

NON-MEAGER FREE SETS AND INDEPENDENT FAMILIES

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ABSTRACT. Our main result is that, given a collection \mathcal{R} of meager relations on a Polish space X such that $|\mathcal{R}| \leq \omega$, there exists a dense Baire subspace F of X (equivalently, a nowhere meager subset F of X) such that F is R -free for every $R \in \mathcal{R}$. This generalizes a recent result of Banach and Zdomskyy. As an application, we show that there exists a non-meager independent family on ω , and define the corresponding cardinal invariant. Furthermore, assuming Martin's Axiom for countable posets, our result can be strengthened by substituting " $|\mathcal{R}| \leq \omega$ " with " $|\mathcal{R}| < \mathfrak{c}$ " and "Baire" with "completely Baire".

1. INTRODUCTION

Given a set X , we say that R is a *relation* on X if $R \subseteq X^n$ for some $n = n_R$ such that $1 \leq n < \omega$. By *space* we mean separable metrizable topological space. A space is *crowded* if it is non-empty and has no isolated points. A subset S of a space X is *meager* if there exist closed nowhere dense subsets C_k of X for $k \in \omega$ such that $S \subseteq \bigcup_{k \in \omega} C_k$. A subset S of a space X is *comeager* if $X \setminus S$ is meager. A space X is *Baire* if every non-empty open subset of X is non-meager in X . A subset S of a space X is *nowhere meager* if $S \cap U$ is non-meager in X for every non-empty open subset U of X . We will be freely using the following easy proposition (see [vM, Exercise A.13.7]).

Proposition 1. *Let X be a space. For a subset S of X , the following conditions are equivalent:*

- S is nowhere meager in X .
- S is dense in X and Baire as a subspace of X .

A relation R on a space X is *meager* if R is a meager subset of X^n , where $n = n_R$. Given a relation R on a set X , we say that $F \subseteq X$ is *R -free* if $x \notin R$ whenever $x : n_R \rightarrow F$ is injective. Given a collection \mathcal{R} consisting of relations on a set X , we say that $F \subseteq X$ is *\mathcal{R} -free* if F is R -free for every $R \in \mathcal{R}$.

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The following result (see [Kur, Section 6] or [Ke, Exercise 8.8 and Theorem 19.1]) has become a standard tool in mathematics. It guarantees the existence of nice free sets for a small number of small relations.

Theorem 2 (Kuratowski). *Let \mathcal{R} be a collection of meager relations on a crowded Polish space X such that $|\mathcal{R}| \leq \omega$.¹ Then there exists $F \subseteq X$ that satisfies the following conditions:*

- F is homeomorphic to 2^ω .
- F is \mathcal{R} -free.

Our main results (Theorems 21 and 22) are in the same vein, except that by “nice” we will mean respectively “nowhere meager” and “dense and completely Baire” instead of “homeomorphic to 2^ω ”. Theorem 21 generalizes (and was inspired by) a recent result of Banach and Zdomskyy, which concerns the case $\mathcal{R} = \{R\}$, where R is a binary relation (see [BZ, Theorem 1]). First, we will give proofs for crowded Polish spaces (see Theorems 10 and 16). As an application, we will show that there exists a non-meager independent family in ZFC (see Theorem 19).

2. MORE NOTATION AND TERMINOLOGY

Assume that a countably infinite set I is given (in most cases $I = \omega$). Throughout this paper, we freely identify subsets of I with their characteristic functions. Accordingly, we say that a collection of subsets of I is *meager* if it is meager as a subset of 2^I . Given a collection $\mathcal{X} \subseteq 2^I$ and $A \subseteq I$, define

$$\mathcal{X} \upharpoonright A = \{x \upharpoonright A : x \in \mathcal{X}\}.$$

A *filter* on I is a collection of non-empty subsets of I that is closed under finite intersections and supersets. Furthermore, we assume that $\{x \subseteq I : |I \setminus x| < \omega\} \subseteq \mathcal{F}$ for every filter \mathcal{F} on I . If the set I is not mentioned, we will assume that $I = \omega$. We will freely use the fact that a filter on I is non-meager if and only if it is Baire as a subspace of 2^I (see for example [MM, Section 2]).

Given $x \subseteq I$, define $x^0 = I \setminus x$ and $x^1 = x$. An *independent family* on I is a collection \mathcal{A} consisting of subsets of I such that $\bigcap_{x \in F} x^{\nu(x)}$ is infinite for every non-empty $F \in [\mathcal{A}]^{<\omega}$ and $\nu : F \rightarrow 2$. Once again, if the set I is not mentioned, we will assume that $I = \omega$. Notice that an independent family might be Baire as a subspace of 2^ω without being non-meager. In fact, it is well known that Theorem 2 implies the existence of independent families that are homeomorphic to 2^ω (see the proof of Theorem 19), and these are necessarily closed nowhere dense in 2^ω .

A space X is *completely Baire*² if every closed subspace of X is Baire. The following classical result (see [Ke, Corollary 21.21] and [vM, Corollary 1.9.13]) collects the most important facts about completely Baire spaces. See also Theorem 23.

Theorem 3 (Hurewicz). *Let X be a space. Consider the following conditions:*

- (1) X is Polish.
- (2) X is completely Baire.
- (3) X does not contain a closed copy of \mathbb{Q} .

¹ In fact, it is clear from the proof that “ $|\mathcal{R}| \leq \omega$ ” can be weakened to “ $|\mathcal{R}| < \text{cov}(\text{meager})$ ”. Furthermore, the example at the beginning of Section 8 shows that the bound $\text{cov}(\text{meager})$ is optimal.

² Some authors use “hereditarily Baire” or even “hereditary Baire” instead of “completely Baire”.

The implications $(1) \rightarrow (2) \leftrightarrow (3)$ hold for every X . If X is a coanalytic subspace of some Polish space, then the implication $(1) \leftarrow (2)$ holds as well.

Recall the following definitions:

- $\text{add}(\text{meager})$ is the minimal size of a collection \mathcal{M} consisting of meager subsets of 2^ω such that $\bigcup \mathcal{M}$ is non-meager.
- $\text{cof}(\text{meager})$ is the minimal size of a collection \mathcal{M} consisting of meager subsets of 2^ω such that for every meager subset N of 2^ω there exists $M \in \mathcal{M}$ such that $N \subseteq M$.
- $\text{cov}(\text{meager})$ is the minimal size of a collection \mathcal{M} consisting of meager subsets of 2^ω such that $\bigcup \mathcal{M} = 2^\omega$.
- $\text{non}(\text{meager})$ is the minimal size of a non-meager subset of 2^ω .
- \mathfrak{d} is the minimal size of a family $\mathcal{F} \subseteq \omega^\omega$ such that for every $f \in \omega^\omega$ there exists $g \in \mathcal{F}$ such that $f(n) \leq g(n)$ for all but finitely many values of n .

Since every crowded Polish space has a dense subspace homeomorphic to ω^ω (see the first paragraph of the proof of [MZ, Theorem 5.4] and [vM, Theorem 1.9.8]), it is easy to see that 2^ω could have been substituted with any other crowded Polish space in the above definitions. Notice that \mathfrak{d} is the minimal size of a family \mathcal{K} consisting of compact subsets of ω^ω such that $\bigcup \mathcal{K} = \omega^\omega$. The inequalities $\text{add}(\text{meager}) \leq \text{non}(\text{meager}) \leq \text{cof}(\text{meager})$ and $\text{add}(\text{meager}) \leq \text{cov}(\text{meager}) \leq \mathfrak{d} \leq \text{cof}(\text{meager})$ are well known (see [Bl, Section 5]). We denote by $\text{MA}(\text{countable})$ the statement that Martin's Axiom holds for countable posets, which is equivalent to $\text{cov}(\text{meager}) = \mathfrak{c}$ (see [Bl, Theorem 7.13]).

3. PRELIMINARIES ON NON-MEAGER FILTERS

In this section, we collect all the preliminaries on non-meager filters that will be needed in the next section. All these results are well known.

Recall that a function $\phi : I \rightarrow J$ is *finite-to-one* if $\phi^{-1}(j)$ is finite for every $j \in J$. Notice that every finite-to-one function $\phi : I \rightarrow \omega$ induces a partition of I into finite sets, namely $\{\phi^{-1}(j) : j \in \omega\} \setminus \{\emptyset\}$. Conversely, given a partition $\{I_j : j \in \omega\}$ of I into finite sets, setting $\phi(i) = j$ for every $i \in I_j$ yields a finite-to-one function $\phi : I \rightarrow \omega$. Given a countably infinite set I , a finite-to-one $\phi : I \rightarrow \omega$ and $x, y \in 2^I$, we will use the notation

$$[[x = y]] = \{j \in \omega : x \upharpoonright \phi^{-1}(j) = y \upharpoonright \phi^{-1}(j)\}.$$

The above set obviously depends on ϕ , but what ϕ is will always be clear from the context. The following two results are immediate consequences of [Bl, Theorem 5.2] and [Bl, Proposition 9.4]. Corollary 5 originally appeared, with a slightly different formulation, as part of [Ta, Théorème 21].

Theorem 4. *Let I be a countably infinite set. For a subset S of 2^I , the following conditions are equivalent:*

- S is meager.
- There exist a finite-to-one $\phi : I \rightarrow \omega$ and $z \in 2^I$ such that $[[x = z]]$ is finite for every $x \in S$.

Corollary 5 (Talagrand). *Let I be a countably infinite set. For a filter \mathcal{F} on I , the following conditions are equivalent:*

- \mathcal{F} is meager.
- There exists a finite-to-one $\phi : I \rightarrow \omega$ such that $[[x = z]]$ is finite for every $x \in \mathcal{F}$, where $z \in 2^I$ is defined by $z(i) = 0$ for every $i \in I$.

Let I and J be countably infinite sets. Given a finite-to-one $\phi : I \rightarrow J$ and a filter \mathcal{F} on J , define

$$\phi^{-1}(\mathcal{F}) = \{A \subseteq I : \phi^{-1}[B] \subseteq A \text{ for some } B \in \mathcal{F}\}.$$

The following three lemmas are simple applications of Corollary 5, and their proofs are left to the reader.

Lemma 6. *Let \mathcal{F} be a non-meager filter, and fix $A \in \mathcal{F}$. Then $\mathcal{F} \upharpoonright A$ is a non-meager filter on A .*

Lemma 7. *Let \mathcal{F}_ℓ be a non-meager filter for $\ell \in \omega$. Then $\bigcap_{\ell \in \omega} \mathcal{F}_\ell$ is a non-meager filter.*

Lemma 8. *Let I and J be countably infinite sets. Fix a finite-to-one function $\psi : I \rightarrow J$. Let \mathcal{F} be a non-meager filter on J . Then $\psi^{-1}(\mathcal{F})$ is a non-meager filter on I .*

4. NOWHERE MEAGER FREE SETS

This section contains our main result, which is Theorem 10. In fact, as Theorem 21 shows, the assumption that X is crowded can be dropped. Lemma 9 is the combinatorial core of this result, and its proof is postponed to the end of the section.

Lemma 9. *Let \mathcal{R} be a collection of meager relations on 2^ω such that $|\mathcal{R}| \leq \omega$. Then there exist nowhere meager subsets E_α of 2^ω for $\alpha \in \mathfrak{c}$ such that $x \notin R$ whenever $R \in \mathcal{R}$ and $x \in \prod_{k \in n_R} E_{\alpha_k}$ for distinct $\alpha_0, \dots, \alpha_{n_R-1} \in \mathfrak{c}$.*

Theorem 10. *Let \mathcal{R} be a collection of meager relations on a crowded Polish space X such that $|\mathcal{R}| \leq \omega$. Then there exists $F \subseteq X$ that satisfies the following conditions:*

- F is dense in X .
- F is Baire.
- F is \mathcal{R} -free.

Furthermore, given any cardinal κ such that $\text{cof}(\text{meager}) \leq \kappa \leq \mathfrak{c}$, it is possible to choose F so that the additional requirement $|F| = \kappa$ will be satisfied.

Proof. Since X is a crowded Polish space, it contains a dense subspace B that is homeomorphic to ω^ω (see the first paragraph of the proof of [MZ, Theorem 5.4] and [vM, Theorem 1.9.8]). Identify B with the subspace of 2^ω consisting of the sequences that are not eventually constant. Given $R \in \mathcal{R}$ with $n = n_R$, let $R' = R \cap (B^n)$, and view each R' as a meager relation on 2^ω . Let $\mathcal{R}' = \{R' : R \in \mathcal{R}\}$.

Since $\kappa \geq \text{cof}(\text{meager})$, it is possible to fix (M_α, U_α) for $\alpha \in \kappa$ so that the following conditions will be satisfied:

- Each M_α is a meager subset of X .
- Each U_α is a non-empty open subset of X .
- Given a meager subset M of X and a non-empty open subset U of X , there exists $\alpha \in \kappa$ such that $M \subseteq M_\alpha$ and $U_\alpha \subseteq U$.

Assume without loss of generality that $\Delta \in \mathcal{R}'$, where $\Delta = \{(z, z) : z \in 2^\omega\}$. Let E_α for $\alpha \in \mathfrak{c}$ be obtained by applying Lemma 9 with $\mathcal{R} = \mathcal{R}'$. Notice that $\Delta \in \mathcal{R}'$ will ensure that $E_\alpha \cap E_\beta = \emptyset$ whenever $\alpha \neq \beta$. Without loss of generality, assume that each $E_\alpha \subseteq B$, and notice that each E_α is nowhere meager in X . Therefore, it is possible to pick

$$z_\alpha \in (E_\alpha \cap U_\alpha) \setminus M_\alpha$$

for $\alpha \in \kappa$. Define $F = \{z_\alpha : \alpha \in \kappa\}$. It is easy to check that F has the desired properties. □

Corollary 11. *Let \mathcal{R} be a collection of meager relations on a crowded Polish space X such that $|\mathcal{R}| < \text{add}(\text{meager})$. Then there exists a nowhere meager $F \subseteq X$ that is \mathcal{R} -free.*

Proof. Define $R'_n = \bigcup\{R : R \in \mathcal{R} \text{ and } n_R = n\}$ for $1 \leq n < \omega$, and notice that each R'_n is a meager relation on X because $|\mathcal{R}| < \text{add}(\text{meager})$. Let $\mathcal{R}' = \{R'_n : 1 \leq n < \omega\}$. Since every \mathcal{R}' -free subset of X is clearly \mathcal{R} -free, an application of Theorem 10 will conclude the proof. □

Proof of Lemma 9. Using Theorem 4, one can easily construct a single finite-to-one function $\phi : \omega \rightarrow \omega$ and $x_R \in (2^\omega)^{n_R}$ for $R \in \mathcal{R}$ so that

$$R \subseteq \left\{ x \in (2^\omega)^{n_R} : \bigcap_{k \in n_R} [[x(k) = x_R(k)]] \text{ is finite} \right\}$$

for every $R \in \mathcal{R}$. Let $\mathcal{S} = \{(k, R) : k \in n_R\}$, and assume without loss of generality that \mathcal{S} is infinite.

Since \mathcal{S} is a countable set, we can apply [Kun, Exercise III.2.12] and fix a family $\mathcal{G} \subseteq \mathcal{S}^\omega$ such that $|\mathcal{G}| = \mathfrak{c}$ and

$$\bigcap_{k \in n} g_k^{-1}(s_k) \text{ is infinite}$$

whenever $1 \leq n < \omega$, $s_0, \dots, s_{n-1} \in \mathcal{S}$, and g_0, \dots, g_{n-1} are distinct elements of \mathcal{G} . Write $\omega = \bigcup_{\ell \in \omega} \Omega_\ell$, where the sets Ω_ℓ are infinite and pairwise disjoint, and fix a non-meager filter \mathcal{F}_ℓ for $\ell \in \omega$ such that $\Omega_\ell \in \mathcal{F}_\ell$.

We claim that the sets

$$E_g = \bigcap_{(k,R) \in \mathcal{S}} \left\{ z \in 2^\omega : [[z = x_R(k)]] \in \bigcap_{\ell \in g^{-1}(k,R)} \mathcal{F}_\ell \right\}$$

for $g \in \mathcal{G}$ have the desired properties. Fix $R \in \mathcal{R}$, distinct $g_0, \dots, g_{n_R-1} \in \mathcal{G}$ and $x \in \prod_{k \in n_R} E_{g_k}$. By the choice of \mathcal{G} , there exists $\ell \in \omega$ such that $\ell \in g_k^{-1}(k, R)$ for every $k \in n_R$. In particular, $[[x(k) = x_R(k)]] \in \mathcal{F}_\ell$ for every $k \in n_R$. Therefore $\bigcap_{k \in n_R} [[x(k) = x_R(k)]] \in \mathcal{F}_\ell$, which implies $x \notin R$.

It remains to show that each E_g is dense in 2^ω and Baire. Fix $g \in \mathcal{G}$, and let $E = E_g$. Define $J_s = \bigcup_{\ell \in g^{-1}(s)} \Omega_\ell$ and $I_s = \phi^{-1}[J_s]$ for $s \in \mathcal{S}$. Since E is closed

under finite modifications of its elements, in order to prove that E is dense in 2^ω , it will be enough to show that $E \neq \emptyset$. It will be sufficient to construct $z \in 2^\omega$ such that $[[z = x_R(k)]] \supseteq J_{(k,R)}$ for every $(k, R) \in \mathcal{S}$. This is easily achieved by setting $z \upharpoonright I_{(k,R)} = x_R(k) \upharpoonright I_{(k,R)}$ for every $(k, R) \in \mathcal{S}$.

Finally, we will show that E is Baire. It is easy to check that $z \in E$ if and only if $z \upharpoonright I_s \in E \upharpoonright I_s$ for every $s \in \mathcal{S}$. It follows that E is homeomorphic to $\prod_{s \in \mathcal{S}} E \upharpoonright I_s$. Since a countable product of Baire spaces is a Baire space (see [Ox, Theorem 3] or [vM, Exercise A.6.11]), it will be enough to show that each $E \upharpoonright I_s$ is Baire.

Fix $s = (k, R) \in \mathcal{S}$, and let $J = J_s, I = I_s$. Define $\mathcal{F} = \bigcap_{\ell \in \mathcal{G}^{-1}(s)} \mathcal{F}_\ell$. Notice that \mathcal{F} is a non-meager filter by Lemma 7. Also define $\psi = \phi \upharpoonright I : I \rightarrow J$. By considering an appropriate homeomorphism of 2^I , we can assume without loss of generality that $x_R(k)(i) = 1$ for every $i \in I$. Under this assumption, it is easy to realize that $E \upharpoonright I = \psi^{-1}(\mathcal{F} \upharpoonright J)$. It follows from Lemmas 6 and 8 that $E \upharpoonright I$ is Baire. \square

5. DENSE COMPLETELY BAIRE FREE SETS

The main result of this section is Theorem 16, which shows that Theorem 10 can be considerably strengthened under the assumption of MA(countable). Once again, this generalizes to arbitrary Polish spaces (see Theorem 22). We will need several preliminary lemmas.

Let X be a set and $1 \leq n < \omega$. Given $A \subseteq X^n$ and $x \in X^{n-1}$, we will use the notation

$$A[x] = \{z \in X : x \frown z \in A\}.$$

Notice that if $n = 1$ and $x = \emptyset$, then $A[x] = A$. The following is a special case of a classical result (see [Ke, Theorem 8.41] for a proof).

Lemma 12 (Kuratowski, Ulam). *Let X be a space and $2 \leq n < \omega$. If A is a comeager subset of X^n , then there exists a comeager subset B of X^{n-1} such that $A[x]$ is comeager in X for every $x \in B$.*

Notice that every meager relation is contained in a meager F_σ relation. Given a bijection $\pi : n \rightarrow n$, define $h_\pi : X^n \rightarrow X^n$ by setting $h_\pi(x)(k) = x(\pi(k))$ for every $x \in X^n$. We say that R is *symmetric* if $h_\pi[R] = R$ for every bijection $\pi : n_R \rightarrow n_R$. Using the fact that each h_π is a homeomorphism, it is easy to see that every meager relation is contained in a symmetric meager F_σ relation.

Assume that R is a symmetric meager F_σ relation on a space X with $n = n_R$. Using Lemma 12, it is easy to recursively construct subsets G_R^ℓ of X^ℓ for $1 \leq \ell \leq n$ such that the following properties are satisfied:

- $G_R^n = X^n \setminus R$.
- Each G_R^ℓ is a symmetric dense G_δ subset of X^ℓ .
- $G_R^{\ell+1}[x]$ is a dense G_δ subset of X for every $x \in G_R^\ell$.

Let \mathcal{R} be a collection consisting of relations on a set X . Given $F \subseteq X$, define the following condition:

$$\textcircled{*}(F, \mathcal{R}) \text{ If } R \in \mathcal{R}, 0 \leq \ell < n_R \text{ and } x : \ell + 1 \rightarrow F \text{ is injective, then } x \in G_R^{\ell+1}.$$

Notice that $\textcircled{*}(F, \mathcal{R})$ implies that F is \mathcal{R} -free.

Lemma 13. *Let \mathcal{R} be a collection of symmetric meager F_σ relations on a space X . Let $F \subseteq X$ be such that $\otimes(F, \mathcal{R})$ holds. Fix $R \in \mathcal{R}$, $0 \leq \ell < n_R$ and an injection $x : \ell \rightarrow F$. Then $G_R^{\ell+1}[x]$ is a dense G_δ subset of X .*

Proof. If $\ell = 0$, then $G_R^{\ell+1}[x] = G_R^1$ is a dense G_δ subset of X by construction. Now assume that $\ell \geq 1$. By applying condition $\otimes(F, \mathcal{R})$ to $x : (\ell - 1) + 1 \rightarrow F$, one sees that $x \in G_R^\ell$. Therefore $G_R^{\ell+1}[x]$ is a dense G_δ subset of X by construction. \square

Lemma 14. *Let \mathcal{G} be a non-empty collection of dense G_δ subsets of a crowded Polish space X such that $|\mathcal{G}| < \mathfrak{d}$. Assume that $\bigcap \mathcal{G}$ is dense in X . Then $\bigcap \mathcal{G}$ is non-meager in X .*

Proof. Since every crowded Polish space has a dense G_δ zero-dimensional subspace (see the first paragraph of the proof of [MZ, Theorem 5.4]), we can assume without loss of generality that X is zero-dimensional. In particular, we can assume that X is a dense (necessarily G_δ) subspace of 2^ω .

Fix a countable dense subset D of $\bigcap \mathcal{G}$. Let $B = 2^\omega \setminus D$, and observe that $X \setminus D$ is a dense G_δ subset of B . Since B is homeomorphic to \mathbb{Z}^ω (see [vM, Theorem 1.9.8]), we can fix a binary operation \cdot on B that makes B a topological group. Let $N = \bigcup \{2^\omega \setminus G : G \in \mathcal{G}\}$, and notice that N can be written as the union of less than \mathfrak{d} compact subsets of B . Assume, in order to get a contradiction, that N is comeager in B . Since B is homeomorphic to ω^ω , it will be enough to show that $N \cdot N^{-1} = B$. To see this, fix an arbitrary $x \in B$ and observe that $(x \cdot N) \cap N$ is comeager in B , hence non-empty. This means that $x \cdot y = z$ for some $y, z \in N$, hence $x = z \cdot y^{-1} \in N \cdot N^{-1}$. In conclusion, we see that $B \setminus N = \bigcap \mathcal{G} \setminus D$ is non-meager in B . It follows that $\bigcap \mathcal{G}$ is non-meager in X . \square

Lemma 15. *Assume that $\mathfrak{d} = \text{cof}(\text{meager})$. Let \mathcal{R} be a collection of meager relations on a crowded Polish space X such that $|\mathcal{R}| < \text{cov}(\text{meager})$. Then there exists a nowhere meager $F \subseteq X$ such that condition $\otimes(F, \mathcal{R})$ holds.*

Proof. Without loss of generality, assume that \mathcal{R} is non-empty and each $R \in \mathcal{R}$ is a symmetric meager F_σ . Let $\kappa = \mathfrak{d} = \text{cof}(\text{meager})$. Fix a collection $\{C_\alpha : \alpha \in \kappa\}$ consisting of comeager subsets of X such that for every comeager subset C of X there exists $\alpha \in \kappa$ such that $C_\alpha \subseteq C$. Fix a countable base $\{U_i : i \in \omega\}$ for X .

We will construct an increasing sequence $\langle F_\alpha : \alpha \in \kappa \rangle$ of subsets of X by transfinite recursion. In the end, set $F = \bigcup_{\alpha \in \kappa} F_\alpha$. By induction, we will make sure that the following requirements are satisfied:

- (1) $|F_\alpha| < \kappa$ for every $\alpha \in \kappa$.
- (2) If $\alpha \in \kappa$ is zero or a limit ordinal and $i \in \omega$, then $U_i \cap C_\alpha \cap F_{\alpha+i+1} \neq \emptyset$.
- (3) Condition $\otimes(F_\alpha, \mathcal{R})$ holds for every $\alpha \in \kappa$.

It is straightforward to check that condition (2) will ensure that F is nowhere meager in X . On the other hand, condition (3) will guarantee that $\otimes(F, \mathcal{R})$ holds.

Start by letting $F_0 = \emptyset$. Take unions at limit stages. At a successor stage $\beta = \alpha + i + 1$, where $\alpha < \kappa$ is zero or a limit ordinal and $i \in \omega$, assume that $F_{\alpha+i}$ is given. Define

$$\mathcal{G} = \{G_R^{\ell+1}[x] \cup \text{ran}(x) : R \in \mathcal{R}, 0 \leq \ell < n_R, x : \ell \rightarrow F_{\alpha+i} \text{ is injective}\},$$

and notice that \mathcal{G} consists of dense G_δ sets by Lemma 13. We claim that it is possible to pick $z \in U_i \cap C_\alpha \cap \bigcap \mathcal{G}$. If $\alpha = 0$, this follows from the fact that $|\mathcal{G}| < \text{cov}(\text{meager})$. Now assume $\alpha > 0$. In this case, it is easy to check that

$F_\omega \subseteq F_{\alpha+i} \subseteq \bigcap \mathcal{G}$. Therefore $U_i \cap \bigcap \mathcal{G}$ is non-meager in U_i by Lemma 14. Put $F_\beta = F_{\alpha+i} \cup \{z\}$. It remains to verify that condition $\textcircled{*}(F_\beta, \mathcal{R})$ holds.

Fix $R \in \mathcal{R}$, $0 \leq \ell < n_R$ and an injection $x : \ell + 1 \rightarrow F_\beta$. We need to show that $x \in G_R^{\ell+1}$. If $z \notin \text{ran}(x)$, this already follows from condition $\textcircled{*}(F_{\alpha+i}, \mathcal{R})$, so assume that $z \in \text{ran}(x)$. Since $G_R^{\ell+1}$ is symmetric by construction, we can assume without loss of generality that $x(\ell) = z$. Let $x' = x \upharpoonright \ell$, and notice that $z \notin \text{ran}(x')$ because x is injective. Therefore $z \in G_R^{\ell+1}[x']$, which implies $x = x' \frown z \in G_R^{\ell+1}$. \square

It is easy to realize that the assumption of MA(countable) in the following theorem can be weakened to $\mathfrak{d} = \mathfrak{c}$, provided $|\mathcal{R}| < \text{cov}(\text{meager})$. However, since it is one of our main results, we preferred to give the following more “quotable” formulation. The same remark holds for Theorem 22.

Theorem 16. *Assume that MA(countable) holds. Let \mathcal{R} be a collection of meager relations on a crowded Polish space X such that $|\mathcal{R}| < \mathfrak{c}$. Then there exists $F \subseteq X$ that satisfies the following conditions:*

- F is dense in X .
- F is completely Baire.
- F is \mathcal{R} -free.

Proof. Without loss of generality, assume that each $R \in \mathcal{R}$ is a symmetric meager F_σ . Enumerate as $\{Q_\alpha : \alpha \in \mathfrak{c}\}$ all copies of \mathbb{Q} in X , making sure to list each one cofinally often. We will construct an increasing sequence $\langle F_\alpha : \alpha \in \mathfrak{c} \rangle$ of subsets of X by transfinite recursion. In the end, set $F = \bigcup_{\alpha \in \mathfrak{c}} F_\alpha$. By induction, we will make sure that the following requirements are satisfied:

- (1) $|F_\alpha| < \mathfrak{c}$ for every $\alpha \in \mathfrak{c}$.
- (2) If $Q_\alpha \subseteq F_\alpha$ for some $\alpha \in \mathfrak{c}$, then $F_{\alpha+1} \cap (\text{cl}(Q_\alpha) \setminus Q_\alpha) \neq \emptyset$.
- (3) Condition $\textcircled{*}(F_\alpha, \mathcal{R})$ holds for every $\alpha \in \mathfrak{c}$.

Using Theorem 3, it is straightforward to check that condition (2) will ensure that F is completely Baire. On the other hand, condition (3) will guarantee that condition $\textcircled{*}(F, \mathcal{R})$ holds.

Start by letting F_0 be a countable dense subset of the \mathcal{R} -free set given by Lemma 15, thus ensuring that F will be dense in X . Take unions at limit stages. At a successor stage $\beta = \alpha + 1$, assume that F_α is given. First assume that $Q_\alpha \not\subseteq F_\alpha$. In this case, simply set $F_\beta = F_\alpha$. Now assume that $Q_\alpha \subseteq F_\alpha$. Apply Lemma 17 with $F = F_\alpha$ and $Q = Q_\alpha$ to get $z \in \text{cl}(Q_\alpha) \setminus Q_\alpha$ such that condition $\textcircled{*}(F_\alpha \cup \{z\}, \mathcal{R})$ is satisfied. Finally, set $F_\beta = F_\alpha \cup \{z\}$. \square

Lemma 17. *Let \mathcal{R} be a collection of symmetric meager F_σ relations on a crowded Polish space X such that $|\mathcal{R}| < \mathfrak{d}$. Assume that F and Q satisfy the following requirements:*

- $|F| < \mathfrak{d}$.
- $Q \subseteq F$ is countable and crowded.
- Condition $\textcircled{*}(F, \mathcal{R})$ holds.

Then there exists $z \in \text{cl}(Q) \setminus Q$ such that condition $\textcircled{}(F \cup \{z\}, \mathcal{R})$ holds.*

Proof. Without loss of generality, assume that \mathcal{R} is non-empty. Define

$$\mathcal{G} = \{(G_R^{\ell+1}[x] \cup \text{ran}(x)) \cap \text{cl}(Q) : R \in \mathcal{R}, 0 \leq \ell < n_R, x : \ell \rightarrow F \text{ is injective}\},$$

and observe that $|\mathcal{D}| < \mathfrak{d}$. By Lemma 13, the collection \mathcal{G} consists of G_δ subsets of the crowded Polish space $\text{cl}(Q)$. Furthermore, it is easy to check that $Q \subseteq F \cap \text{cl}(Q) \subseteq \bigcap \mathcal{G}$. Therefore $\bigcap \mathcal{G}$ is non-meager in $\text{cl}(Q)$ by Lemma 14, and it is possible to pick $z \in \bigcap \mathcal{G} \setminus Q$.

It remains to verify that condition $\otimes(F \cup \{z\}, \mathcal{R})$ holds. Fix $R \in \mathcal{R}$, $0 \leq \ell < n_R$ and an injection $x : \ell + 1 \rightarrow F \cup \{z\}$. We need to show that $x \in G_R^{\ell+1}$. If $z \notin \text{ran}(x)$, this already follows from condition $\otimes(F, \mathcal{R})$, so assume that $z \in \text{ran}(x)$. Since $G_R^{\ell+1}$ is symmetric by construction, we can assume without loss of generality that $x(\ell) = z$. Let $x' = x \upharpoonright \ell$, and notice that $z \notin \text{ran}(x')$ because x is injective. Therefore $z \in G_R^{\ell+1}[x']$, which implies $x = x' \frown z \in G_R^{\ell+1}$. \square

6. APPLICATIONS TO INDEPENDENT FAMILIES

Given a certain kind of combinatorial object on ω (such as a filter, an almost disjoint family, or an independent family), it is natural to ask how “big” such an object can be (in the sense of cardinality, Baire category, or Lebesgue measure). In keeping with the rest of the paper, by “big” we will mean “big in the sense of Baire category”. For example, it is easy to see that every maximal filter (that is, ultrafilter) is non-meager, and the existence of such filters is guaranteed by Zorn’s Lemma. On the other hand, every almost disjoint family must be meager. Furthermore, it is an easy exercise to show that no filter (hence no independent family) can be comeager.

However, to the best of our knowledge, the existence of non-meager independent families in ZFC was an open problem. In analogy with the case of filters, one might wonder whether every maximal independent family is non-meager. The following proposition shows that this is not the case.

Proposition 18 (Milovich). *There exists a meager maximal independent family.*

Proof. Fix an infinite coinfinite $I \subseteq \omega$. Let \mathcal{A} be a maximal independent family on I . Define $x^* = x \cup (\omega \setminus I)$ for $x \subseteq I$; then let $\mathcal{A}^* = \{x^* : x \in \mathcal{A}\}$. Since $\{z \subseteq \omega : (\omega \setminus I) \subseteq z\}$ is closed nowhere dense in 2^ω , it is clear that \mathcal{A}^* is nowhere dense, hence meager. Furthermore, it is easy to check that \mathcal{A}^* is an independent family. Before continuing the proof, we clarify one bit of notation. Given $x \subseteq I$, we let $x^0 = I \setminus x$ and $x^1 = x$. On the other hand, given $x \subseteq I$, we let $(x^*)^0 = \omega \setminus (x^*)$ and $(x^*)^1 = x^*$.

It remains to show that \mathcal{A}^* is maximal. Fix $z \subseteq \omega$ such that $z \notin \mathcal{A}^*$. We need to show that $\mathcal{A}^* \cup \{z\}$ is not an independent family. Let $z' = z \cap I$. First assume that $z' \in \mathcal{A}$. Then $(z')^* \in \mathcal{A}^*$. The fact that $z \cap (\omega \setminus (z')^*) = \emptyset$ concludes the proof in this case. Now assume that $z' \notin \mathcal{A}$. Since \mathcal{A} is maximal, by adding an element to F if necessary, we can fix $F \in [\mathcal{A}]^{<\omega}$, $\delta \in 2$ and $\nu : F \rightarrow 2$ satisfying the following conditions:

- There exists $x \in F$ such that $\nu(x) = 0$.
- $w \cap ((z')^\delta)$ is finite, where $w = \bigcap \{x^{\nu(x)} : x \in F\}$.

Let $w' = \bigcap \{(x^*)^{\nu(x)} : x \in F\}$, and observe that $w' \subseteq I$ by the first condition. It is clear that $(x^*)^\varepsilon \cap I = x^\varepsilon$ for every $x \subseteq I$ and $\varepsilon \in 2$. Hence $w' = w' \cap I = w$. Furthermore, one readily sees that $z^\delta \cap I = (z')^\delta$, where $z^0 = \omega \setminus z$ and $z^1 = z$. It follows that $w' \cap (z^\delta) = w \cap ((z')^\delta)$, which concludes the proof. \square

A straightforward application of Theorem 10 shows that big independent families do exist in ZFC. In fact, the following result has been the main motivation for the research contained in this article.

Theorem 19. *There exists a non-meager independent family.*

Proof. Given $1 \leq n < \omega$ and $\nu : n \rightarrow 2$, define

$$R_\nu = \left\{ x \in (2^\omega)^n : \bigcap_{k \in n} x(k)^{\nu(k)} \text{ is finite} \right\}.$$

Let $\mathcal{R} = \{R_\nu : 1 \leq n < \omega, \nu : n \rightarrow 2\}$, and observe that an \mathcal{R} -free subset of 2^ω is simply an independent family. Furthermore, it is easy to check that each R_ν is a meager relation. An application of Theorem 10 concludes the proof. \square

Similarly, it is clear that the following result can be deduced from Theorem 16 and the remark that precedes it. Theorem 20 slightly improves [KMZ, Theorem 26], where “ $\mathfrak{d} = \mathfrak{c}$ ” is substituted by “MA(countable)”.

Theorem 20. *Assume $\mathfrak{d} = \mathfrak{c}$. Then there exists an independent family that is dense in 2^ω and completely Baire.*

7. ARBITRARY POLISH SPACES

In this section, we will show that the results of Sections 4 and 5 generalize to arbitrary (that is, not necessarily crowded) Polish spaces. We will use a straightforward adaptation of the method used in [BZ].

Theorem 21. *Let \mathcal{R} be a collection of meager relations on a Polish space X . Assume that $|\mathcal{R}| \leq \omega$. Then there exists $F \subseteq X$ that satisfies the following conditions:*

- F is dense in X .
- F is Baire.
- F is \mathcal{R} -free.

Furthermore, if X is uncountable and κ is a cardinal such that $\text{cof}(\text{meager}) \leq \kappa \leq \mathfrak{c}$, then it is possible to choose F so that the additional requirement $|F| = \kappa$ will be satisfied.

Proof. Let E be the set of isolated points of X . If X is countable, then E is dense in X , hence we can simply set $F = E$. Now assume that X is uncountable. Then it is possible to find a crowded closed subspace Z of X such that $C = X \setminus Z$ is countable (see [Ke, Theorem 6.4]). Notice that E is a dense subset of C and Z is a crowded Polish space.

Given a relation R on X with $n = n_R$ and a function p such that $\text{dom}(p) \subsetneq n$ and $\text{ran}(p) \subseteq E$, define

$$R[p] = \{q \in Z^{n \setminus \text{dom}(p)} : p \cup q \in R\}.$$

Identify each $Z^{n \setminus \text{dom}(p)}$ with $Z^{|n \setminus \text{dom}(p)|}$ through the unique increasing bijection $n \setminus \text{dom}(p) \rightarrow |n \setminus \text{dom}(p)|$. Notice that each $R[p]$ will be a meager relation on Z whenever R is meager on X . Define

$$\mathcal{R}' = \{R[p] : R \in \mathcal{R} \text{ and } p \text{ is a function such that } \text{dom}(p) \subsetneq n_R \text{ and } \text{ran}(p) \subseteq E\}.$$

Let F' be the \mathcal{R}' -free subset of Z of size κ given by Theorem 10. Set $F = E \cup F'$.

It is clear that F is dense in X and Baire. In order to check that F is \mathcal{R} -free, fix $R \in \mathcal{R}$ with $n = n_R$ and an injective $x \in F^n$. Assume, in order to get a contradiction, that $x \in R$. Let $p = x \upharpoonright \{i \in n : x(i) \in E\}$ and $q = x \setminus p$, and notice that $\text{dom}(p) \subsetneq n$ because R is meager. Observe that $q \in (F')^{|q|}$ is injective and $q \in R[p]$. This contradicts the fact that F' is \mathcal{R}' -free. \square

Theorem 22. *Assume that $\text{MA}(\text{countable})$ holds. Let \mathcal{R} be a collection of meager relations on a Polish space X . Assume that $|\mathcal{R}| < \mathfrak{c}$. Then there exists $F \subseteq X$ that satisfies the following conditions:*

- F is dense in X .
- F is completely Baire.
- F is \mathcal{R} -free.

Proof. Proceed as in the proof of Theorem 21. \square

We remark that, by the following theorem (see [MZ, Theorem 9.9]), the free set given by Theorem 22 will have size \mathfrak{c} whenever X is uncountable.

Theorem 23 (Medini, Zdomskyy). *Every uncountable completely Baire space has size \mathfrak{c} .*

8. OPEN QUESTIONS

The first two questions aim at improving Theorem 10 (hence Theorem 21 as well). Let \mathcal{R} be a collection of meager subsets of 2^ω such that $|\mathcal{R}| = \text{cov}(\text{meager})$ and $\bigcup \mathcal{R} = 2^\omega$, viewed as a collection of unary relations on 2^ω . It is clear that the only \mathcal{R} -free subset of 2^ω is the empty set. However, we do not know the answer to the following question. Notice that, by Corollary 11 and Lemma 15 respectively, any model showing that the answer to Question 1 is “no” would have to satisfy $\text{add}(\text{meager}) < \text{cov}(\text{meager})$ and $\mathfrak{d} < \text{cof}(\text{meager})$.

Question 1. Is it possible to prove in ZFC that for every collection \mathcal{R} consisting of meager relations on a crowded Polish space X such that $|\mathcal{R}| < \text{cov}(\text{meager})$ there exists a nowhere meager \mathcal{R} -free subset of X ?

Question 2. Is it possible to substitute “ $\text{cof}(\text{meager})$ ” with “ $\text{non}(\text{meager})$ ” in the statement of Theorem 10?

As Corollary 25 shows, it is consistent that $\text{non}(\text{meager}) < \text{cof}(\text{meager})$ and the answer to the above question is “yes”. Proposition 24 can be safely assumed to be folklore.

Proposition 24. *It is consistent that $\text{non}(\text{meager}) < \text{cof}(\text{meager})$ and every non-meager subset of 2^ω has a non-meager subset of size ω_1 .*

Proof. Assume that CH holds in V . We will force with the usual poset \mathbb{P}_{ω_2} for adding ω_2 Cohen reals. More precisely, given an ordinal α , we will denote by \mathbb{P}_α the poset of finite partial functions from α to 2, ordered by reverse inclusion. It is well known that $\text{non}(\text{meager}) < \text{cof}(\text{meager})$ in $V^{\mathbb{P}_{\omega_2}}$ (see [BL, Section 11.3]).

Now work in $V^{\mathbb{P}_{\omega_2}}$. Fix a non-meager subset A of 2^ω . Using the fact that CH holds in V , it is straightforward to construct a strictly increasing sequence $\langle \delta_\alpha : \alpha \in \omega_1 \rangle$ consisting of elements of ω_2 such that, given any dense \mathbb{G}_δ subset G of 2^ω coded in $V^{\mathbb{P}_{\delta_\alpha}}$, there exists $x \in 2^\omega \cap V^{\mathbb{P}_{\delta_{\alpha+1}}}$ such that $x \in A \cap G$.

Let $\delta = \sup\{\delta_\alpha : \alpha \in \omega_1\}$. Let $B = A \cap V^{\mathbb{P}_\delta}$, and notice that $|B| = \omega_1$. It is clear that $B \cap G \neq \emptyset$ for every dense \mathbf{G}_δ subset G of 2^ω coded in $V^{\mathbb{P}_\delta}$.

Finally, since $V^{\mathbb{P}_{\omega_2}}$ is the same as $V^{\mathbb{P}_\delta * \mathbb{P}_{\omega_2}}$, the arguments from [Bl, Section 11.3] show that B is a non-meager subset of 2^ω in $V^{\mathbb{P}_{\omega_2}}$. \square

Corollary 25. *It is consistent that $\text{non}(\text{meager}) < \text{cof}(\text{meager})$ and every nowhere meager subset of 2^ω has a nowhere meager subset of size ω_1 .*

Observe that the assumption of $\text{MA}(\text{countable})$ in Theorem 16 cannot be altogether dropped. To see this, consider the example at the beginning of this section in any model of $\text{cov}(\text{meager}) < \mathfrak{c}$, such as the Sacks model (see [Bl, Section 11.5]). However, we do not know the answer to the following question.

Question 3. Is it possible to prove in ZFC that for every collection \mathcal{R} consisting of meager relations on a crowded Polish space X such that $|\mathcal{R}| \leq \omega$ there exists a dense completely Baire \mathcal{R} -free subspace of X ?

Given Theorem 19, it makes sense to define the following cardinal invariant. Let

$$i(\text{meager}) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a non-meager independent family}\}.$$

It is easy to see that $\text{non}(\text{meager}) \leq i(\text{meager}) \leq \text{cof}(\text{meager})$. In fact, the first inequality is trivial, while the second one follows from Theorem 10, as in the proof of Theorem 19. Furthermore, by Corollary 25, it is consistent that $i(\text{meager}) < \text{cof}(\text{meager})$. However, we do not know if the first inequality can be strict. It is clear that a positive answer to Question 2 would also give a positive answer to the following question.

Question 4. Is $i(\text{meager}) = \text{non}(\text{meager})$ provable in ZFC?

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