea of cellularity.

Note that cellularity depends on the embedding of K in M, rather than merely on K itself. For example there exist wild arcs (i.e. embeddings of the interval $[0,1]$) in $\mathbf{R}^n$ for $n \geq 3$ ([20], [79]) which have non-simply-connected complement and so are non-cellular, while the standard embeddings manifestly are cellular in $\mathbf{R}^n$. Nor is this dependence on embedding only a phenomenon associated with wildness. The dunce hat, a celebrated contractible 2-complex, can be PL embedded in $\mathbf{R}^4$ in several ways, some of which are cellular, some of which are not [159].

In view of this dependence on embedding, it is convenient to use a wider concept, introduced by R.C. Lacher in [104] — see also [97], which does not suffer from this defect. A non-empty compactum K is said to be *cell-like* if for some embedding of K in an ANR M, the following property holds:

UV^∞: *For every neighbourhood U of K in M, there exists a neighbourhood V such that $K \subset V \subset U$ and the inclusion $i : V \to U$ is nullhomotopic.*

Here and in what follows ANR denotes an absolute neighbourhood retract for the class of separable metric spaces. The usual definition of an ANR Y is that whenever Y is embedded as a closed subset of a separable metric space, then it is a retract of some neighbourhood in that space. However an equivalent, and more useful, version of the definition is in terms of absolute neighbourhood extension, namely that given a map f of a closed subset A of a separable metric space Z to Y, there exists an extension of f to a map $F : U \to Y$, where U is some neighbourhood of A in Z. The class of ANRs includes all polyhedra, and is roughly the largest convenient class of spaces which homotopically resemble polyhedra. For details of the properties of these spaces, see [23].

Scholium 1.1. *The following are equivalent for a finite dimensional compactum K.*

(a) K is cell-like.
(b) For any embedding of K in an ANR M, property UV^∞ holds.
(c) K has the shape of a point.
(d) There exists some embedding of K in a manifold P such that K is cellular in P.
(e) Every map of K into an ANR is nullhomotopic.
(f) For every embedding of K in an ANR, K is contractible in each of its neighbourhoods.

For a proof of these statements a convenient source is Lacher's survey article [105]. Notice that (b) reveals that cell-likeness (unlike cellularity) is independent of the embedding of K in a manifold.

Given a map $f : X \to Y$, we say f is *cell-like*, respectively *cellular*, if for each $y \in Y$, the inverse image $f^{-1}(y)$ is cell-like, respectively cellular in (the manifold!) X. Careful study of the definitions reveals that a cellular or cell-like compactum cannot be empty, so f is necessarily onto. It is clear that both for maps and spaces, cellular implies cell-like. The converse is false, since by (f) above, any contractible polyhedron is cell-like; thus the wild arcs mentioned earlier provide examples of cell-like, non-cellular spaces and maps. Readers should note that cell-like maps are called CE-maps by certain authors. We shall explain at the end of section 3 why this is a very unfortunate name, and refrain from using it again.

What exactly is the difference between cellularity and cell-likeness, when considering subsets of an n-manifold? It follows easily from the Schoenflies Theorem that the concepts agree for $n \leq 2$ [44]. In higher dimensions, we have the following result.

Cellularity Criterion. *Let $K \subset M$ be a compact subset of an n-manifold, $n \geq 3$. If $n = 3$ assume that some neighbourhood of K in M contains no fake 3-cells. Then K is cellular in M if and only if K is cell-like and satisfies the following cellularity criterion:*

> *For every neighbourhood $U \subset M$ of K there exists a neighbourhood $V \subset U$ of K such that the inclusion-induced homomorphism $i_* : \pi_1(V - K) \to \pi_1(U - K)$ is trivial.*

This was proved for $n \neq 4$ by D.R. McMillan, Jr. [114]; for $n = 4$ it was shown by D. Repovš [130], using work of F.S. Quinn [126] and M.H. Freedman [80].

As evidence that cell-likeness is not only a more convenient generalisation of homeomorphism than cellularity, but also the correct one, we state the following result. A proof may be found in [105]; see also the end of section 3.

Theorem 1.2. *Let $f : X \to Y$ be an onto map between locally compact ANRs which is proper (i.e. all point inverses are compact). Then f is cell-like if and only if for every non-empty open subset U of Y, $f|f^{-1}(U) : f^{-1}(U) \to U$ is a homotopy equivalence.*

Thus cell-like maps on locally compact ANRs are hereditary homotopy equivalences, and so a natural homotopy theoretic generalization of homeomorphisms. It is interesting that such hereditary homotopy equivalences appeared in the work of Sullivan [141] on the Hauptvermutung in 1966 before the concept of cell-likeness was fully evolved.

As further evidence of the value of all this, suppose that M^n and N^n are n-manifolds. Let $H(M^n, N^n)$ denote the (possibly empty) space of homeomorphisms of M to N, topologized as a subspace of the space of all continuous maps, with the usual compact-open topology [71]. Let $CELL(M, N)$ and $CE(M, N)$ denote similarly the spaces of cellular and cell-like maps respectively. It is obvious (using the earlier remark about limits of homeomorphisms being cellular) that:

$$H(M,N) \subset \overline{H(M,N)} \subset CELL(M,N) \subset CE(M,N).$$

Cell-like Approximation Theorem 1.3. *For every cell-like map $f : M^n \to N^n$ between topological n-manifolds, and every $\epsilon > 0$, there is a homeomorphism $h : M \to N$ such that $d(f,h) < \epsilon$ in the sup-norm metric on the space of all continuous maps. [If $n = 3$ one must assume that f is cellular.]*

Corollary 1.4.

(i) For every n, $CELL(M,N) = \overline{H(M,N)}$.

(ii) Provided $n \neq 3$, $CE(M,N) = CELL(M,N) = \overline{H(M,N)}$.

These results have a varied history. For $n \leq 2$ they were proved by J.H. Roberts and N.E. Steenrod **[131]**, and J.W.T. Youngs **[158]**; much earlier R.L. Moore had settled the special case where $M \cong N \cong S^2$ **[121]**. For $n = 3$, following proofs in certain special cases by V.N. Kyong **[103]** and McMillan **[116]**, the general case was obtained independently by S. Armentrout **[11]** and L.C. Siebenmann **[137]**. The last-mentioned paper also contains a proof of the case $n \geq 5$, while the case $n = 4$ was tackled by Quinn **[126]**.

The restrictions here and elsewhere on the case $n = 3$ arise because of the unresolved status of the Poincaré conjecture. If this conjecture is false, let F be a fake 3-cell, i.e. a compact contractible 3-manifold with boundary S^2 such that $F \not\cong B^3$; such a space is obtained by removing the interior of a 3-simplex from a counter-example to the Poincaré conjecture. Then let $M = (S^3 -$ the interior of a 3-simplex $) \cup F$ and let $f : M \to S^3$ be a map which collapses F to a point but is otherwise a homeomorphism. Then f is cell-like since F is contractible, but it cannot be approximated by homeomorphisms since $M \not\cong S^3$ **[31]**.

A thorough survey of the state of the art for cell-like maps up to 1977 is contained in the paper of Lacher **[105]**. Related later survey papers are **[35]**, **[38]**, **[41]**, **[70]**, **[76]** and **[129]**.

2. WHY ARE CELL-LIKE MAPS IMPORTANT?

We give examples showing how cell-like maps have played an essential role in geometric topology in the last three decades. One particularly rich way of viewing cell-like maps is via decomposition theory. Given a closed, proper, onto map $f : X \to Y$, there is an associated decomposition $G(f) = \{f^{-1}(y) \mid y \in Y\}$ of X into compact subsets. Note that the map f is cell-like precisely if $G(f)$ consists of cell-like sets. One may form the quotient space $X/G(f)$, in which each element of $G(f)$ is identified to a (separate) point, and it is easy to see that $Y \cong X/G(f)$. Thus one may study the map f via X and a decomposition G of X, which has the advantage that one is working purely inside one space X. We shall see that this is a natural and useful procedure.

A. Exotic Factors Of $\mathbf{R}^n$. Suppose that X is a space such that, for some n, $X \times \mathbf{R} \cong \mathbf{R}^{n+1}$. If $n \leq 2$, then X is necessarily homeomorphic to $\mathbf{R}^n$, and in particular is a manifold [155]. For many years the corresponding result in higher dimensions was unknown. The first counterexample, for $n = 3$, is due to R.H. Bing. His celebrated dog-bone space [16] is a decomposition G of $\mathbf{R}^3$ consisting of single points plus an uncountable family of tame arcs. The arcs are so entangled with each other that the decomposition space $\mathbf{R}^3/G$ fails to be a 3-manifold around the points corresponding to each arc. Nevertheless, as Bing showed in [17], $\mathbf{R}^3/G \times \mathbf{R} \cong \mathbf{R}^4$. The decomposition space is thus very nearly a manifold, and indeed the proof that it is not is intricate and delicate. Many more such examples are now known. It is always possible to find in $\mathbf{R}^n$ (for $n \geq 3$) an arc A such that $\mathbf{R}^n - A$ is not simply connected [20], [79]. The quotient map $p : \mathbf{R}^n \to \mathbf{R}^n/A$ is cell-like, but $\mathbf{R}^n/A$ cannot be a manifold, because of the lack of simple connectivity on removing the point corresponding to A. However, as shown by Andrews and Curtis [9], $\mathbf{R}^n/A \times \mathbf{R} \cong \mathbf{R}^{n+1}$.

B. The Double Suspension Problem. At the 1963 Conference on Differential and Algebraic Topology in Seattle, J.W. Milnor gave a list of seven of the toughest and most important problems in geometric topology [108]. At the top of the list was the following problem. Let M^3 be a non-simply-connected homology 3-sphere (i.e. a closed topological 3-manifold with the integral homology of S^3). An example of such a space is provided by the Poincaré dodecahedral space [94]. Its suspension ΣM is necessarily simply connected and so by the Whitehead theorem homotopy equivalent to S^4, but cannot be homeomorphic to S^4, since the links of the suspension points are not simply-connected [113 ; 2.4.5]. [By the suspension of the space X we mean the unreduced suspension, obtained by erecting two cones on X, formally $\Sigma X = X \times [-1,1]/\{X \times -1, X \times 1\}$.] However the double suspension $\Sigma^2 M$ disarms such an objection, and Milnor's problem is whether $\Sigma^2 M \cong S^5$.

An affirmative answer has surprising consequences. Clearly $\Sigma^2 M$ contains a circle, joining up the suspension points, which is a subcomplex of the obvious triangulation. In this triangulation, 1-simplexes of the circle are linked to copies of M, whereas in any triangulation of S^5 obtained by subdividing the usual one, links of 1-simplexes are simply-connected. Thus a homeomorphism of $\Sigma^2 M$ with S^5 provides a weird triangulation of a sphere, totally unrelated to the usual one. The homeomorphism, if it exists, cannot be a

PL one, but must be 'wild'. After a partial solution in some cases by R.D. Edwards in 1975, J.W. Cannon in 1977 [34], [36] showed that the double suspension of any homology n-sphere is homeomorphic to S^{n+2}.

Cannon's proof proceeds in two main stages. First he obtains a resolution of $\Sigma^2 M$, that is a proper, cell-like map $f : S^{n+2} \to \Sigma^2 M$. Then he shows that such a map f can be approximated by homeomorphisms, if its target has the so-called *disjoint discs property*, denoted by DDP. A space X has the DDP if for any two maps $f_1, f_2 : B^2 \to X$ of the standard two-dimensional disc, and any $\epsilon > 0$, there exist maps $g_1, g_2 : B^2 \to X$, such that for $i = 1, 2$ we have $d(f_i, g_i) < \epsilon$, and $g_1(B^2) \cap g_2(B^2) = \emptyset$. The DDP is exploited to produce self-homeomorphisms of S^{n+2} which shrink the elements in $G(f)$ in a controlled way, so as to produce the approximating homeomorphisms. This shrinking technique, invented and pioneered by R.H. Bing, shows the gain that results from working with the understandable space (here S^{n+2}) rather than the more mysterious one (here $\Sigma^2 M$). An excellent discussion of the proof can be found in the survey article by Cannon [35].

C. Edwards' Approximation Theorem. Building on the work of Cannon, Edwards proved in 1977 the following theorem [73], which generalizes Siebenmann's Cell-like Approximation Theorem [137].

Theorem 2.1: Edwards' Approximation Theorem. *A proper, cell-like map f : $M \to X$ from a topological n-manifold M^n, $n \geq 5$, onto an ANR X can be approximated by homeomorphisms if and only if X has the DDP.*

Edwards' original manuscript [73] was never published. He wrote an outline of the proof for the Proceedings of the 1978 International Congress of Mathematicians in Helsinki [76]. Later F. Latour presented a proof at the Séminaire Bourbaki [109], but a complete proof (for $n \geq 6$ only) appeared for the first time in R.J. Daverman's book [44]. The proof is one of the most impressive in the whole subject, intricate yet elegant.

The theorem provides an effective way of identifying manifolds. For example it is easy to verify that a double suspension $\Sigma^2 M$ has the DDP, and so the theorem replaces the second (harder) half of Cannon's argument in the double suspension theorem. More generally it provides a powerful tool for recognising higher dimensional manifolds. There are two snags in its use to identify a space X as a manifold. The first is the need to have a cell-like map from a manifold to X. The work of Quinn [125], [126], [127], [128] provides powerful machinery which to some extent guarantees that for a suitable space X, such a resolution does exist. The second snag (the requirement that X is finite dimensional) we shall discuss in section 3.

Theorem 2.2: Recognition Theorem. *An n-dimensional, locally compact, separable metric space X is a topological n-manifold ($n \geq 5$) if and only if X satisfies the following conditions:*

(1) X is an ANR;
(2) For every $x \in X$, $H_*(X, X - x; \mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$;
(3) X has the DDP;

(4) *A certain integer-valued surgery obstruction $\sigma(X)$ (the local signature) vanishes.*

Condition (2) implies that X is a homology n-manifold, and condition (1) implies that it is an ENR (Euclidean neighbourhood retract) and hence a generalized n-manifold [23], [44]. Condition (4) then guarantees the resolution needed to apply Edward's theorem. For more information on the recognition of topological manifolds, see the surveys of Cannon [35] and Repovš [129]. We shall return to this problem in the epilogue.

D. Topology Of 4-manifolds. One of the three Fields medals awarded in 1986 went to M.H. Freedman for his epoch-making work on the topology of 4-manifolds. His work culminates in a classification of all simply-connected 4-manifolds in terms of the intersection form on the second homology group and the Kirby-Siebenmann obstruction [95]. This follows from a topological theory of surgery in dimension 4. One of the main results is the proof of the 4-dimensional Poincaré conjecture, that every 4-manifold homotopy equivalent to the 4-sphere S^4 is homeomorphic to it [80]. The key step in the proof is that the so-called Casson handle CH [85] is homeomorphic to the standard open 2-handle $H^\circ = (B^2 \times \text{Int } B^2, \partial B^2 \times \text{Int } B^2)$. The Casson handle arises after an infinite process of adding smooth standard handles so as to control the fundamental group, during which control over diffeomorphism type is lost. The proof of homeomorphism uses two cell-like maps

$$H^\circ \to CH/\{\text{gaps}^+\} \leftarrow CH$$

where $\{\text{gaps}^+\}$ is a certain cell-like decomposition. Freedman's proof establishes, using dazzingly beautiful arguments, that both these maps are in fact approximable by homeomorphisms. We recommend a survey of this work in a forthcoming book by R.C. Kirby [93], while full details are contained in the book by Freedman and Quinn [81].

E. The Bing Shrinking Criterion. As already indicated, one of the crucial ideas in this subject was introduced by Bing. A map $f : X \to Y$ of complete metric spaces is said to *satisfy the Bing shrinking criterion* if for every continuous positive-valued function $\epsilon : Y \to \mathbf{R}_+$, there exists a homeomorphism $h : X \to X$ such that, for all $y \in Y$, diam $hf^{-1}(y) < \epsilon(y)$, and for all $x \in X$, $d(fh(x), f(x)) < \epsilon(f(x))$. [If Y is compact, the function ϵ may be replaced by a single positive number.] Under these hypotheses on X and Y, it is true that f satisfies the Bing shrinking criterion if and only if it is approximable by homeomorphisms [112]. The first use of this was made by Bing, when he proved that the sum of two Alexander horned spheres is S^3 [15]. Normally the space X will be a manifold while Y will be more mysterious, and the criterion allows problems about Y (such as proving that it is a manifold, homeomorphic to X) to be tackled within X. This approach is dominant in decomposition space theory [44]. We shall not concern ourselves directly with the shrinking process here, but the reader should be aware that it is a vital ingredient on the technical side of many proofs.

F. The Borsuk Conjecture. This conjecture [22] is that every compact ANR has the homotopy type of a polyhedron. For simply-connected ANRs it was proved in 1957 by de Lyra [49]. The first proof of the general case used the methods of Q-manifolds, that is spaces locally homeomorphic to the Hilbert cube Q. Using a theorem of R.T. Miller and Edwards [117] to the effect that the cone over a compact ANR is the cell-like image of the Hilbert cube Q, J.E. West proved [151] that every compact ANR X is the cell-like image of some compact Q-manifold. Now since cell-like maps between ANRs are homotopy equivalences (see 1.2 above), X has the homotopy type of a Q-manifold, and so by the earlier triangulation theorem for Q-manifolds, that every Q-manifold is homeomorphic to the product of Q with a polyhedron [38], the Borsuk conjecture follows.

G. Maps Between ANRs And Simple Homotopy Theory. Further evidence of the close connection of cell-like maps with homeomorphisms is provided by the following theorem due to Chapman [38].

Chapman's Cell-like Approximation Theorem. *A map $f : X \to Y$ between locally compact ANRs is cell-like if and only if $f \times id_Q : X \times Q \to Y \times Q$ is approximable by homeomorphisms.*

This result, apparently of a rather abstract nature, has startling consequences. The ANR theorem of R.D. Edwards [38] states that for any compact ANR X, $X \times Q$ is a Q-manifold, and hence by the triangulation theorem mentioned in the previous section, $X \times Q \cong K \times Q$ for some polyhedron K. It follows naturally from the theory of Q-manifolds that K is determined up to *simple* homotopy equivalence [38]. Thus to X we may associate a well-defined simple homotopy type. This is summarised in the following result:

Chapman–West Theorem. *If $f : X \to Y$ is a cell-like map between compact ANRs, then f is a simple homotopy equivalence.*

Notice that as a trivial consequence a homeomorphism between compact polyhedra is thus necessarily a simple homotopy equivalence, thereby demonstrating for example the topological invariance of simple homotopy type and of Whitehead torsion [40], [118].

3. THE CELL–LIKE MAPPING PROBLEM

Here we explain the cell-like mapping problem and analyse its history and connections with other problems. Before tackling the main topic of this section, we need to review some results from dimension theory. We remind the reader that in this paper we restrict attention to separable metric spaces and continuous maps.

First we give a definition of Lebesgue or covering dimension. Although not the classical one, it is convenient for our purposes. We say that dim $X \leq n$ if for every closed subset A of X, and every map $f : A \to S^n$, there exists an extension $F : X \to S^n$ of f. [The reader who dislikes thinking of S^{-1} as the empty set may regard it as a convention that dim $\emptyset = -1$.] We define dim $X = n$ if dim $X \leq n$ but it is false that dim $X \leq n-1$. It must of course be verified that dim $X \leq n$ implies dim $X \leq n+1$, and for this we refer the reader to J.J. Walsh's elegant survey [148]. We record the following equivalent characterizations.

Theorem 3.1. *For a separable metric space X the following are equivalent.*

(i) dim $X \leq n$

(ii) For every cover $\{U_\alpha\}$ of X, there exists a refinement $\{V_\beta\}$ of $\{U_\alpha\}$ (i.e. for every β there is an α such that $V_\beta \subset U_\alpha$) such that every point in X has a neighbourhood meeting at most $n+1$ elements of the cover $\{V_\beta\}$.

(iii) Every point of X has arbitrarily small neighbourhoods whose frontiers have dimension less than or equal to $n-1$.

In (iii) the dimension referred to is defined by induction in the obvious way. For more information on dimension refer to R. Engelking's book [78]. We content ourselves with stating some of the properties we shall use most often.

Theorem 3.2. *Let X be a separable metric space. Then the following properties of dimension hold.*

(i) If $X = \bigcup_{i=1}^{\infty} X_i$, with X_i closed in X and dim $X_i \leq n$ for all i, then dim $X \leq n$.

(ii) If $X = X_1 \cup X_2$ then dim $X \leq$ dim $X_1 +$ dim $X_2 + 1$.

(iii) If $f : X \to Y$ is a closed onto map between separable metric spaces, and for some integer k and all $y \in Y$ it is true that card $f^{-1}(y) \leq k$, then dim $Y \leq$ dim $X + (k-1)$.

For details of these results, and their original authors, see the books by W. Hurewicz and H. Wallman [90], and Engelking [78].

To define the cohomological dimension of a space, we replace S^n in the above definition of dimension with the Eilenberg–MacLane complex $K(\mathbb{Z}, n)$. We recall some properties of these spaces. For any abelian group G, an *Eilenberg–MacLane space of type* (G, n) is a connected CW complex X such that

$$\pi_q(X) \cong \begin{cases} G & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

An Eilenberg–MacLane space of type (G, n) will be denoted by $K(G, n)$, a forgiveable abuse of notation in view of 3.3 below. We shall be particularly concerned with complexes of type $(\mathbf{Z}, n)$. It is clear that S^1 is of type $(\mathbf{Z}, 1)$, and $\mathbf{CP}^\infty$ is of type $(\mathbf{Z}, 2)$. In general an Eilenberg–MacLane space of type $(\mathbf{Z}, n)$ may be made from S^n by attaching cells of dimension greater than n so as to kill the higher homotopy groups of S^n. We record the following properties of Eilenberg–MacLane spaces.

Theorem 3.3. *Eilenberg–MacLane spaces have the following properties.*

(i) For any G and n, an Eilenberg–MacLane space of type (G, n) exists, and any two are homotopy equivalent.

(ii) For any G and n, and any space X, the set of homotopy classes $[X, K(G, n)]$ is naturally isomorphic to the n^{th} cohomology group $H^n(X; G)$.

For proofs of these facts see [140]. In (ii) the cohomology is Alexander–Čech–sheaf cohomology, which coincides with singular cohomology if X is a CW complex [89], [140]. A good introduction to properties of Eilenberg–MacLane spaces is the paper of D.B. Fuks [82].

Throughout this paper, we shall write H^* to denote Alexander–Čech–sheaf cohomology. Coefficients will invariably be taken in the integers $\mathbf{Z}$, and suppressed from notation. We now define the *cohomological dimension of a space* X, denoted by c–dim X. We say c–dim $X \leq n$ if for every closed subset $A \subset X$ and map $f : A \to K(\mathbf{Z}, n)$, there exists an extension to a map of X to $K(\mathbf{Z}, n)$. We make the convention that $K(\mathbf{Z}, -1) = \emptyset$ (or equivalently that c–dim $X \leq -1$ if and only if $X = \emptyset$). Again we define c–dim $X = n$ if c–dim $X \leq n$ but c–dim $X \not\leq (n-1)$. As before this definition requires a verification that c–dim $X \leq n$ implies c–dim $X \leq n+1$; the most elegant proof of this uses the observation (implied by 3.3 (i)) that $K(\mathbf{Z}, n-1)$ is homotopy equivalent to the loop space $\Omega K(\mathbf{Z}, n)$ — see [148]. We remark that it is easy to define cohomological dimension over any group G by substituting $K(G, n)$ for $K(\mathbf{Z}, n)$. We shall not need this idea, but in cases where this is done it is usual then to refer to our definition as *integral* cohomological dimension, and denote it by c–dim$_{\mathbf{Z}}$.

Theorem 3.4. *For a separable metric space X the following are equivalent.*

(i) c–dim $X \leq n$.

(ii) For every closed subset A of X, $i^* : H^n(X) \to H^n(A)$ is onto.

(iii) For every open set U in X, $H_c^{n+1}(U) \cong 0$.

(iv) For every open set U in X, and every $q \geq n+1$, $H_c^q(U) \cong 0$.

In this theorem $H_c^*(U)$ denotes cohomology with compact supports. For U with compact closure this is isomorphic to both the reduced cohomology of the one–point compactification, and to $H^*(X, X - U)$. For more information on this theorem and on cohomological dimension, see [3], [26], [69], [86], [102].

It took some while before a satisfactory definition of dimension was written down. Before G. Peano produced his examples of space-filling curves, an informal notion of "number of parameters required to describe the space" had been used. Following work by (among others) H. Poincaré, L.E.J. Brouwer and H. Lebesgue, a satisfactory exposition (using a different definition, equivalent to that above) was given by P.S. Uryson [145] in 1924. Difficulties in using this led P.S. Aleksandrov to introduce the notion of (co)homological dimension. He proved the following theorem in 1932. (More precisely he proved an equivalent statement about homological dimension [1].)

Theorem 3.5 (Aleksandrov's Theorem). *Let X be a separable metric space. Then*

(i) c–dim $X \leq$ dim X.

(ii) If dim $X < \infty$ then c–dim X = dim X.

Aleksandrov then posed the obvious problem [2].

ALEKSANDROV'S PROBLEM. *Is there an infinite–dimensional compactum which has finite cohomological dimension ?*

Note that in one case the problem has a straightforward answer. Since S^1 is a $K(\mathbf{Z}, 1)$, dimension one and cohomological dimension one coincide. However the general case of this formidable problem resisted many attacks. Its significance was enhanced by work of R.D. Edwards [74], who showed in 1978 that the problem had a connection with another major problem, namely the Cell-like Mapping problem.

GENERAL CELL-LIKE MAPPING PROBLEM. *Does there exist a cell-like map $f : X \to Y$ between compact metric spaces such that dim $Y >$ dim X?*

For maps on 3-manifolds this problem first appeared (in an equivalent form) in the works of Bing [18] and Armentrout [10]. The solution in this case is given in section 4 below. By Aleksandrov's Theorem, and the Vietoris–Begle Theorem (which applies since cell-like maps induce isomorphisms on cohomology), Y must either have the same dimension as X, or be infinite-dimensional [12], [154]. In many results about cell-like maps, a restriction to a finite-dimensional range was needed, but it was hoped that it could be removed. In particular it is known that if X is a finite-dimensional ANR, then Y is an ANR if and only if Y is finite-dimensional; this will be taken up later in the section.

Theorem 3.6 (Edwards' Theorem). *The general cell-like mapping problem and Aleksandrov's problem are equivalent. More precisely, if X is a compactum of cohomological dimension n, there exists a compactum Z of dimension n and a cell-like map $f : Z \to X$.*

We shall refer to a compactum such as Z as an *Edwards' resolution* of X. Edwards' Theorem supplies the hard half of the equivalence, the easy part following from the Vietoris–Begle Theorem as above. As with so many of his profound results, Edwards never published his proof. Walsh's excellent survey, to which we have already referred [148], contains an independent proof. In 1987 L.R. Rubin and P.J. Schapiro showed that the equivalence holds good if one considers arbitrary separable metric spaces rather than compacta [**135**].

In November 1987 A.N. Dranišnikov announced in Moscow the following outstanding result.

Theorem 3.7 (Dranišnikov's Theorem [68]). *There exists an infinite-dimensional compactum with cohomological dimension 3.*

It seems fitting that the solution should in the end be obtained by a Russian, bearing in mind the contribution of Russians to the theory of dimension; Dranišnikov is a student of E.V.Ščepin, who in turn was Aleksandrov's last student. Dranišnikov had worked consistently on the problem for the last six years, although as can be seen from the bibliography, which includes a complete list of his publications [**50–70**], during this time he has obtained many other outstanding results.

We shall devote an entire chapter to Dranišnikov's result, and most of the remainder of the paper will be devoted to the cell-like mapping problem. It is convenient to consider various specialisations of the problem, since in practice cell-like maps arise on nice spaces such as manifolds.

CE($\mathcal{C}, n$): THE CELL-LIKE MAPPING PROBLEM IN DIMENSION n FOR A CLASS $\mathcal{C}$ OF SPACES. *For any space $X \in \mathcal{C}$ with dim $X \leq n$, does there exist a cell-like map $f : X \to Y$ with dim $Y >$ dim X?*

We note various relationships between these problems, in the following theorem. Let $\mathcal{C}$ denote the class of compacta, $\mathcal{ANR}$ the class of compact ANRs, and $\mathcal{M}$ the class of compact topological manifolds.

Theorem 3.8. *The following relationships exists between the various problems (where = denotes that problems are equivalent, $A \subset B$ denotes that an affirmative solution of A yields an affirmative solution of B, etc.)*

(i) $CE(\mathcal{C}, n) \subset CE(\mathcal{M}, 2n+1) \subset CE(\mathcal{C}, 2n+1)$.
(ii) $CE(\mathcal{C}, n) = CE(\mathcal{ANR}, n+1)$.
(iii) *If* $n \geq 4$, $CE(\mathcal{M}, n) \subset CE(\mathcal{C}, n-2)$.
(iv) *For any class* $\mathcal{D}$, $CE(\mathcal{D}, n-1) \subset CE(\mathcal{D}, n)$.

Proof (sketch): In (i) the second inclusion is obvious. For the first, suppose there is a cell-like dimension-raising map $f : X \to Y$ of an n-dimensional compactum X. Then X embeds in $\mathbf{R}^{2n+1}$ by the Menger-Pontrjagin embedding theorem [**124**]. Take the decomposition G of $\mathbf{R}^{2n+1}$ whose only non-trivial elements are $\{f^{-1}(y) \mid y \in Y\}$. Then the projection map $p : \mathbf{R}^{2n+1} \to \mathbf{R}^{2n+1}/G$ is cell-like and must raise dimension since its image contains a copy of Y.

That $CE(\mathcal{C}, n) \subset CE(\mathcal{ANR}, n+1)$ follows by a similar argument from the result of H. Bothe [**24**] that any n-dimensional compactum embeds in an $(n+1)$-dimensional ANR. For the reverse inclusion, suppose that dimension cannot be raised on n-dimensional compacta, and let $f : X \to Y$ be a map on an $(n+1)$-dimensional ANR. We use a result of K. Sieklucki [**138**] that a collection of pairwise disjoint m-dimensional closed subsets of an m-dimensional ANR must be countable. If F is the frontier of an open set in Y, by this result only countably sets of the form $f^{-1}F$ can be $(n+1)$-dimensional, while for any F for which this is not true, $F = f\left(f^{-1}F\right)$ is by hypothesis at most n-dimensional. By 3.1 (iii) it follows that Y is at most n-dimensional.

Given a cell-like dimension-raising map on an n-manifold M^n, Daverman [43] has shown how one may choose a dense countable family of arcs and a (possibly different) cell-like map $\overline{f}$ which is one-to-one over the (1-dimensional) union A of the arcs. Then we have a cell-like map on $M - A$, and dim $M - A \leq n-2$ — see the remarks in section 6 about duality and cohomological dimension. By 3.2(i), $\overline{f}(A)$ is at most 1-dimensional and so $\overline{f}$ must raise dimension on $M - A$.

Finally (iv) is trivial, as one simply converts $f : M \to X$ to $f \times id_{S^1} : M \times S^1 \to X \times S^1$.

It can be seen that the problems for various classes of spaces are intricately connected. In what follows we will consider the problem for the special case of manifolds. Thus by the **cell-like mapping problem in dimension** n we mean (in the above notation) the problem CE($\mathcal{M}, n$).

We conclude this section with a discussion of cell-like maps in general. As evidence of their good categorical properties, we mention the following.

Proposition 3.9.

(i) If (f_n) is a uniformly convergent sequence of cell-like maps between ANRs, the limit map is also cell-like.

(ii) If $f : X \to Y$ and $g : Y \to Z$ are cell-like maps of locally compact ANRs, then so is gf.

(iii) Let $\{X_i, p_{i+1,i}\}$ be an inverse sequence of ANRs, and assume that each map $p_{i+1,i}$ is cell-like. Then the limit map $p_1 : X = \varprojlim X_i \to X_1$ is cell-like.

Proof: We leave (i) as an easy exercise, while (ii) follows from 1.2, since gf is a hereditary homotopy equivalence. For (iii), note that X must be an ANR. For on crossing the inverse system with the Hilbert cube Q, by Edwards' ANR Theorem the spaces in the new system are Q-manifolds. Hence the cell-like bonding maps are approximable by homeomorphisms by [38]. Thus by a theorem of M. Brown [30], $p_1 \times id_Q : X \times Q \to X_1 \times Q$ is also approximable by homeomorphisms. Thus as $X \times Q$ is a Q-manifold, by Edwards' ANR Theorem again, X is an ANR: Then as $p_1 \times id_Q$ is approximable by homeomorphisms, it is a cell-like and so its factor p_1 must be too.

Remark: 3.9 (iii) solves a problem of Lacher (private communication).

Following Edwards' equivalence theorem in 1978, there was renewed interest in Aleksandrov's problem. Attempts were made to show that various infinite-dimensional spaces had finite cohomological dimension. It was known that a space with finite-dimensional subspaces of arbitrarily high finite dimension could not be the target of a cell-like map which raised dimension. Spaces with few finite-dimensional subspaces were also known

— in 1979 Walsh [147] had constructed an infinite-dimensional space with *no* subspaces of positive dimension!

Although powerful results, theorems 1.2 and 2.1 suffer from the serious limitation that a hypothesis is made on the image. Since in practice most cell-like maps of interest arise via decomposition spaces, such hypotheses are not easy to verify. The focus of much work prior to Dranišnikov's example was directed towards cell-like maps on general spaces.

When considering a cell-like map $f : X \to Y$ between general spaces, we cannot expect 1.2 to hold, due to possible local pathology in the spaces. Indeed the proof of 1.2 shows that f is a weak homotopy equivalence and then appeals to the Whitehead theorem, valid for ANRs. However one might hope that in general f would be a hereditary shape equivalence, i.e. that for any closed subset K of Y, $f|f^{-1}K : f^{-1}K \to K$ would be a shape equivalence. We recall that a map $g : A \to B$ is a shape equivalence if $g^* : [B, P] \to [A, P]$ is an isomorphism for every polyhedron P; this implies that g induces isomorphisms of cohomology and more generally of any representable functor. We have the following result, due to Kozlowski [98].

Theorem 3.10. *Let $f : X \to Y$ be a cell-like map and suppose that* $\dim Y < \infty$. *Then f is a hereditary shape equivalence.*

At the same time one should mention a related theorem of Kozlowski [98] (see also [88]).

Theorem 3.11. *Let $f : X \to Y$ be a hereditary shape equivalence of metric compacta. If X is an ANR, then Y is also an ANR.*

The finite-dimensional hypothesis on Y in 3.10 cannot be removed. An important example due to J.L. Taylor exhibits the Hilbert Cube Q as the cell-like image of a map f, defined on a compactum, such that f is not even a shape equivalence [142]. (This is the reason for our dislike of the term 'CE' for cell-like maps, since it suggests incorrectly that they are equivalences.) This example has been very fertile, and later authors have produced yet more subtle counterexamples from it [48], [111]. At the heart of Taylor's example is an example of J.F. Adams and independently H. Toda (see [48]) of a map $\beta : \Sigma^N L \to L$ between finite complexes such that all the compositions $\beta \cdot \Sigma^N\beta \cdot \Sigma^{2N}\beta \ldots \Sigma^{kN}\beta$ are essential. The example essentially does little more than take the inverse limit of the associated sequence; it is interesting to note that Dranišnikov's example in section 5 also contains an inverse system in which all compositions of bonding maps are essential.

In the light of this example a great many reductions of the cell-like mapping problem to equivalent problems were made. These are now of minor interest, following Dranišnikov's example. However there is one which is definitely worth recording. This is an unpublished result of Edwards and Henderson [74], [84].

Assertion 3.12. *The answer to the general cell-like mapping problem is affirmative (i.e dimension can be raised) if and only if there exists an inverse sequence of spheres*

$$S^{n_1} \xleftarrow{f_1} S^{n_2} \xleftarrow{f_2} S^{n_3} \xleftarrow{f_3} \ldots S^{n_k} \xleftarrow{f_k} S^{n_{k+1}} \xleftarrow{f_{k+1}} \ldots$$

such that

(a) *All the compositions* $f_1 f_2 \cdots f_k$ *are essential.*
(b) *There exists an integer* m *such that no* f_k *is homotopic to an* m*-fold suspension* $\Sigma^m g$, *where* $g : S^{n_{k+1}-m} \to S^{n_k - m}$.

For remarks on the status of the proof of this assertion, see [70]. Condition (a) above is reminiscent of the property of the map used in the construction of the Taylor example. However it is known that the maps f_k cannot all be suspensions of a single map, since by a celebrated result of G. Nishida [122] the stable homotopy ring is nilpotent. Thus it would appear that Dranišnikov's example has highly non–obvious consequences for the stable homotopy ring of spheres.

4. CE($\mathcal{M}, n$) FOR $n \leq 3$: DIMENSION IS PRESERVED.

We first deal with the cases where $n \leq 2$. In each of these cases, it follows from well-known results or obvious remarks that cell–like maps cannot raise dimension. For $n = 0$, there really is nothing to be said, since cell–like maps are homeomorphisms. For $n = 1$, the requirement that point-inverses be connected forces them to be closed intervals. From this it speedily follows that cell–like maps are limits of homeomorphisms, which shrink the non–trivial interval point inverses very small. Of course in these dimensions cell-like maps are obviously cellular.

For $n = 2$, the argument is not quite trivial. Firstly note that a cell–like subset of the plane $\mathbf{R}^2$ is connected and cannot separate the plane, hence it is cellular. This is easily seen by the Schoenflies Theorem. Hence the notions of cell-like and cellular coincide. To see that the cell–like image of a 2–manifold must be a 2–manifold, it is easiest to use the classical characterizations of such spaces [155]. These can be phrased entirely in homologically terms, since they involve only notions like separating arcs and circles. Since a cell–like map definitely induces isomorphisms on homology (see 1.2), it must follow that the characterization as a 2–manifold of the source M must pass down to the target $f(M)$ if f is cell–like. Details may be found in a paper of R.L. Wilder [154] — for a more modern treatment, see Cannon's paper [35].

In the absence of a characterization of 3–manifolds, the first non–trivial case of the problem occurs in this dimension. A proof that dimension cannot be raised in this case was first obtained by G. Kozlowski and Walsh [100]; the proof appeared several years later [101]. Variants have also been given by F.D. Ancel [5] and Dranišnikov and Ščepin [70]. In 1984 Walsh found a very slick proof of the more general fact that a $\mathbf{Z}$–homology 3–manifold must be 3–dimensional [149].

Theorem 4.1. *Let* X *be a* $\mathbf{Z}$*-homology 3–manifold (i.e. a locally compact separable metric space of finite cohomological dimension such that for any* $x \in X$, $H_q(X, X - x) \cong H_q(\mathbf{R}^3, \mathbf{R}^3 - 0)$.*) Then* $\dim X = 3$.

Corollary 4.2. *Cell–like maps on 3–manifolds cannot raise dimension.*

Proof of corollary: If X is the cell-like image of a 3-manifold, by the Vietoris-Begle theorem it follows that X inherits the homological properties of a 3-manifold, and the result is immediate.

Proof of 4.1: Since X is locally connected, any map $\alpha : I \to X$ may be approximated by a map $\overline{\alpha}$ with dim $\overline{\alpha}(I) \leq 1$. To achieve this, simply split $\alpha(I)$ into finitely many maps into path-connected neighbourhoods and use the fact that path-connected implies arc-connected, plus the sum theorem for dimension (3.2 (i)). The metric space of all maps of I to X is second countable, and so we may find a countable dense set of maps $\{\alpha_i : I \to X\}$ with dim $\alpha_i(I) \leq 1$. Let $A = \bigcup_{i=1}^{\infty} \alpha_i(I)$. Then by Theorem 3.2 (i), dim $A \leq 1$. Now for any closed set K in X, c-dim $K > 1$ implies that either Int $K \neq \emptyset$ or that K separates X locally (see section 6). If K is a closed set in $X - A$, it is easy to see that for any open connected set U, the set $U - K$ is non-empty and connected, by definition of A. Hence c-dim $K \leq 1$ and so dim $K \leq 1$, by the coincidence of dimension and cohomological dimension in this case. By a theorem of Tumarkin [**143** — see also 1.5.11 of [**78**]], there exists a G_δ set (i.e. an intersection of countably many open sets) G such that $A \subset G$ and dim G = dim $A = 1$. Hence $X - G$ is a countable union of closed sets: by the above each must be at most 1-dimensional and so by Theorem 3.2 (i), dim $X - G \leq 1$. Hence by Theorem 3.2 (ii), we have

$$\dim X \leq \dim G + \dim (X - G) + 1 \leq 3.$$

5. CE($\mathcal{M}, n$) FOR $n \geq 7$: DIMENSION CAN BE RAISED.

As is clear, the positive solution of this problem (i.e. that dimension-raising maps exist) follows from Dranišnikov's example of a compactum X with infinite dimension and cohomological dimension 3. For by embedding an Edwards' resolution of X in $\mathbf{R}^7 \subset \mathbf{R}^n$ ($n \geq 7$), and taking the quotient map as in Theorem 3.8, it is easy to produce dimension-raising maps on $\mathbf{R}^n$.

The construction of such an example involves a judicious mixture of geometric techniques and algebraic ones. Apparently Dranišnikov initially believed that there were no dimension-raising maps. The turning point, and the revelation that it was necessary to use algebraic machinery beyond ordinary homology, came from a proof of the fact that there is a map $f : S^n \to PS^n$ which is a weak homotopy equivalence but induces the **zero** map in K-theory. Here PS^n is the inverse limit of a Postnikov system for S^n. Moreover Dranišnikov had shown that if the compactum X has cohomological dimension at most n, then for every closed subset $A \subset X$ and every continuous map $g : A \to PS^n$, there exists an extension to X. If PS^n were S^n, then this latter result would solve the problem negatively, i.e. would show dimension cannot be raised. Clearly these two results suggest that the way to detect differences between PS^n and S^n is via an extraordinary cohomology theory like K-theory, as opposed to ordinary cohomology or homotopy. With the key insight that extraordinary cohomology theories had a role to play, Dranišnikov soon found the necessary example. Certain natural constructions involve Eilenberg-MacLane complexes, and for success appropriate cohomology groups must be zero. Fortunately the Eilenberg-MacLane complexes $K(\mathbf{Z}, n)$ for $n \geq 3$ behave in K-theory like points [8], [32]. In fact it turns out to be necessary to use K-theory with finite coefficients, to ensure that finite subcomplexes of Eilenberg-MacLane complexes can be pieced together into an inverse system with a compact limit. Dranišnikov has developed these ideas further [67], suggesting that for each generalized cohomology theory between the Eilenberg-MacLane spectrum and the sphere spectrum, there will be a different class of compacta which have infinite dimension but are finite-dimensional in the eyes of the extraordinary theory under consideration. In this section we confine ourselves to explaining the construction of Dranišnikov's example.

We describe Dranišnikov's example in the reverse order to the usual logical one, in the hope that this process will enable the reader to understand the key points in the construction. One by one, the necessary details will be revealed.

The example X is the inverse limit of a sequence of simplicial complexes,

$$X = \varprojlim \{X_i, p_{i+1,i}\},$$

such that $\dim X > 3$ while c-$\dim X \leq 3$. By Aleksandrov's theorem (see 3.5(ii)), it follows that $\dim X = \infty$. It is trivial to modify the construction to produce infinite dimensional examples with larger finite cohomological dimension.

The bonds $p_{j,i} : X_j \to X_i$ $(j > i)$ are all essential, and $X_0 = S^4$. Then $p_{\infty,0} : X \to X_0$ must be essential, since as X_0 is an absolute neighbourhood extensor, and ϵ–close maps to it are homotopic, any nullhomotopy of $p_{\infty,0}$ forces $p_{j,0}$ to be nullhomotopic for large j. We show that if dim $X \leq 3$, then $p_{\infty,0}$ must be nullhomotopic. For then $\dim(X \times I) \leq 4$ and so by the definition of dimension we can extend $p_{\infty,0} \mid X \times 0$ and a constant map of $X \times 1$ to a map of $X \times I$ to S^4, contradicting the essential nature of $p_{\infty,0}$. Hence dim $X > 3$.

That c–dim $X \leq 3$ follows from the following condition satisfied by the maps $p = p_{i+1,i}$:

for every subcomplex A of X_i, Im $i^* : [X, K(\mathbf{Z},3)] \to [p^{-1}A, K(\mathbf{Z},3)]$ *contains* Im $(p|p^{-1}(A))^* : [A, K(\mathbf{Z},3)] \to [p^{-1}A, K(\mathbf{Z},3)]$, *where $i : p^{-1}A \to X$ is the inclusion.*

This roughly says that the definition of cohomological dimension no more than 3 holds for those closed sets which are inverse images of subcomplexes. This suffices to achieve the result for all closed subsets of X, since in the construction the X_i are embedded in the Hilbert cube so that any closed subset A is a limit (in the Hilbert cube) of polyhedra of the form $p_{i+1,i}^{-1}(J_i)$, where J_i are subcomplexes. A routine application of neighbourhood extensors and homotopy extension reveals that this yields the necessary extension to X of any map of A to $K(\mathbf{Z},3)$.

Notice that it follows from this that the maps $p_{j,0}$ will be zero on cohomology in degrees above 3. Since the cohomology of $X_0 = S^4$ vanishes in degrees 3 and below, one cannot use cohomology to detect that $p_{j,0}$ is essential. To do this Dranišnikov employs an extraordinary cohomology theory, which gives the proof its unique mixture of non–trivial algebraic and geometric ingredients. The particular theory used is reduced complex K–theory with mod p coefficients. This theory is represented by $k^*(X) = [X \wedge Y_p^2, BU]$, where Y_p^2 is the Moore space $S^1 \cup_p B^2$ (2–cell attached by a map of degree p, p prime). The bonds $p_{i+1,i}$ will all induce injective maps on k^*, in particular $p_{i,0}^* : k^*(X_0) \cong \mathbf{Z}_p \to k^*(X_i)$ will be injective.

Having surveyed the features of the construction necessary for success, we now begin to explain the details. As a final preliminary, we introduce the notion of an *Edwards–Walsh modification* of a finite–dimensional simplicial complex K. This construction is called the Edwards modification by Dranišnikov in [68]. In the proof of Edwards' theorem published by Walsh [148] the construction appears anonymously. Apparently Edwards' own proof applied obstruction theory without using this useful device, whose introduction is due to Walsh.

As usual let $K^{(n)}$ denote the n-skeleton of the simplicial complex K. We construct stepwise on the skeleta a map $\phi_n : Ed^3(K)^{<n>} \to K^{(n)}$, where the complexes $Ed^3(K)^{<n>}$ are an increasing sequence of subcomplexes of the final complex $Ed^3(K) = Ed^3(K)^{<k>}$, where $k = \dim K$. Moreover the maps ϕ_n are compatible, in the obvious sense that $\phi_n \mid Ed^3(K)^{<n-1>} = \phi_{n-1}$. [Warning: $Ed^3(K)^{<n>}$ is not the n-skeleton of $Ed^3(K)$, but in general is an infinite–dimensional complex.]

To begin, set $Ed^3(K)^{<3>} = K^{(3)}$ and $\phi_3 = id_K$. For every 4-simplex σ of K, regard its 3-skeleton (which is homeomorphic to S^3) as the 3-skeleton of the bottom cell in a copy K_σ of some fixed simplicial version of the Eilenberg-MacLane space $K(\mathbf{Z},3)$. Let $j_\sigma : \sigma^{(3)} \to K_\sigma$ be the inclusion, and M_σ the mapping cylinder (made into a simplicial complex) of j_σ. Then $Ed^3(K)^{<4>}$ is obtained from $Ed^3(K)^{<3>}$ by attaching (for every 4-simplex σ of K) a copy of M_σ along $\sigma^{(3)}$. Extend ϕ_3 to ϕ_4 by collapsing each K_σ to the barycentre $\hat{\sigma}$ of σ and extending linearly along the mapping cylinder.

The construction of $Ed^3(K)^{<5>}$ from $Ed^3(K)^{<4>}$ is typical of the steps needed thereafter to construct $Ed^3(K) = Ed^3(K)^{<\dim K>}$. Let σ be a 5-simplex of K. Consider $\phi_4^{-1}(\sigma) = \phi_4^{-1}\left(\sigma^{(4)}\right)$. This is an Edwards-Walsh modification of (some triangulation of) S^4, and it is clear that

$$\pi_3\left(\phi_4^{-1}(\sigma)\right) \cong \pi_3\left(\phi_4^{-1}\left(\sigma^{(3)}\right) - \bigcup_{\tau<\sigma} K_\tau\right) \cong \pi_3\left(\sigma^{(3)}\right) \cong \bigoplus_1^{15} \mathbf{Z},$$

while the lower homotopy groups vanish. (15 is the number of 3-simplexes in σ.) Thus, attaching simplexes of dimension more than 4 to $\phi_4^{-1}(\sigma)$, we may construct an Eilenberg-MacLane space $K\left(\bigoplus_1^{15}\mathbf{Z},3\right)$. Let M_σ be the simplicial mapping cylinder of the inclusion $\phi_4^{-1}(\sigma) \subset K\left(\bigoplus_1^{15}\mathbf{Z},3\right)$. Attach, for each 5-simplex σ, this mapping cylinder to $\phi_4^{-1}(\sigma)$. As before extend ϕ_4 to ϕ_5 by collapsing the added Eilenberg-MacLane complexes to the barycentres of the 5-simplexes. Continue inductively in this way for 6-simplexes, $\ldots$, $\dim K$-simplexes.

Note that there are many choices in the above construction, so it is not remotely canonical. However we may assume that $Ed^3(K)$ is a countable simplicial complex. Also there is a kind of naturality, in that if L is a subcomplex of K, then $\phi|\phi^{-1}(L) : \phi^{-1}(L) \to L$ is an Edwards-Walsh modification of L. Moreover for every simplex $\tau \in K$ of dimension more than 3, $\phi^{-1}(\tau)$ is an Eilenberg-MacLane complex $K\left(\pi_n\left(\tau^{(3)}\right),3\right)$.

The key property of the construction is that for every simplex ρ, of dimension more than 3 and with boundary $\partial\rho$, the inclusion $\phi^{-1}(\partial\rho) \subset \phi^{-1}(\rho)$ induces an *isomorphism* on 3rd cohomology. Of course in K itself, $i^* : H^3(\rho) \to H^3(\partial\rho)$ fails to be an isomorphism when ρ is a 4-simplex, and this is why c-dim $K \geq 4$.

This key property allows us to verify that the 'cohomological dimension less than 3' criterion is satisfied for inverse images under ϕ of subcomplexes. Indeed let L be a subcomplex of K, and $\xi : L \to K(\mathbf{Z},3)$ a map. Recalling that $[\ ,K(\mathbf{Z},3)] \cong H^3(\ ,\mathbf{Z})$, there are no obstructions to extending ξ to a map $\overline{\xi} : L \cup K^{(3)} \to K(\mathbf{Z},3)$ say. We require an extension to $Ed^3(K)$ of the map $\xi\phi|\phi^{-1}(L) : \phi^{-1}(L) \to K(\mathbf{Z},3)$. We already have an extension $\overline{\xi}\phi$ to $\phi^{-1}\left(L \cup K^{(3)}\right)$. Now in attempting to construct a extension from $\phi^{-1}\left(L \cup K^{(n)}\right)$ to $\phi^{-1}\left(L \cup K^{(n+1)}\right)$, $(n \geq 3)$, the vanishing of the relevant obstructions follows from the key property. Indeed the obstructions in question lie in $H^{n+1}(\phi^{-1}(\sigma), \phi^{-1}(\partial\sigma); \pi_n(K(\bigoplus \mathbf{Z}, 3))$. Only when $n = 3$ can these be non-zero. However from the exact sequence of the pair and the key property, it follows that $H^4(\phi^{-1}(\sigma), \phi^{-1}(\partial\sigma)) \cong H^4(\phi^{-1}\sigma)$ and it is easy to see that the latter group is zero.

The observant reader will be becoming increasingly worried that the Edwards–Walsh modification is patently a non-compact complex. To restore compact complexes we make use of properties of the extraordinary cohomology theory k^*.

Firstly we note that ϕ^* induces an isomorphism

$$\phi^* : k^*(K) \to k^*(Ed^3(K)).$$

To see this we use induction on dim $K = k$ say. If $k \le 3$ there is nothing to prove. For the inductive step, a standard Mayer–Vietoris technique allows reduction to the addition of a single $(k+1)$–simplex σ to K. Given the inductive hypothesis, by the 5–lemma ϕ^* will induce an isomorphism on $k^*(K \cup \sigma)$ provided that

$$(\phi \mid \phi^{-1}(\sigma))^* : k^*(\sigma) \to k^*(\phi^{-1}(\sigma))$$

is an isomorphism. Since the left hand group is of course trivial, and $\phi^{-1}(\sigma) \simeq K(\bigoplus \mathbf{Z}, 3)$, we require that $k^*(K(\bigoplus \mathbf{Z}, 3)) \cong 0$. By the Künneth formula for k^* [153], since by 3.3(i) $K(G \oplus H, n) \simeq K(G, n) \times K(H, n)$, it is enough to show that $k^*(K(\mathbf{Z}, 3)) \cong 0$ and this result can be found in the works of Buhštaber and Miščenko [32] or Anderson and Hodgkin [8].

Secondly we note that k^* has compact supports. More precisely let J be an increasing union of finite subcomplexes $J = \bigcup_1^\infty J_n$. Then since all the groups $k^*(J_n)$ are finite, the Mittag–Leffler condition is automatically satisfied, the $\lim^1$ term vanishes and so $\varprojlim k^*(J_n) \cong k^*(J)$ [153]. It follows that since $Ed^3(K)$ is a countable complex, for any element $\lambda \in k^*(K)$, there exists a *finite* subcomplex J_λ of $Ed^3(K)$ such that, if $i : J_\lambda \to Ed^3(K)$ denotes inclusion, $(\phi i)^* : k^*(K) \to k^*(J_\lambda)$ is non-zero. We remark that this is the reason for introducing $\mathbf{Z}_p$ coefficients. With ordinary K–theory, the $\lim^1$ term might well be non–zero, and so non–trivial classes could restrict to zero on every finite subcomplex, causing difficulties in securing a compact inverse limit.

We can now describe the construction of the inverse sequence producing the example. Define $X_0 = S^4$. Pick α_0, a non–zero element of $k^*(S^4) \cong \mathbf{Z}_p$ (exercise!). By the above, there exists a finite subcomplex, X_1 say, of $Ed^3(X_0)$ such that $(\phi \mid X_1)^*(\alpha_0) = \alpha_1$ say is a non–zero element of $k^*(X_1)$. The inductive construction of a finite subcomplex X_n of $Ed^3(X_{n-1})$ such that $(\phi \mid X_n)^*(\alpha_{n-1}) = \alpha_n$ is a non–zero element of $k^*(X_n)$ is now clear, noting the 'naturality' condition on subcomplexes of the Edwards–Walsh modification. The bonding maps are simply restrictions of the appropriate ϕ. Since all the α_n are non–zero, no composition of the bonds can be inessential.

This completes our survey of Dranišnikov's construction. Note that the construction is at least in principle an explicit one *except* for the selection of finite subcomplexes of the Edwards–Walsh modifications.

6. CE($\mathcal{M}, n$) FOR $n = 4,5,6$: THE OPEN CASES.

The answer to the problem is unknown for these three values of n. The cases $n = 5,6$ seem to differ from $n = 4$. Little progress seems possible for them. Obviously if an infinite-dimensional compactum of cohomological dimension 2 exists, then it could be embedded in $\mathbf{R}^5$, showing that the problems for $n = 5,6$ would have affirmative solutions (*cf* 3.8 (i)). Unfortunately Dranišnikov's methods cannot produce such a compactum. The reason is that if one uses the Edwards–Walsh 2–modification in place of the 3–modification, the construction runs into problems since $k^*(K(\mathbf{Z},2)) \cong \bigoplus \mathbf{Z}_p$ is no longer zero.

The case $n = 4$ does look more tractable. In this case Dranišnikov and Ščepin [70] conjecture that cell–like maps cannot raise dimension. As evidence for this we note the following result of the authors and Ščepin.

Theorem 6.1. [120] *Let X be a $\mathbf{Z}$-homology 4-manifold which is lc^1_s. Then dim $X < \infty$ (equivalently dim $X = 4$) if and only if for some $n \geq 3$, X has the disjoint Pontrjagin n-tuples property.*

Corollary 6.2. *Let M^4 be a topological 4-manifold and $f : M \to X$ be a proper cell–like map. Then dim $X < \infty$ if and only if for some $n \geq 3$, X has the disjoint Pontrjagin n-tuples property.*

In the Theorem, lc^1_s denotes locally 1-connected with respect to singular homology. A metric space X is said to have the *disjoint Pontrjagin n-tuples property*, denoted dd_n, if for every $\epsilon > 0$ and every collection of maps $f_1, f_2, \ldots, f_n : \mathbf{D}^2 \to X$ of the Pontrjagin disc $\mathbf{D}^2$ into X, there exist maps $g_1, g_2, \ldots, g_n : \mathbf{D}^2 \to X$ such that (i) for every i, $\mathrm{d}(f_i, g_i) < \epsilon$, and (ii) $\bigcap_{i=1}^n g_i(\mathbf{D}^2) = \emptyset$. The definition of the Pontrjagin disc $\mathbf{D}^2$ is roughly as follows. It is obtained from the standard 2-cell by repeatedly subdividing and replacing the interior of each 2-simplex by a small punctured torus. Thus it is the inverse limit of a sequence of multiple tori of increasing genus. One can map $\mathbf{D}^2$ onto any surface, and there are results relating maps of surfaces to maps of $\mathbf{D}^2$. More details may be found in [120].

Before explaining the proof, we remark that the Pontrjagin disc does have geometric interest. Very often in decomposition space theory, the fact that a surface has finite genus is significant. For example in Bing's hooked rug (a certain wild 2–sphere in $\mathbf{R}^3$ — see [19]) a loop round the stem of an eye–bolt cannot span a surface in the exterior of the sphere, since infinite genus is required to miss the chain of eyebolts. Clearly the Pontrjagin disc does have infinite genus around every point. (Although it is not a surface, it is homogeneous modulo the boundary.) One may see Pontrjagin discs lurking in the background in the survey of Cannon [35]. It is also worth remarking that the ghastly generalized n-manifolds of Daverman and Walsh [46] satisfy the disjoint Pontrjagin triples property, but do not have the corresponding disjointness property for maps of ordinary discs.

Outline of the proof: To show X is finite-dimensional if it satisfies dd_n, we proceed as follows. By a routine Baire category argument, one shows that the space of all maps of

D^2 to X contains a dense countable subset $\{g_i\}$, where each g_i is $(n-1)$ to 1. By 3.2 (iii) and 3.2 (i), $A = \bigcup_1^\infty g_i(D^2)$ is finite dimensional. The aim is now to mimic the argument of Walsh in the case $n = 3$, by proving $X - A$ has cohomological dimension 1. By the remark after 3.5 it is then 1-dimensional, and so by 3.2(ii) it follows that X has finite dimension.

Using the Poincaré duality spectral sequence [26, V], one can show that in a homology n-manifold, for a closed set K, c-dim $K < k$ if and only if for all open sets U in X, the map $i_* : \tilde{H}_j^c(U - K) \to \tilde{H}_j^c(U)$ is injective for $j \leq n - k - 1$. It follows that (for $n = 4$) c-dim $A \leq 1$ if and only if i_* is injective for $j \leq 1$ and Int $(X - A) = \emptyset$. Under the lc_s^1 hypothesis, we may take the homology to be singular, and the density of A means that only the case $j = 1$ is non-obvious. Now (replacing A by a G_δ set as in section 4) a class in ker $i_* : \tilde{H}_1(A \cap U) \to \tilde{H}_1(U)$ may be represented by a map of a surface $f : (F, \partial F) \to (U, U \cap A)$. Then after mapping a Pontrjagin disc onto F, one uses the density of A to obtain a nearby map $g : D^2 \to U \cap A$, and approximates this by a map of a surface, to show the original class was zero in $\tilde{H}_1(A \cap U)$.

The converse is a routine exercise in dimension theory. First one filters X by a sequence F_i of 1-dimensional sets such that $X - (\bigcup_1^\infty F_i)$ is 2-dimensional. Again using duality, one can approximate a map of D^2 to X by a map missing each F_i, and so obtain a map with 2-dimensional image. Given a second map, one can similarly arrange that the 2-dimensional images of near approximations have 0-dimensional intersection, and it is then simple to avoid this with a third map. This completes the outline of the proof.

Note finally that by 3.8 (ii), if every compactum of cohomological dimension 2 is in fact 2-dimensional, then cell-like maps cannot raise dimension on 4-manifolds.

7. EPILOGUE.

We conclude this survey by listing some of the remaining open problems related to cell-like maps.

1. The Cell-like Mapping Problem For Manifolds.

The following remains one of the central problems of the subject.

PROBLEM 7.1. *Is there a cell-like map $f : M \to X$, defined on a topological n-manifold M, $n \in \{4,5,6\}$, such that dim $X = \infty$?*

As was shown in chapter 6, this reduces for $n = 4$ to the following problem.

PROBLEM 7.2. *Let $f : M \to X$ be a cell-like map defined on a topological 4-manifold M^4. Does X have the disjoint Pontrjagin triples property?*

An alternative way of looking at the problem CE($\mathcal{M}$, 4) is via compacta of cohomological dimension 2 — see 3.8(iii).

PROBLEM 7.3. *Determine which of the following hold:*

(i) All compacta of cohomological dimension 2 are 2-dimensional.

(ii) There exists an infinite-dimensional compactum of cohomological dimension 2, but no Edwards' resolution of it embeds in any 4-manifold.

(iii) There exists an infinite-dimensional compactum of cohomological dimension 2, and an Edwards' resolution of it embeds in some 4-manifold.

Note that (i) and (ii) settle CE($\mathcal{M}$, 4) negatively, while (iii) implies a positive solution.

Turning to the higher-dimensional cases, let $f : Z \to X$ be an Edwards' resolution of Dranišnikov's compactum X, i.e. Z is a compact 3-dimensional metric space and f is a cell-like map. We obtained dimension-raising maps by embedding Z in $\mathbf{R}^7$.

PROBLEM 7.4. *Does Z embed in $\mathbf{R}^n$ for some $n \leq 6$?*

If so, there is obviously a cell-like dimension-raising map on $\mathbf{R}^n$. As a supplementary problem, one might ask about the embedding dimension of Z. Notice that $\mathbf{R}^7/G(f)$ is a homology manifold and so by [42], $\mathbf{R}^7/G(f) \times \mathbf{R}^2$ has the DDP.

PROBLEM 7.5. *Does $\mathbf{R}^7/G(f)$ have the DDP? the DADP? (disjoint arc-disc property)*

At any rate it is clear that the DDP cannot prevent dimension being raised. Another obvious problem is

PROBLEM 7.6. *Give an explicit description (rather than a mere proof of existence) of* **either** *an infinite-dimensional compactum of finite cohomological dimension,* **or** *a cell-like dimension-raising map.*

We also mention two problems due to Toruńczyk.

PROBLEM 7.7. *Suppose that in $I^4 = I \times I^3$ (the 4-cell) there are disjoint continua K and L such that $p(K) = p(L) = I$, where $p : I \times I^3 \to I$ is projection on the first factor. Let $C \subset I$ be a Cantor set. Does there exist an upper-semicontinuous cell-like decomposition G of I^4 such that for every $c \in C$ and every non-degenerate element $g \in G$, it is true that $g \cap p^{-1}(c) \cap K \neq \emptyset \neq g \cap p^{-1}(c) \cap L$?*

If the answer is yes for every such K, L and C, then Toruńczyk can show that dimension-raising maps exist on 4-manifolds.

Recall that a dendrite D is the closure (in $\mathbf{R}^2$ say) of a direct limit of trees. Every 2-dimensional compactum embeds in $D \times D \times I$ [**25**].

PROBLEM 7.8. *Can cell-like maps raise dimension on $D \times D \times I$?*

Clearly it follows as in 3.8 from the existence of Edwards' resolutions that if the answer is no, then compacta of cohomological dimension 2 are 2-dimensional. Kozlowski, Row and Walsh [**99**] have studied cell-like maps with 1-dimensional fibres on polyhedra (see also [**150**]), and it may be that this is relevant to the special form of $D \times D \times I$.

We have said little about the 3-dimensional case. As noted earlier, the unsettled status of the Poincaré conjecture puts limits on the way theorems may be stated. Assuming the Poincaré conjecture is false, generalized 3-manifolds homotopy equivalent to S^3 and without a resolution (i.e. a cell-like map from a genuine 3-manifold) have been constructed by Wilder [**155**], Brin [**28**], Brin and McMillan [**29**] and Jakobsche [**91**]. By a result of Jakobsche and Repovš [**92**], the last of these examples embeds in $\mathbf{R}^4$. It seems likely that the other examples behave in the same way.

PROBLEM 7.9. *Let M be one of the above generalized 3-manifolds. Does $M \times \mathbf{R}$ embed in S^4?*

2. Resolution And Recognition Problems

The obstruction to the existence of a resolution of a generalized n-manifold was mentioned earlier (see 2.C). Apart from its obvious use in the theorem characterizing manifolds, resolutions are of general utility, providing a way of desingularizing the space. The word originates in algebraic geometry, where systematic use is made of the concept. Very roughly Quinn's obstruction comes about as follows. One takes a map of a manifold to the generalized manifold to be resolved and attempts by means of surgery to make it more and more like a (hereditary) homotopy equivalence. As usual various surgery obstructions arise connected with intersection forms. By passing to coverings most of the obstructions can be shown to vanish, but the very last one persists and gives rise to the obstruction of the local signature.

PROBLEM 7.10. *Does there exist a generalized n-manifold X ($n \geq 5$) with non-trivial local signature $\sigma(X)$?*

It is worth remarking that this is unconnected with the DDP. For Daverman [42] has shown that $X \times \mathbf{R}^2$ has the DDP, while on the other hand by a result of Quinn [128] $\sigma(X \times \mathbf{R}^2) = \sigma(X)$.

In view of Edwards' Approximation Theorem, the problem of recognising topological manifolds among (say) the class of generalized manifolds is intimately bound up with the resolution problem. In 1977 Cannon posed the recognition problem in the following form.

PROBLEM 7.11. *Determine easy-to-verify geometric properties which characterize topological n-manifolds among the class of generalized n-manifolds.*

For $n \leq 2$ the empty property suffices. For $n \geq 5$ the DDP plus the Quinn local signature provide a reasonably satisfactory answer, although further clarification of the exact status of the local signature is desirable. For $n = 3$ the problem is intimately tied up with the Poincaré conjecture, as the example at the end of the introduction suggests. The case $n = 4$ appears completely untouched. On the one hand, there seems no obvious general position property to replace DDP. (An interesting exercise for the reader is to prove that for $n \geq 5$ every topological n-manifold possesses the DDP.) On the other hand, the shrinking theorems available appear less powerful, the best available (due to Bestvina and Walsh [14]) having some restrictions not analogous to those in higher dimensional cases.

3. Edwards' Approximation Theorem. Neither in Edwards' original manuscript [73], nor in the subsequent expositions of the proof [44], [109], did the case $n = 5$ get addressed. This dimension poses extra general position problems, and requires further ideas.

PROBLEM 7.12. *Prove the 5-dimensional case of Edwards' Approximation Theorem.*

4. The Moore Problem.

The following problem is named in honour of R.L. Moore, who (albeit in a different form) posed and solved the 2-dimensional case.

PROBLEM 7.13. *Let G be a cell-like upper semicontinuous decomposition of $\mathbf{R}^n$. If $\mathbf{R}^n/G$ is finite-dimensional, is $\mathbf{R}^n/G \times \mathbf{R} \cong \mathbf{R}^{n+1}$?*

The finite-dimensionality assumption is of course necessary in view of Dranjšnikov's example [68]. By Edwards' Approximation Theorem it is only necessary to verify that $\mathbf{R}^n/G \times \mathbf{R}$ has the DDP. Daverman has proved [42] that this space does has the 'disjoint arc-disc property' (DADP). Even the ghastliest of examples have turned out to support this conjecture [37], [41], [46], but despite considerable efforts the conjecture seems no nearer solution.

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