World Scientific
www.worldscientific.com

n-QUASI-ISOTOPY: II. COMPARISON

SERGEY A. MELIKHOV

Steklov Mathematical Institute, Division of Geometry and Topology,
ul. Gubkina 8, Moscow 119991, Russia
sergey@melikhov.mccme.ru

and

University of Florida, Department of Mathematics,
358 Little Hall, PO Box 118105, Gainesville, FL 32611-8105, USA
melikhov@math.ufl.edu

DUŠAN REPOVŠ

Institute of Mathematics, Physics and Mechanics, University of Ljubljana,
19 Jadranska cesta, 1001 Ljubljana, Slovenija
dusan.repovs@fmf.uni-lj.si

ABSTRACT

Geometric aspects of the filtration on classical links by k-quasi-isotopy are discussed, including the effect of Whitehead doubling, relations with Smythe's n-splitting and Kobayashi's k-contractibility. One observation is: ω-quasi-isotopy is equivalent to PL isotopy for links in a homotopy 3-sphere (respectively, contractible open 3-manifold) M if and only if M is homeomorphic to S^3 (respectively, $\mathbb{R}^3$). As a byproduct of the proof of the "if" part, we obtain that every compact subset of an acyclic open set in a compact orientable 3-manifold M is contained in a PL homology 3-ball in M.

We show that k-quasi-isotopy implies $(k + 1)$-cobordism of Cochran and Orr. If $z^{m-1}(c_0 + c_1 z^2 + \cdots + c_n z^{2n})$ denotes the Conway polynomial of an m-component link, it follows that the residue class of c_k modulo $\gcd(c_0, \ldots, c_{k-1})$ is invariant under k-quasi-isotopy. Another corollary is that each Cochran's derived invariant β^k is also invariant under k-quasi-isotopy, and therefore assumes the same value on all PL links, sufficiently C^0-close to a given topological link. This overcomes an algebraic obstacle encountered by Kojima and Yamasaki, who "became aware of impossibility to define" for wild links what for PL links is equivalent to the formal power series $\sum \beta^n z^n$ by a change of variable.

Keywords: PL isotopy; k-cobordism; Conway polynomial; Milnor $\bar{\mu}$-invariants; Cochran derived invariants β^i; Whitehead double; Whitehead manifold; Fox–Artin arc; n-splitting.

Mathematics Subject Classification 2000: Primary 57M25; Secondary 57Q37, 57Q60

1. Introduction

This paper can be read independently of "*n-Quasi-isotopy I*". We relate k-quasi-isotopy to k-cobordism in Sec. 3, which is entirely independent of Sec. 2 and only

uses the following definition from Sec. 1. Section 2 is concerned with ω-quasi-isotopy. This section illustrates various versions of k-quasi-isotopy by examples and simple geometric observations, and reduces them to k-splitting and k-contractibility. Unless the contrary is explicitly stated, everything is understood to be in the PL category.

1.1. *Definition*

Let N be a compact 1-manifold, M an orientable 3-manifold, and k a nonnegative integer for purposes of this section and Sec. 3, or an ordinal number for purposes of Sec. 2. Two PL embeddings L, $L'\colon(N,\partial N)\hookrightarrow(M,\partial M)$ will be called (*weakly*) [*strongly*] *k-quasi-isotopic* if they can be joined by a generic PL homotopy where every singular level is a (weak) [strong] k-quasi-embedding. A PL map $f\colon(N,\partial N)\to(M,\partial M)$ with precisely one double point x will be called a (*weak*) [*strong*] *k-quasi-embedding* if there exist an arc J_0 with $\partial J_0=f^{-1}(x)$ and chains of subpolyhedra $\{x\}=P_0\subset\cdots\subset P_k\subset M$ and $J_0\subset\cdots\subset J_k\subset N$ such that

(i) $f^{-1}(P_i)\subset J_i$ for each $i\leq k$;
(ii) $P_i\cup f(J_i)\subset P_{i+1}$ for each $i<k$;
(iii) the inclusion $P_i\cup f(J_i)\hookrightarrow P_{i+1}$ is null-homotopic (respectively, induces zero homomorphisms on reduced integral homology) for each $i<k$;
(iv) P_{i+1} is a compact PL 3-manifold [respectively, closed PL 3-ball] and J_{i+1} is a closed arc for each $i<k$.

We could let the P_i's with finite indices be arbitrary compact subpolyhedra of M (as we did in "*n-Quasi-isotopy I*"); the above situation is then restored by taking regular neighborhoods. Also, by [33] or [13], condition (iii) can be weakened for finite i to

(iii$'$) the inclusion $P_i\cup f(J_i)\hookrightarrow P_{i+1}$ is trivial on π_1 (respectively, H_1) for finite $i<k$.

Let us consider small values of k in more detail, in the case where $M=S^3$ and N is $mS^1:=S_1^1\sqcup\cdots\sqcup S_m^1$. Let $f\colon mS^1\to S^3$ be a map with precisely one double point $f(p)=f(q)$. Clearly, f is a 0-*quasi-embedding*, in either of the 3 versions, iff it is a *link map*, i.e. $f(S_i^1)\cap f(S_j^1)=\emptyset$ whenever $i\neq j$. Hence all 3 versions of 0-quasi-isotopy coincide with *link homotopy*, i.e. homotopy through link maps. Note that if f is a link map, its singular component splits into two *lobes*, i.e. the two loops with ends at $f(p)=f(q)$.

The map f is a (*weak*) [*strong*] 1-*quasi-embedding* iff it is a link map, and at least one of the two lobes of the singular component (namely, J_0) is null-homotopic (respectively, null-homologous) [respectively, contained in a PL 3-ball] in the complement X to the other components. See [25, Sec. 2] for some observations on 1-quasi-isotopy.

Finally, f is a 2-*quasi-embedding* iff it is a 1-quasi-embedding, so $f(J_0)$ is null-homotopic in X, and, for some arc $J_1 \subset S_i^1$ containing $f^{-1}(F(D^2))$, this null-homotopy $F \colon D^2 \to X$, which we assume PL and generic, can be chosen so that every loop in $F(D^2) \cup f(J_1)$ is null-homotopic in X. Note that a point of the finite set $f^{-1}(F(\operatorname{Int} D^2))$ may be outside J_0 as well as inside it (compare examples (i) and (iii) below).

1.2. *Examples*

The reader is encouraged to visualize

  (i) a strong null-$(k-1)$-quasi-isotopy for the Milnor link $\mathcal{M}_k$ [28] (for $k=4$ see Fig. 1 ignoring the four disks), and for any of its twisted versions (in Fig. 1, imagine any number of half-twists along each disk);

 (ii) a strong k-quasi-isotopy between any two twisted versions of $\mathcal{M}_k$ that only differ by some number of full twists in the rightmost clasp;

(iii) a null-$(k-1)$-quasi-isotopy for the link $\mathcal{W}_k$, the k-fold untwisted left handed Whitehead double of the Hopf link (for $k=3$ see Fig. 2(a) ignoring the three disks);
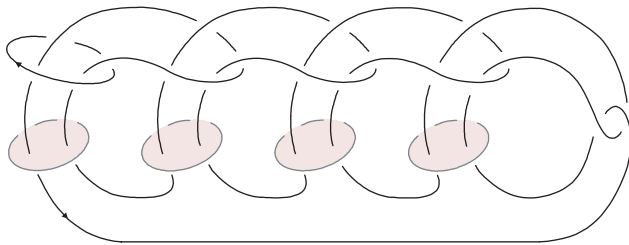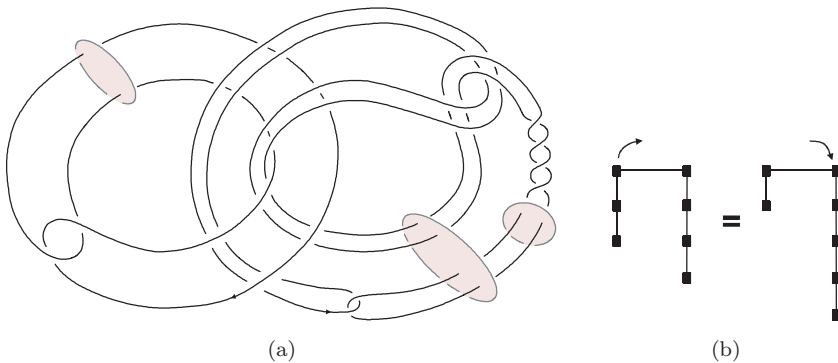

Fig. 1.

(a)   (b)

Fig. 2.

(iii$'$) a null-$(k-1)$-quasi-isotopy for any k-fold Whitehead double of the Hopf link that is untwisted (i.e. differs from $\mathcal{W}_k$ by at most one positive half-twist) at all stages except possibly for the first or the last, and is arbitrarily twisted at that stage (in Fig. 2(a), imagine any number of half-twists along one of the two smaller disks and possibly one clockwise 180 degree rotation of the visible side of the other smaller disk and/or the larger disk).

 (iv) a weak null-$(k-1)$-quasi-isotopy for an arbitrarily twisted k-fold Whitehead double of the Hopf link (any number of half-twists along each disk in Fig. 2(a));

 (v) a (weak) k-quasi-isotopy between any versions of $\mathcal{W}_k$, untwisted at the first $k-1$ stages and arbitrarily twisted at the last stage (respectively, arbitrarily twisted at all stages), that only differ from each other at the last stage.

Note that it does not matter which component is doubled in an iterated left handed untwisted Whitehead doubling of the Hopf link (and therefore $\mathcal{W}_k$ is well-defined) by the symmetry of $\mathcal{W}_1$, i.e. realizability of its components' interchange by an ambient isotopy, applied inductively as shown in Fig. 2(b).

Remark. We show in Sec. 3 that $\mathcal{M}_k$ is not null-k-quasi-isotopic, for each $k \in \mathbb{N}$.

Conjecture 1.1. (a) *$\mathcal{W}_k$ is not null-k-quasi-isotopic, even weakly, for each $k \in \mathbb{N}$.*
  (b) *$\mathcal{W}_k$ is not strongly null-1-quasi-isotopic, for each $k \in \mathbb{N}$.*

Certainly, the obvious null-$(k-1)$-quasi-isotopy in (iii) is not a strong 1-quasi-isotopy, since $\mathcal{W}_{k-1}$ (for $k \geq 1$) is not a split link [36] (see also [1, Sec. 3]), and is not a weak k-quasi-isotopy, since the components of $\mathcal{W}_{k-1}$ are $(k-1)$-linked [32].

1.3. *n-Linking, n-splitting and n-contractibility*

In 1937, Eilenberg termed two disjoint knots $K_1 \cup K_2 \subset S^3$ 0-linked if they have nonzero linking number, and *n-linked* if every subpolyhedron of $S^3 \backslash K_2$, in which K_1 is null-homologous, contains a knot, $(n-1)$-linked with K_2. Note that if K_1 and K_2 are not n-linked, K_1 bounds a map of a grope of depth $n+1$ into $S^3 \backslash K_2$, i.e. represents an element of the $(n+1)$th derived subgroup of $\pi_1(S^3 \backslash K_2)$.

Having proved that the components of $\mathcal{W}_n$ are n-linked (as it had long been expected), Smythe proposed in 1970 that the relation of being not n-linked should be replaced by a stronger relation with substantially lower quantifier complexity. Compact subsets A, $B \subset S^3$ are said to be *n-split* [32, p. 277], if there exists a sequence of compact subpolyhedra $A \subset P_0 \subset \cdots \subset P_{n+1} \subset S^3 \backslash B$ such that each inclusion $P_i \subset P_{i+1}$ is trivial on reduced integral homology groups. Thus "(-1)-split" means "disjoint", and two disjoint knots are 0-split iff they have zero linking number. It is also not hard to see that the following are equivalent for a 2-component PL link $K_1 \cup K_2$:

(1) K_1 and K_2 are 1-split;
(2) K_1 and K_2 are not 1-linked;
(3) $K_1 \cup K_2$ is a *boundary link*, i.e. K_1 and K_2 bound disjoint Seifert surfaces.

On the other hand, it turns out (see Theorem 2.8 below) that if A and B are compact subpolyhedra of S^3, there is an $n \in \mathbb{N}$ such that if A and B are n-split, they are split by a PL embedded S^2.

An advantage of working in S^3 is that the relation of being n-split is symmetric for compact subpolyhedra of S^3 [32]. However, as in [33], we can consider the same notion of n-splitting for compact subsets $A, B \subset M$ of an orientable 3-manifold M. Its interest for us stems from

Theorem 1.2. *For each $n \in \mathbb{N}$, links $L_1, L_2 : (I \sqcup \cdots \sqcup I, \partial) \hookrightarrow (M, \partial M)$ are (weakly) n-quasi-isotopic if and only if they can be joined by a generic PL homotopy h_t such that for every t, each component of the image of h_t is $(n-1)$-contractible in the complement to the others (respectively, is $(n-1)$-split from the union of others).*

The notion of n-contractibility was introduced recently by Kobayashi [16]. We say that a compact subset $A \subset M$ is *n-contractible* in M if there exists a sequence of compact subpolyhedra $A \subset P_0 \subset \cdots \subset P_{n+1} \subset M$ such that each inclusion $P_i \subset P_{i+1}$ is null-homotopic.[a]

Proof. The "if" part follows immediately from the definitions by taking regular neighborhoods. The converse is also easy: just replace the manifolds P_i from the definition of (weak) n-quasi-embedding by their unions with the entire singular component. $\square$

This result does not hold for links in S^3 (see Remark (iii) below), for which we could only find a faint version of it (see next proposition). However, a link of m circles in S^3 uniquely corresponds to a link of m arcs in the exterior of m PL 3-balls in S^3, and this bijection descends to n-quasi-isotopy classes (as well as ambient isotopy classes and PL isotopy classes). This avoids the non-uniqueness of presentation of a link in S^3 as the closure of a string link.

1.4. *Brunnian links and Whitehead doubling*

We now turn back to links in S^3. Clearly, if f is a weak n-quasi-embedding with a single double point, the lobe J_0 of f is $(n-1)$-split from the union of the non-singular components. Conversely, if the ith component of a link is null-homotopic in a polyhedron, $(n-1)$-split from the union of the other components, then the link is evidently weakly n-quasi-isotopic to a link with the ith component split by a PL embedded S^2 from the other components. In particular:

Proposition 1.3. *If a component of a Brunnian link is $(n-1)$-split from the union of the other components, then the Whitehead doubling of the link on this component (with arbitrary twisting) is weakly null-n-quasi-isotopic.*

---

[a]In the definition of n-contractibility (n-splitting) it is enough to require that each homomorphism $\pi_1(P_i) \to \pi_1(P_{i+1})$ (respectively, $H_1(P_i) \to H_1(P_{i+1})$) be zero, provided that A is a 1-dimensional subpolyhedron of M with no edges in ∂M [13, 33]; compare [32, argument on p. 278].

Remark. (i) As shown by the Borromean rings, Proposition 1.3 cannot be "desuspended", that is, a Brunnian link with a component n-split from the union of the other components is not necessarily weakly null-n-quasi-isotopic.

(ii) Assuming Conjecture 1.1, $n-1$ cannot be replaced with $n-2$ in Proposition 1.3, since the components of $\mathcal{W}_{n-1}$ are $(n-2)$-split.

(iii) Neither version of null-n-quasi-isotopy implies even 1-splitting, since $\mathcal{M}_{k+1}$ is strongly null-n-quasi-isotopic, but is not a boundary link, as detected by Cochran's invariants (cf. Sec. 3).

Once Whitehead doubling appeared above, it is tempting to study the behavior of our filtration under this operation. It is easy to see that if two links are weakly n-quasi-isotopic with support in the ith component, then the links, obtained by Whitehead doubling (with arbitrary twisting) on all components except for the ith one, are weakly $(n + 1)$-quasi-isotopic with support in the ith component. Two possible ways to deal with the Whitehead double K' of the ith component K (drag K' along the given homotopy of K or unclasp K' in a regular neighborhood of K) result in the following two corollaries.

Proposition 1.4. (a) *Weakly 1-quasi-isotopic links have weakly 2-quasi-isotopic Whitehead doubles. Weakly 2-quasi-isotopic links with trivial pairwise linking numbers have weakly 3-quasi-isotopic Whitehead doubles.*

(b) *The Whitehead double of the Borromean rings is weakly null-2-quasi-isotopic.*

Here the Whitehead doubling is performed on all components, with an arbitrary (but fixed) twisting on each. Part (a) does not generalize, since the property of being a boundary link is not preserved under 3-quasi-isotopy (see Remark (iii) above).

Proof. Let us verify e.g. the second part of (a). If $f \colon mS^1 \to S^3$ is a singular link in a generic PL weak 2-quasi-isotopy between the given links, let P_1 be as in the definition, and $N \sqcup N'$ be a regular neighborhood of $f(mS^1)$ in S^3 such that $f^{-1}(N)$ is the singular component, say S_i^1, and $N' \cap P = \emptyset$. We can assume that $P_1 \cap N$ is connected, and that N (respectively, N') contains the Whitehead doubles of the two resolutions K_+, K_- of $f(S_i^1)$ (respectively, $Wh(f(S_j^1))$ for all $j \neq i$). Then each of the four singular knots in the obvious homotopy between $Wh(K_+)$ and $Wh(K_-)$ is a 1-quasi-embedding, if regarded as a map into $N \cup P_1$. The hypotheses imply that every cycle in either P_1 or N is null-homologous in $S^3 \backslash N'$, hence so is every cycle in $P_1 \cup N$. Finally, every cycle in $S^3 \backslash N'$ is null-homologous in $S^3 \backslash \bigcup_{j \neq i} Wh(S_j^1)$. □

2. ω-Quasi-Isotopy

In this section we study how the relations of k-quasi-isotopy approximate the relation of *PL isotopy*, i.e. PL homotopy through embeddings.

2.1. *Basic observations*

First of all, we get rid of infinite ordinals in the definition of k-quasi-isotopy for $k = w$, $w + 1$ and w_1.

Proposition 2.1. *Let $f: N \to M$, its double point $\{x\} = P_0$, the 3-manifolds $P_1 \subset P_2 \subset \cdots$ and the arcs $J_0 \subset J_1 \subset \cdots$ be as in the definition of (weak) [strong] k-quasi-embedding in Sec.* 1. *Then*

(a) *f is a (weak) [strong] ω-quasi-embedding iff the P_i, J_i's exist for all finite i.*
(b) *f is a (weak) $(\omega + 1)$-quasi-embedding iff the P_i, J_i's exist for all finite i and $\bigcup_{i<\omega} P_i$ is contained in a compact 3-manifold $Q \subset M$.*
(c) *f is a (weak) [strong] ω_1-quasi-embedding iff x is contained in a contractible (acyclic) compact 3-manifold [closed 3-ball] P_* such that $f^{-1}(P_*)$ is an arc J_*.*
(d) *Two links $L, L': (N, \partial N) \hookrightarrow (M, \partial M)$ are strongly ω_1-quasi-isotopic iff they are PL isotopic.*

By (d), PL isotopy implies either version of k-quasi-isotopy for any k; conversely, each k-quasi-isotopy is supposed to give an approximation of PL isotopy. This explains why the P_i's were required to be compact in Sec. 1 when $i = i' + 1$ for some i'; the point of our dropping this requirement when this is not the case (i.e. when i is a *limit* ordinal) is that the interesting relation of ω-quasi-isotopy would otherwise fall out of the hierarchy due to the resulting shift of infinite indices.[b]

Proof. (a) Set $P_\omega = \bigcup_{i<\omega} P_i$ and $J_\omega = \bigcup_{i<\omega} J_i$. Then the only condition $f^{-1}(P_\omega) \subset J_\omega$ imposed on P_ω and J_ω holds due to $f^{-1}(P_i) \subset J_i$ for finite i.

(b) Set tentatively $P_\omega = \bigcup_{i<\omega} P_i$ and $J_\omega = \bigcup_{i<\omega} J_i$. If the closure of J_ω happens to be the entire singular component S_i^1, we puncture S_i^1 at the singleton $pt = S_i^1 \backslash J_\omega$ and M at $f(pt)$, and sew in a trivial closed PL ball pair (B^3, B^1). (This in effect redefines $J_k, \ldots, J_\omega$ and $P_k, \ldots, P_\omega$ starting from some k, similarly to the explicit redefinition that follows.) Next, consider a map $F: M \to M$ shrinking a regular neighborhood R of the 1-manifold $Z := \overline{f(N \backslash J_1)}$ rel ∂J_1 onto Z so that all non-degenerate point inverses are closed PL 2-disks, and sending $M \backslash R$ homeomorphically onto $M \backslash Z$. Redefine each P_i to be $F^{-1}(P_i)$ for $2 \leq i \leq \omega$; it is easy to see that the conditions (i)–(iv) from Sec. 1 still hold for these. Set $P_{\omega+1} = \overline{F^{-1}(Q) \backslash F^{-1}(Z')}$, where $Z' = \overline{f(N \backslash J_\omega)}$. This is a compact 3-manifold (since Q is), and $J_{\omega+1} := f^{-1}(P_{\omega+1}) = \overline{J_\omega}$ is a closed arc. Moreover, the inclusion $P_\omega \cup f(J_\omega) \subset P_{\omega+1}$ holds (since $f^{-1}(P_\omega) = J_\omega$ due to conditions (i) and (ii) for

---

[b] The extra care about limit ordinals $> \omega$ will turn out to be superfluous in light of the following results. Note, however, that the k-quasi-concordance filtration on classical links [25, Remark (ii) at the end of Sec. 1] may well be highly nontrivial when similarly extended for infinite ordinals.

finite i, and since $P_\omega \subset Q$), and is null-homotopic/null-homologous (since P_ω is contractible/acyclic due to condition (iii) for finite i).

(c) The 3-manifolds P_i and the arcs J_i have to stabilize at some countable stage since M and N are separable.

(d) Let h_s be a singular level in a strong ω-quasi-isotopy $h_t \colon (N, \partial N) \to (M, \partial M)$, and let the 3-ball P_* and the arc J_* be as in (c) corresponding to $f = h_s$. Without loss of generality, there is an $\varepsilon > 0$ such that for each $t \in [s - \varepsilon, s + \varepsilon]$, the level h_t is an embedding, coinciding with h_s outside J_*. The transition between the links $h_{s-\varepsilon}$ and $h_{s+\varepsilon}$ can now be realized by a PL isotopy that is conewise on J_* and first shrinks the local knot of $h_{s-\varepsilon}$ occurring in P_* to a point and then inserts the local knot of $h_{s+\varepsilon}$ occurring in P_* by an inverse process. $\qquad\square$

Lemma 2.2 [38]. *A knot in a 3-manifold M, PL isotopic to the unknot, is contained in a ball in M.*

Under the unknot in a 3-manifold M we understand the ambient isotopy class in M of the unknot in some ball in M (since all balls in M are ambient isotopic). A PL isotopy may first create a local knot K, then push, say, a homology ball "through a hole" in K, and finally shrink K back to a point, so Lemma 2.2 is not obvious. In fact, its higher-dimensional analogue is false (for PL isotopy of S^n in $S^{n+2} \backslash S^n$) [37]. There also exist two PL links $L, L'\ 2S^1 \hookrightarrow S^1 \times S^2$ that are PL isotopic but not ambient isotopic, even though each component of L is ambient isotopic with the corresponding component of L' [37].

For convenience of the reader we shall give an alternative proof of Lemma 2.2, based on an approach different from Smythe's.

Proof. This is in the spirit of the uniqueness of knot factorization. Suppose that a knot $K' \subset \operatorname{Int} M$ is obtained from a knot $K \subset \operatorname{Int} M$ by a PL isotopy with support in a ball $B \subset \operatorname{Int} M$, which intersects K (hence also K') in an arc. Assuming that K is contained in a ball, we show that so is $B \cup K$, hence also K'. We may view ball as a special case of punctured ball, i.e. a homeomorph of the exterior in S^3 of a nonempty collection of disjoint balls. Let n be the minimal number of components in the intersection of ∂B with the boundary of a punctured ball $P \subset \operatorname{Int} M$, containing K.

We claim that $n = 0$. Let C be a circle in $\partial B \cap \partial P$ that is innermost on ∂B (with $\partial B \cap K$ considered "outside"). Then C bounds a disk $D \subset \partial B$ with $D \cap \partial P = C$. If D lies outside P, we attach to P the embedded 2-handle with core D, thus increasing the number of components in ∂P, but decreasing the number of components in $\partial B \cap \partial P$. If D is inside P, we subtract the 2-handle from P, which by Alexander's Schönflies theorem splits P into two punctured balls, and discard the one which is disjoint from K.

Thus we may assume that each component of ∂P is disjoint form ∂B. By Alexander's theorem the components of ∂P, contained in B, bound balls in B.

Since B contains only an arc of K, P itself cannot be contained in B, so these balls are disjoint from Int P. Adding them to P makes P have no boundary components in B, hence contain B. If this makes P have no boundary whatsoever, M must be S^3, and there is nothing to prove then. Otherwise we may connect the components of ∂P by drilling holes in $P \backslash (B \cup K)$, so as to obtain a ball containing $B \cup K$. $\square$

Remark.   Rolfsen asked [37] whether every two-component link in a 3-manifold that is isotopic to the unlink is a split link (i.e. a link whose components are contained in disjoint balls). In fact, this had been established earlier by Smythe [38]. The above proof of Lemma 2.2 yields a different proof of this fact. Indeed, if L is isotopic to a split link L' by a PL isotopy with support in the ith component, by the proof of Lemma 2.2, the ith component of L is contained in a ball B disjoint from the other components. Although B may intersect the disjoint balls containing the other components by the hypothesis, this can be remedied via the same argument.

2.2. *Simply-connected 3-manifolds and finiteness results*

Recall the construction of the Whitehead contractible open manifold W. Consider a nested sequence of solid tori $\cdots T_1 \subset T_0 \subset S^3$, where each $T_{i+1} \cup S^3 \backslash T_{i+1}$ is equivalent, by a homeomorphism of S^3, to a regular neighborhood of the Whitehead link $\mathcal{W}_1$ in S^3. Then W is the union of the ascending chain of solid tori $S^3 \backslash T_0 \subset S^3 \backslash T_1 \subset \cdots$. Now W is not homeomorphic to $\mathbb{R}^3$, since a meridian $K_0 := \partial D^2 \times \{pt\}$ of $T_0 \cong D^2 \times S^1$ is not contained in any (piecewise-linear) 3-ball in W; otherwise the ball would be disjoint from some T_n, hence $\mathcal{W}_n$ (see Fig. 2 above) would be a split link, which is not the case [36] (see also [1, Sec. 3]).

The preceding lemma implies that K_0 is not PL isotopic to the unknot in W. On the other hand, since the Whitehead manifold W is contractible, it is not hard to see that any knot K in W is ω-quasi-isotopic to the unknot. Indeed, if f is a singular level in a generic PL homotopy between K and the unknot, we set J_0 to be any of the two lobes, and construct the polyhedra P_j and the arcs J_j, $1 \le j < \omega$, as follows. Assuming that P_j is a handlebody, we set P_{j+1} to be a regular neighborhood of P_j union the track of a generic PL null-homotopy of a wedge of circles onto which $P_j \cup f(J_j)$ collapses. By [33] or [13], the null-homotopy can be chosen so that P_{j+1} is again a handlebody. Since the track is 2-dimensional and generic, $f^{-1}(P_{j+1})$ is not the whole circle, and we set J_{j+1} to be the smallest arc containing $f^{-1}(P_{j+1})$. Then f is an ω-quasi-embedding by Proposition 2.1(a).

In fact, any contractible open 3-manifold other than $\mathbb{R}^3$ contains a knot which is not contained in any 3-ball [8]. Thus the above argument proves

Proposition 2.3. *Any contractible open* 3-*manifold other than* $\mathbb{R}^3$ *contains a knot,* ω-*quasi-isotopic but not PL isotopic to the unknot.*

By Bing's characterization of S^3 (see [9] for a proof and references to 4 other proofs), any closed 3-manifold M other than S^3 contains a knot K that is not

contained in any ball, hence is not PL isotopic to the unknot. If M is a homotopy sphere, clearly K is ω_1-quasi-isotopic to the unknot. (Indeed, let f be a singular level in a generic PL homotopy between K and the unknot, and let B be a PL 3-ball in M meeting $f(mS^1)$ in an arc; then $P_* := \overline{M \backslash B}$ is contractible and $f^{-1}(P_*)$ is an arc.) This proves

Proposition 2.4. *Any homotopy* 3*-sphere other than* S^3 (*if exists*) *contains a knot,* ω_1*-quasi-isotopic but not PL isotopic to the unknot.*

Let us now turn to positive results.

Theorem 2.5. *Two links in a compact orientable* 3*-manifold are weakly* ω*-quasi-isotopic if and only if they are weakly* ω_1*-quasi-isotopic.*

This is in contrast with the existence of compact orientable 3-manifolds such that the lower central series of the fundamental group does not stabilize until the 2ωth stage [7] (see also [26]). However, it is an open problem, to the best of our knowledge, whether a compact 3-manifold group or indeed a finitely presented group may have derived series of length $> \omega$. (Theorem 2.8 below implies that embedded loops in a compact 3-manifold M that bound *embedded* gropes of depth exceeding certain finite number $n(M)$ are contained in the intersection of the transfinite derived series of $\pi_1(M)$.)

Corollary 2.6. *Let* M *be a compact orientable* 3*-manifold containing no nontrivial homology balls. Then the relations of* ω*-quasi-isotopy, weak* ω*-quasi-isotopy, strong* ω*-quasi-isotopy and PL isotopy coincide for links in* M.

Note that by Alexander's Schönflies theorem any 3-manifold, embeddable in S^3, contains no nontrivial homology balls.

Corollary 2.7. *The assertions of Theorem* 2.5 *and Corollary* 2.6 *hold for links in a non-compact orientable* 3*-manifold if* ω *is replaced by* $\omega + 1$.

The first of the two corollaries above follows by part (d) and the second by part (a) along with the (trivial) implication "only if" of part (b) of Proposition 2.1.

Proof of Theorem 2.5. Let Q^3 be the manifold and $f : mS^1 \to Q$ a weak ω-quasi-embedding. Let S_i^1 be the singular component, and J_0 be as in the definition of k-quasi-embedding. Let D^3 be a regular neighborhood of some $y \in f(S_i^1 \backslash J_0)$ in N^3, meeting the image of f in an arc D^1. Then $A := \overline{f(S_i^1) \backslash D^1}$ is n-split from $B := f(mS^1 \backslash S_i^1)$ in $M := \overline{Q^3 \backslash D^3}$ for every $n \in \mathbb{N}$ (cf. Theorem 1.2). Thus the assertion follows from

Theorem 2.8. *Let* M *be a compact connected orientable* 3*-manifold and* A, B *nonempty compact subpolyhedra of* M. *Then there exists an* $n \in \mathbb{N}$ *such that if* A *is* n*-split from* B, *then* A *is contained in a PL homology ball in* $M \backslash B$.

In the case $M = S^3$ this was claimed in [32], but the proof there appears to be valid only if both A and B are connected (Lemma on p. 279 is incorrect if A is disconnected, due to a misquotation on p. 280 of Theorem from p. 278, and no proof is given when B is disconnected). The idea of our proof is close to Smythe's, but essential technical changes are needed in case A or B is disconnected, $\partial M \neq \emptyset$ or $H_1(M) \neq 0$. Interplay of these cases brings additional complications: it is certainly unnecessary to invoke Ramsey's theorem if either B is connected or $\partial M = \emptyset$ (argue instead that for each s there is an i such that $H_1(M_i^s) = 0$). Note that in proving Theorem 2.5 for 3-component links in S^3 one already encounters both nonempty ∂M and disconnected B in Theorem 2.8.

Note that the knot K_0 in the Whitehead manifold W (see above) and the Whitehead continuum $S^3 \backslash W$ are mutually n-split in S^3 for all $n \in \mathbb{N}$, but no PL embedded S^2 splits them, since each $\mathcal{W}_n$ is not a split link. Another such example is based on Milnor's wild link $\mathcal{M}_\infty$ [28, p. 303], which can be approximated by the links $\mathcal{M}_n$ (see Fig. 1). Remove from S^3 a PL 3-ball, disjoint from the tame component and meeting the wild component in a tame arc, and let $\mu \colon (S^1 \sqcup I, \partial I) \hookrightarrow (B^3, \partial B^3)$ denote the resulting wild link. Then $\mu(I)$ is n-split from $\mu(S^1)$ in B^3 for all $n \in \mathbb{N}$, but is not contained in any PL 3-ball in $B^3 \backslash \mu(S^1)$, since each $\mathcal{M}_n$ is not PL isotopic to the unlink (cf. Sec. 3).

2.3. *Proof of the finiteness results*

The proof of Theorem 2.8 is based on the Kneser–Haken Finiteness theorem, asserting that there cannot be infinitely many disjoint incompressible, ∂-incompressible pairwise non-parallel surfaces[c] in a compact 3-manifold M [12], [14, III.20]. (Erroneous theorem III.24 in [14], which was concerned with weakening the condition of ∂-incompressibility, is corrected in [11]). More precisely, we will need a slightly stronger version of this result, where the manifold M is obtained by removing a compact surface with boundary from the boundary of a compact 3-manifold $\hat{M}$. (A proper surface in M, ∂-incompressible in M, need not be ∂-incompressible in $\hat{M}$.) This strengthening is proved in [21, 6.3.10] in the case where M is irreducible and ∂-irreducible; the general case can be reduced to this case by the same argument [14].

Notation. Fix an $n \in \mathbb{N}$, and let M_0 and $\overline{M \backslash M_{n+1}}$ be disjoint regular neighborhoods of A and B in M. A collection $(M_1, \ldots, M_n)$ of 3-dimensional compact PL

---

[c]A connected 2-manifold F, properly PL-embedded in an orientable 3-manifold M, is called *compressible* if either $F = \partial B$ for some 3-ball $B \subset M$ or there exists a 2-disk $D \subset M$ such that $D \cap F = \partial D$ and $D \cap F \neq \partial D'$ for any 2-disk $D' \subset F$. Next, F is *∂-compressible* if either $F = \overline{\partial B \backslash \partial M}$ for some 3-ball $B \subset M$, meeting ∂M in a disk, or there exists a 2-disk $D \subset M$, meeting ∂M in an arc and such that $D \cap F = \overline{\partial D \backslash \partial M}$ and $D \cap F \neq \overline{\partial D' \backslash \partial M}$ for any 2-disk $D' \subset M$. Finally, F and F' are *parallel* in M if $(F', \partial F')$ is the other end of a collar $h(F \times I, \partial F \times I)$ of $(F, \partial F) = h(F \times \{0\}, \partial F \times \{0\})$ in $(M, \partial M)$.

submanifolds of M such that $M_i \cap \overline{M \setminus M_{i+1}} = \emptyset$ and $i_*\colon H_1(M_i) \to H_1(M_{i+1})$ is zero for each $i = 0, \ldots, n$ will be called a *pseudo-splitting*. Its *complexity* is defined to be

$$c(M_1, \ldots, M_n) := \sum_{i=1}^{n} \sum_{F \in \pi_0(\mathrm{Fr}\, M_i)} \big(\mathrm{rk}\, H_1(F) + \mathrm{rk}\, H_{-1}(\partial F)\big)^2,$$

where $H_{-1}(X) = 0$ or $\mathbb{Z}$ according as X is empty or not, and π_0 stands for the set of connected components.

Lemma 2.9. (compare [22, p. 130]) *If $(M_1, \ldots, M_n)$ is a pseudo-splitting of minimal complexity, each component of $\mathrm{Fr}\, M_1 \cup \cdots \cup \mathrm{Fr}\, M_n$, which is not a sphere or a disk, is incompressible and ∂-incompressible in $M \setminus (A \cup B)$.*

Proof. Suppose that a component of some $\mathrm{Fr}\, M_i$, $1 \le i \le n$, which is not a sphere or a disk, is $(\partial\text{-})$compressible in $M \setminus (A \cup B)$. By the innermost circle argument, a component F of some $\mathrm{Fr}\, M_j$, $1 \le j \le n$, which is not a sphere or a disk, is compressible (respectively, compressible or ∂-compressible) in $M \setminus (M_{j-1} \cup \overline{M \setminus M_{j+1}})$. If the $(\partial\text{-})$compressing disk D lies outside M_j, attach to M_j an embedded 2-handle with core D (respectively, cancelling 1-handle and 2-handle with cores $D \cap \partial M$ and D) and denote the result by M_j'. By Mayer–Vietoris, $i_*\colon H_1(M_j) \to H_1(M_j')$ is epic. If D lies inside M_j, remove the handle from M_j, and denote the result again by M_j'; by Mayer–Vietoris, $i_*\colon H_1(M_j') \to H_1(M_j)$ is monic. In any case, it follows that $(M_1, \ldots, M_j', \ldots, M_n)$ is a pseudo-splitting.

If ∂D (respectively, $\overline{\partial D \setminus \partial M}$) is not null-homologous in $(F, \partial F)$, the result $F' \subset \mathrm{Fr}\, M_j'$ of the surgery on F is connected, and $\mathrm{rk}\, H_1(F') < \mathrm{rk}\, H_1(F)$. Since $\partial F' = \emptyset$ iff $\partial F = \emptyset$, we have that $c(M_1, \ldots, M_j', \ldots, M_n) < c(M_1, \ldots, M_n)$ in this case. In the other case F' consists of two connected components F_+' and F_-' such that $\mathrm{rk}\, H_1(F_+') + \mathrm{rk}\, H_1(F_-') \le \mathrm{rk}\, H_1(F)$. If one of the summands, say $\mathrm{rk}\, H_1(F_+')$, is zero, we must be in the case of non-∂ compression, and F_+' must be a disk. Hence if c_F denotes $\mathrm{rk}\, H_1(F) + \mathrm{rk}\, H_{-1}(\partial F)$, both $c_{F_+'}$ and $c_{F_-'}$ are always nonzero, whereas $c_{F_+'} + c_{F_+'} \le c_F + 1$. Thus again $c(M_1, \ldots, M_j', \ldots, M_n) < c(M_1, \ldots, M_n)$. □

Lemma 2.10 (compare [32, p. 278]). *Let $(M_1, \ldots, M_n)$ be a pseudo-splitting.*

(a) *If a component N of M_i is disjoint from M_{i-1}, $(M_1, \ldots, M_i \setminus N, \ldots, M_n)$ is a pseudo-splitting. If a component N of $\overline{M \setminus M_i}$ is disjoint from $\overline{M \setminus M_{i+1}}$ and either $i < n$ or $H_1(M) \to H_1(N, \mathrm{Fr}\, N)$ is zero, $(M_1, \ldots, M_i \cup N, \ldots, M_n)$ is a pseudo-splitting.*

(b) *There exists a pseudo-splitting $(M_1^*, \ldots, M_n^*)$ such that*

(i) *each $\mathrm{Fr}\, M_i^* \subset \mathrm{Fr}\, M_i$;*

(ii) *each $i_*\colon H_0(M_{i-1}^*) \to H_0(M_i^*)$ and $i_*\colon H_0(M \setminus M_{i+1}^*) \to H_0(M \setminus M_i^*)$ are onto;*

(iii) *no component of M_1^* is disjoint from A, and $H_1(M) \to H_1(N, \partial N)$ is nonzero for each component N of $\overline{M \setminus M_n^*}$, disjoint from B.*

Proof. First note that (b) follows by an inductive application of (a). The first assertion of (a) is obvious; we prove the second. Assume that $H_1(M) \to H_1(N, \mathrm{Fr}\, N)$ is zero. $H_1(M_i \cup N) \to H_1(N, \mathrm{Fr}\, N)$ factors through $H_1(M)$, hence is zero. Then $i_*\colon H_1(M_i) \to H_1(M_i \cup N)$ is onto, and the assertion follows.

If $i < n$, $H_1(M_{i+1}) \to H_1(N, \mathrm{Fr}\, N)$ factors through $H_1(M_{i+2})$, hence is zero. On the other hand, $H_1(M_{i+1}) \to H_1(M_{i+1}, M_i) \simeq H_1(N, \partial N) \oplus H_1(M_{i+1}, M_i \cup N)$ is monic. So $H_1(M_{i+1}) \to H_1(M_{i+1}, M_i \cup N)$ is monic and $H_1(M_i \cup N) \to H_1(M_{i+1})$ is zero. $\qquad\square$

Proof of Theorem 2.8. Let D be $\mathrm{rk}\, H_1(M)$ plus the number of elementary divisors of $\mathrm{Tors}\, H_1(M)$, and set $S = \mathrm{rk}\, H_0(A) + \mathrm{rk}\, H_0(B) + D + \mathrm{rk}\, H_1(M)$. Recall the simplest case of Ramsey's theorem: for each $k, l \in \mathbb{N}$ there exists an $R(k, l) \in \mathbb{N}$ such that among any $R(k, l)$ surfaces in a 3-manifold either some k are pairwise parallel, or some l are pairwise non-parallel. Let h be the number given by the Haken Finiteness Theorem for the 3-manifold $M \backslash (A \cup B)$. Set $r_0 = 2$, $r_{i+1} = R(r_i, h)$ (so $r_1 = h + 1$) and finally $n = Sr_S$.

Since A is n-split from B, there exists a pseudo-splitting $(M_1', \ldots, M_n')$ of minimal complexity. Feed it into Lemma 2.10(b), and let $(M_1, \ldots, M_n)$ denote the resulting pseudo-splitting. By Lemma 2.9 and (i) of Lemma 2.10(b), each component of $\mathrm{Fr}\, M_1 \cup \cdots \cup \mathrm{Fr}\, M_n$, which is not a sphere or a disk, is incompressible and ∂-incompressible in $M \backslash (A \cup B)$. By (ii) and (iii) of Lemma 2.10(b), the same holds for sphere and disk components.

If $N_1, \ldots, N_d$ are the components of $\overline{M \backslash M_n}$, disjoint from B, the image I_i of $H_1(M)$ in each $H_1(N_i, \mathrm{Fr}\, N_i)$ is nontrivial. $H_1(M)$ surjects onto the subgroup $\oplus I_i$ of $H_1(\bigcup N_i, \bigcup \partial N_i)$, so $d \le D$. On the other hand, $i_*\colon H_1(M_i) \to H_1(M)$ is zero for each $i \le n$, and the image of $H_1(M \backslash M_i)$ in $H_1(M)$ contains that of $H_1(M \backslash M_{i+1})$. Hence the Mayer–Vietoris image of $H_1(M)$ in $H_0(\mathrm{Fr}\, M_i)$ changes at most $\mathrm{rk}\, H_1(M)$ times as i runs from 1 to n. Since $n = Sr_S$, by the pigeonhole principle we can find an i_0 such that whenever $i_0 \le i < i_0 + r_S$, each component of $M_{i+1} \backslash M_i$ is adjacent to the unique component of $\mathrm{Fr}\, M_i$ and also to the unique component of $\mathrm{Fr}\, M_{i+1}$.

Let $\mathrm{Fr}\, M_i^s$ denote the component of $\mathrm{Fr}\, M_i$, corresponding to $s \in \{1, \ldots, S\}$ under the epimorphism $H_0(A) \oplus H_0(B \cup \bigcup N_i) \oplus H_1(M) \to H_0(\mathrm{Fr}\, M_i)$. Since $r_S = R(r_{S-1}, h)$, by Ramsey's theorem either some r_{S-1} of the surfaces $\mathrm{Fr}\, M_i^S$, $i_0 \le i \le i_0 + r_S$, are pairwise parallel in $M \backslash (A \cup B)$, or some h are pairwise non-parallel; but the latter cannot be by Haken's Theorem. Proceeding by induction, we eventually boil down to $r_0 = 2$ indices i, j such that $i_0 \le i < j \le i_0 + r_S$ and $\mathrm{Fr}\, M_i^s$, $\mathrm{Fr}\, M_j^s$ are parallel for each s. Since each component of $M_j \backslash M_i$ is $M_j^s \backslash M_i^s$ for some s, we obtain that $\mathrm{Fr}\, M_i$ and $\mathrm{Fr}\, M_j$ are parallel in $M \backslash (A \cup B)$. Then $i_*\colon H_1(M_i) \to H_1(M_j)$ is an isomorphism. Since $(M_1, \ldots, M_n)$ is a pseudo-splitting, it also is the zero map, so $H_1(M_i) = 0$. Then $H_1(\partial M_i)$ is a quotient of $H_2(M_i, \partial M_i) = H^1(M_i) = 0$, hence ∂M_i is a disjoint union of spheres.

Since A is 0-split from B, all components of A are contained in the same component of $M\backslash B$, and by Lemma 2.11 below all components of $B\cup\partial M\backslash A$ are contained in the same component of $M\backslash A$. If a component of ∂M were contained in A, triviality of $i_*\colon H_2(A) \to H_2(M\backslash B)$ would imply $B=\emptyset$, contradicting the hypothesis. Hence the components of ∂M_i can be connected by tubes in $M\backslash(A\cup B)$, first by subtracting 1-handles avoiding A from the components of M_i and then by joining the amended components of M_i by 1-handles avoiding B. The resulting 3-manifold M_i^+ is connected, ∂M_i^+ is a sphere, and clearly $H_1(M_i^+) = H_1(M_i) = 0$, hence M_i^+ is a homology ball.   □

Lemma 2.11. *If A, B are disjoint subpolyhedra of a compact connected orientable 3-manifold M and $H_2(A) \xrightarrow{i_*} H_2(M\backslash B)$ is zero, $\tilde{H}^0(M\backslash A) \xrightarrow{i_*} \tilde{H}^0(B \cup \partial M\backslash A)$ is zero.*

Proof. By the hypothesis, $H_2(M\backslash B) \to H_2(M\backslash B, A \cap \partial M)$ is monic. Hence $i_*\colon H_2(A, A \cap \partial M) \to H_2(M\backslash B, A \cap \partial M)$ is zero, and by Lefschetz duality so is $H^1(M, M\backslash A) \to H^1(M, B \cup \partial M\backslash A)$. On the other hand, $\tilde{H}^0(M) = 0$, so $\delta^*\colon \tilde{H}^0(B \cup \partial M\backslash A) \to H^1(M, B \cup \partial M\backslash A)$ is monic, and the assertion follows.   □

2.4. *Acyclic open sets and strong ω-quasi-isotopy*

There is another way to view Theorem 2.5 and its proof. Showing that ω-quasi-isotopy implies PL isotopy for links in a compact 3-manifold Q amounts[d] to replacing a contractible open neighborhood U of an arc J, properly PL immersed with one double point into a compact 3-manifold M, by a closed PL ball neighborhood V. The natural absolute version of this problem is to do the same for a PL knot $K \subset \text{Int}\,M$. Note that the latter is clearly impossible if one wishes to additionally demand either $V \subset U$ (by letting U be the Whitehead manifold W and K be the meridian K_0 of T_0) or $V \supset U$ (by letting U be Alexander's horned ball in S^3 and M be $\overline{S^3\backslash T}$, where T is an essential solid torus in $S^3\backslash\bar{U}$). However, it is possible to do this without such additional restrictions if M embeds in S^3, since Theorem 2.8 (with $B = $ point) implies

Corollary 2.12. *If U is an acyclic open subset of a compact orientable 3-manifold M, every compact subset of U is contained in a PL-embedded homology 3-ball.*

This was proved by McMillan in the case where M is irreducible, $\partial M \neq \emptyset$ and $U \subset \text{Int}\,M$ [23, Theorem 2, Remark 1] (see also [24, Proposition] for the above

---

[d]Indeed, let $f\colon mS^1 \to Q$ be an ω-quasi-embedding. By taking increasingly loose regular neighborhoods in Q, we may assume without loss of generality that P_i lies in the interior of P_{i+1} in Q for each $i < \omega$. Then $\bigcup_{i<\omega} P_i$ is contractible and open in Q, and this property will preserve under the construction from the proof of Proposition 2.1(b). Using the final notation from that proof, $U := P_\omega \cup (D\backslash\partial D)$, where D denotes the pair of disks $F^{-1}(\partial J_{\omega+1})$, is a contractible open subset of $M := P_{\omega+1}$ and contains $J := f(J_{\omega+1})$.

formulation and [22, Theorem 2] for the essential part of the argument; note that Lemma 4 of [24] is incorrect if $U \cap \partial M \neq \emptyset$, for in this case their $(\partial Q^3) - (\partial M^3)$ must have nonempty boundary if V is sufficiently small). McMillan's proof is also based on Haken's Finiteness theorem; in fact, our proof of Theorem 2.8 can be used to simplify his argument by omitting the reference to Waldhausen's result from the proof of [22, Theorem 2].

Remarks. (i) Corollary 2.12 fails if compactness of M is weakened to the assumption that $M \backslash \partial M$ is the interior of a compact 3-manifold and ∂M is the interior of a compact 2-manifold. Figure 3 depicts a wedge of a PL knot V (dashed) and a disguised version of the Fox–Artin wild arc $Z \subset S^3$ from [10, Example 1.3]; the only points of wildness of Z are its endpoints $\{p, q\}$. If J is a closed sub-arc of $Z \backslash \{p, q\}$, its exterior in Z consists of two wild arcs Y_p and Y_q, and it follows from [10, Example 1.2] that $S^3 \backslash (Y_p \cup Y_q)$ is homeomorphic to $S^3 \backslash \{p, q\}$. Let $R_p \sqcup R_q$ be a regular neighborhood of $Y_p \cup Y_q \backslash \{p, q\}$ in $S^3 \backslash \{p, q\}$ rel ∂. The exterior M of $R_p \cup R_q$ in $S^3 \backslash \{p, q\}$ is a partial compactification of $S^2 \times \mathbb{R}$ with $\partial M \cong \mathbb{R}^2 \sqcup \mathbb{R}^2$.

The compact subset $J \cup V$ of M is contained in the contractible open subset $U := M \backslash X$, where X is any PL arc in S^3, connecting p and q and disjoint from $J \cup V$ and from $R_p \cup R_q$. (Indeed, arguments similar to those in [10, Example 2] show that $U \backslash \partial U$ is homeomorphic to $S^3 \backslash X$.) On the other hand, suppose that $J \cup V$ is contained in a PL ball B^3 in M. Then B^3 meets Z precisely in J, so $\partial B^3 \backslash Z$ is a twice punctured 2-sphere. Let γ_0 be a generator of $\pi_1(\partial B^3 \backslash Z) \simeq \mathbb{Z}$. Since $\pi_1(S^3 \backslash Z) = 1$ by [10], but no power of the meridian γ_0 is trivial in the tame knot group $\pi_1(B^3 \backslash Z)$ (even modulo commutators), by Seifert–van Kampen γ_0 has to be trivial in $\pi_1(\overline{S^3 \backslash B^3} \backslash Z)$. Let W be any pushoff of V, disjoint from Z, such as the one in [10, Fig. 8]. Then without loss of generality $W \subset B^3$, so γ_0 is trivial in $\pi_1(S^3 \backslash (W \cup Z))$, contradicting [10].

(ii) The set $J \cup V$ in Fig. 3 is the image of a PL map $f \colon (I, \partial I) \to (M, \partial M)$ with one double point. This f is not a strong ω_1-quasi-embedding (i.e. there exists no PL 3-ball B^3 in M such that $f^{-1}(B^3)$ is an arc) by the preceding discussion.
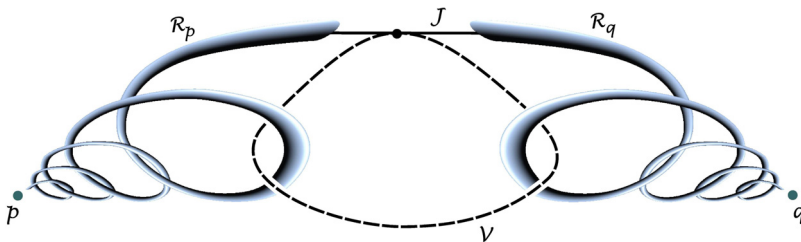

Fig. 3.

However, f is a strong ω-quasi-embedding. Indeed, the manifold U from (i) is a partial compactification of $\mathbb{R}^3$ with $\partial U \cong \mathbb{R}^2 \sqcup \mathbb{R}^2$. For $1 \leq j < \omega$ let P_j be any PL ball in $U \backslash \partial U$ containing the compact set $P_{j-1} \cup f(J_{j-1})$, and let J_j be the smallest subarc of I containing $f^{-1}(P_j)$. The assertion now follows from Proposition 2.1(a).

Theorem 2.13. *There exists an open 3-manifold containing two PL knots that are strongly ω-quasi-isotopic but not PL isotopic.*

Proof. We use the notation of Remark (i) above. Let $h \colon \partial R_p \to \partial R_q$ be a homeomorphism between the boundary planes of M that identifies the endpoints of J. The proper arcs J and J' in M, shown in Figs. 3 and 4 respectively (where $\partial J' = \partial J$) descend to knots K and K' in the open 3-manifold $M_h := M/h$. Since M is simply-connected, there exists a generic PL homotopy $H_t \colon I \to M$ such that $H_0(I) = J$, $H_1(I) = J'$ and $H_t(\partial I) = H_0(\partial I)$ for each $t \in I$. By the argument of Remark (ii) above, H_t descends to a strong ω-quasi-isotopy between K and K'.

If K_1 is a PL knot in M_h, let $\Gamma(K_1)$ denote the kernel of the inclusion induced homomorphism $\pi_1(M_h \backslash K_1) \to \pi_1(M_h)$. If K_2 is obtained from K_1 by insertion of a local knot, the argument in Remark (i) above shows that $\Gamma(K_1) = 1$ iff $\Gamma(K_2) = 1$. Now $\Gamma(K) = 1$ since a meridian of J is obviously null-homotopic in M. So to prove that K and K' are not PL isotopic it suffices to show that $\Gamma(K') \neq 1$. The arc J' was chosen so that the mirror-symmetric halves of the representation in [10, Example 1.3] extend to a nontrivial representation $\rho \colon \pi_1(M \backslash J') \to A_5$ in the alternating group; the images of the additional Wirtinger generators are indicated in Fig. 4. Moreover, ρ factors as $\rho \colon \pi_1(M \backslash J') \to \pi_1(M_h \backslash K') \to A_5$, where the first homomorphism is induced by the quotient map and the second trivializes an additional generator of the HNN-extension $\pi_1(M_h \backslash K')$ of $\pi_1(M \backslash J')$. Since $M \backslash J'$ is acyclic, $\pi_1(M_h \backslash K')$ is non-abelian. But $\pi_1(M_h) = \mathbb{Z}$, so $\Gamma(K') \neq 1$. $\qquad\square$
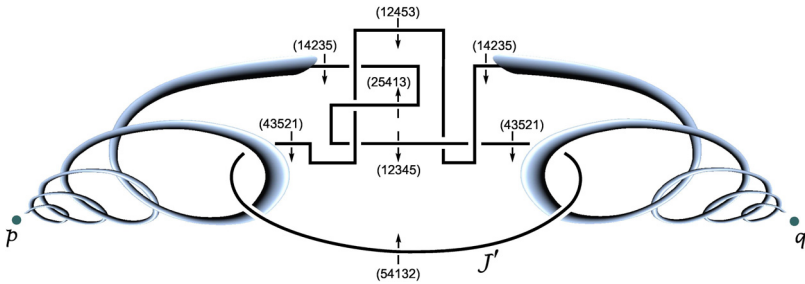

Fig. 4.

3. *k*-Cobordism and Cochran's Invariants

3.1. *Preliminaries*

The results of this section are based on the following

Proposition 3.1. *Suppose that links L and L' differ by a single crossing change on the ith component so that the intermediate singular link f is a k-quasi-embedding. Let ℓ denote the lobe J_0 of f, let $\hat{\mu} \in \pi(f)$ be a meridian of ℓ which has linking number $+1$ with ℓ, and let $\hat{\lambda} \in \pi(f)$ denote the corresponding longitude of ℓ (see definition in [25]). Let μ, λ denote the images of $\hat{\mu}, \hat{\lambda}$ in $\pi(L)$, and set $\tau = \mu^{-l}\lambda$, where l is the linking number of the lobes. Then*

(a) [25, Lemma 3.1] $\lambda \in \underbrace{\langle\mu\rangle^{\langle\mu\rangle^{\cdot^{\cdot^{\cdot^{\langle\mu\rangle^{\pi(L)}}}}}}}_{k \text{ of } \langle\mu\rangle\text{'s}}$; (b) $\tau \in [\ldots [[\pi(L), \underbrace{\langle\mu\rangle], \langle\mu\rangle] \ldots, \langle\mu\rangle}_{k \text{ of } \langle\mu\rangle\text{'s}}]$.

Here $\pi(L)$ denotes the fundamental group $\pi_1(S^3 \backslash L(mS^1))$, and for a subgroup H of a group G, the normal closure $\langle g^{-1}hg \mid h \in H, g \in G\rangle$ is denoted by H^G. Part (b) can be proved analogously to (a), or deduced from it as follows.

Proof of (b). Let us denote the subgroups in (a) and (b) by A_k and B_k, respectively. Suppose that $\lambda = \mu^{n_1 g_1} \cdots \mu_r^{n_r g_r}$, where each $n_i \in \mathbb{Z}$ and each $g_i \in A_{k-1}$. Then $n_1 + \cdots + n_r = l$, so using the identity $ab = b[b, a^{-1}]a$, we can write

$$\mu^{-l}\lambda = \mu^{-n_1}\mu^{n_1 g_1}h_1\mu^{-n_2}\mu^{n_2 g_2} \cdots h_{r-1}\mu^{-n_r}\mu^{n_r g_r},$$

where $h_i = [\mu^{n_i g_i}, \mu^{l-(n_1+\cdots+n_i)}]$. But each $\mu^{-n_i}\mu^{n_i g_i} = [\mu^{n_i}, g_i] \in B_k$ and each $h_k^{-1} \in B_{k+1} \subset B_k$ by the following lemma. $\square$

Lemma 3.2. [2, Lemma 2] *Let G be a group, $g \in G$, and $n \in \mathbb{N}$. Then*

$$\left[\langle g\rangle, \underbrace{\langle g\rangle^{\langle g\rangle^{\cdot^{\cdot^{\cdot^{\langle g\rangle^G}}}}}}_{n \text{ of } \langle g\rangle\text{'s}}\right] = [\ldots [[G, \underbrace{\langle g\rangle], \langle g\rangle] \ldots, \langle g\rangle}_{n+1 \text{ of } \langle g\rangle\text{'s}}].$$

Since we could not find an English translation of [2], we provide a short proof for convenience of the reader.

Proof. By induction on n. Since $[b, a] = [a, b]^{-1}$,

$$\underbrace{[[\ldots [G, \langle g\rangle]] \ldots, \langle g\rangle], \langle g\rangle]}_{n+1} = [\langle g\rangle, [\langle g\rangle, \ldots \underbrace{[\langle g\rangle, G] \ldots]]}_{n+1}.$$

So it suffices to prove that $[\langle g\rangle, [\langle g\rangle, A_{n-1}]] = [\langle g\rangle, A_n]$, where A_n denotes the expression in the left hand side of the statement of the lemma, after the comma. Indeed, pick any $h_1, \ldots, h_r \in A_{n-1}$ and $m, r, m_1, \ldots, m_r, n_1, \ldots, n_r \in \mathbb{Z}$, then

$$[g^m, [g^{m_1}, h_1]^{n_1} \cdots [g^{m_r}, h_r]^{n_r}] = [g^m, (g^{-m_1}g^{m_1 h_1})^{n_1} \cdots (g^{-m_r}g^{m_r h_r})^{n_r}].$$

For some $s, m'_1, \ldots, m'_s, m''_0, \ldots, m''_s \in \mathbb{Z}$ and some $h'_1, \ldots, h'_s \in A_{n-1}$, the latter expression can be rewritten as

$$
\begin{aligned}
[g^m, &\, g^{m''_0} g^{m'_1 h'_1} \cdots g^{m''_{s-1}} g^{m'_s h'_s} g^{m''_s}] \\
&= [g^m, g^{m''_0} g^{m'_1 h'_1} \cdots g^{m''_{s-1} + m''_s} g^{m'_s h'_s} g^{m''_s}] \\
&= \cdots = [g^m, g^{m''_0 + \cdots + m''_s} g^{m'_1 h'_1} g^{m''_1 + \cdots + m''_s} \cdots g^{m'_s h'_s} g^{m''_s}] \\
&= [g^m, g^{m'_1 h''_1} \cdots g^{m'_s h''_s}]
\end{aligned}
$$

for some new $h''_1, \ldots, h''_s \in A_{n-1}$. Clearly, this procedure is reversible. $\qquad\square$

3.2. *k-cobordism*

We can now establish a relation with k-cobordism of Cochran [5] and Orr [29]. We recall that the *lower central series* of a group G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$. If V is a properly embedded compact orientable surface in $S^3 \times I$, the *unlinked pushoff* of V is the unique section v of the spherical normal bundle of V such that $(v|_C)_* \colon H_1(C) \to H_1(S^3 \times I \setminus C)$ is zero for each component C of V.

Two links $L_0, L_1 \colon mS^1 \hookrightarrow S^3$ are called *k-cobordant* if they can be joined by m disjointly embedded compact oriented surfaces $V = V_1 \sqcup \cdots \sqcup V_m \subset S^3 \times I$ (meaning that $V_j \cap S^3 \times \{i\} = L_i(S^1_j)$ for $i = 0, 1$ and each j) so that if $v \colon V \hookrightarrow S^3 \times I \setminus V$ denotes the unlinked pushoff of V then the image of $v_* \colon \pi_1(V) \to \pi_1(S^3 \times I \setminus V)$ lies in the subgroup generated by $v_*(\pi_1(\partial V))$ and $\gamma_k \pi_1(S^3 \times I \setminus V)$.

Theorem 3.3. *k-quasi-isotopy implies $(k+1)$-cobordism.*

Proof. Let $h_t \colon mS^1 \to S^3$ be a k-quasi-isotopy, viewed also as $H \colon mS^1 \times I \to S^3 \times I$. The (combinatorial) link of each double point p of H in the pair $(S^3 \times I, H(mS^1 \times I))$ is a copy of the Hopf link in S^3. Let us replace the star of p in $H(mS^1 \times I)$, which is the image of two disks $D^2 \times S^0$ in the same component of $mS^1 \times I$, by an embedded twisted annulus $A_p \simeq S^1 \times D^1$ cobounded in S^3 by the components of the Hopf link. This converts $H(mS^1 \times I)$ to an embedded surface $V \subset S^3 \times I$ with m components and genus equal to the number of double points of H. We may assume that the cylinder $A_p \simeq S^1 \times I$ corresponding to a double point $p = (t_p, x_p) \in S^3 \times I$ meets the singular level $S^3 \times \{t_p\}$ in two generators $\{a, b\} \times I$, moreover $\{a\} \times I$ has both ends on the lobe $\ell_p = J_0$ of h_{t_p}. Now $\pi_1(V)$ is generated by $\pi_1(\partial V)$ and the homotopy classes of the loops $\tilde{\ell}_p := (\ell_p \setminus H(D^2 \times S^0)) \cup \{a\} \times I$ and $l_p := S^1 \times \{\frac{1}{2}\} \subset A_p$, connected to the basepoint of V by some paths.

Since $l_p \subset A_p \subset S^3$ bounds an embedded disk in $p * S^3$ whose interior is disjoint from V, the pushoff $v(l_p)$ is also contractible in the complement to V. By Proposition 3.1(b), the free homotopy class of $v(\tilde{\ell}_p)$ lies in $\gamma_{k+1} \pi(h_{t_p - \varepsilon})$ and therefore in $\gamma_{k+1} \pi_1(S^3 \times I \setminus V)$. $\qquad\square$

Remarks. (i) When $k = 1$, the above proof works for weak k-quasi-isotopy in place of k-quasi-isotopy, since the loop $v(\tilde{\ell}_p)$ is null-homologous both in the complement to V_i, where $p \in H(S_i^1 \times I)$ (by the definition of the pushoff v) and in the complement to $\bigcup_{j \neq i} V_i$ (by the definition of weak 1-quasi-isotopy). In the case of two-component links with vanishing linking number this is not surprising, since the Sato–Levine invariant $\beta = \bar{\mu}(1122)$, which is well-defined up to weak 1-quasi-isotopy [25, Sec. 2], is a complete invariant of 2-cobordism in this case [31].

(ii) By Dwyer's theorem (see [29, Theorem 5]), $\pi(L)/\gamma_{k+2}$ is invariant under $(k+1)$-cobordism and hence under k-quasi-isotopy. But an easier argument shows that even the $(k+3)$rd lower central series quotient is invariant under k-quasi-isotopy. Indeed, the quotient of $\pi(L)$ over the normal subgroup

$$\mu_k \pi(L) = \left\langle [m, m^g] \,\middle|\, m \text{ is a meridian of } L, \ g \in \underbrace{\langle m \rangle^{\langle m \rangle^{\cdot^{\cdot^{\cdot^{\langle m \rangle^{\pi(L)}}}}}}}_{k \text{ of } \langle m \rangle\text{'s}} \right\rangle$$

is invariant under k-quasi-isotopy by [25, Theorem 3.2], and $\mu_k \pi(L) \subset \gamma_{k+3} \pi(L)$ by Lemma 3.2.

3.3. $\bar{\mu}$-invariants

Applying the result of [20], we get from Theorem 3.3 the following statement with $2k + 2$ in place of $2k + 3$.

Theorem 3.4. *Milnor's $\bar{\mu}$-invariants [28] of length $\leq 2k + 3$ are invariant under k-quasi-isotopy.*

To take care of the remaining case (length equals $2k+3$), we give a direct proof of Theorem 3.4, which is close to the above partial proof but avoids the reference to Dwyer's theorem in [19].

Proof. Consider a $\bar{\mu}$-invariant $\bar{\mu}_I$ where the multi-index I has length $\leq 2k+3$. If L is a link, let $D_I(L)$ denote the link obtained from L by replacing the ith component by n_i parallel copies, labelled $i_1, \ldots, i_{n_i}$, where n_i is the number of occurrences of i in I. Then $\bar{\mu}_I(L) = \bar{\mu}_J(D_I(L))$, where J is obtained from I by replacing the n_i occurrences of each index i by single occurrences of $i_1, \ldots, i_{n_i}$ in some order [28]. Since $\bar{\mu}_J$ is an invariant of link homotopy [27, 28], it suffices to show that if I has length $\leq 2k + 3$ and L_0, L_1 are k-quasi-isotopic, then $D_I(L_0), D_I(L_1)$ are link homotopic.

Given a k-quasi-isotopy h_t between L_0, L_1, we will again convert it into a $(k+1)$-cobordism V between L_0, L_1. (Although this cobordism will be isotopic to the one constructed in the proof of Theorem 3.3, here we are interested in attaching the handles of V instantaneously.) To this end, we emulate each crossing $L_- \mapsto L_+$ in h_t by taking a connected sum of L_- with the boundary of a punctured torus
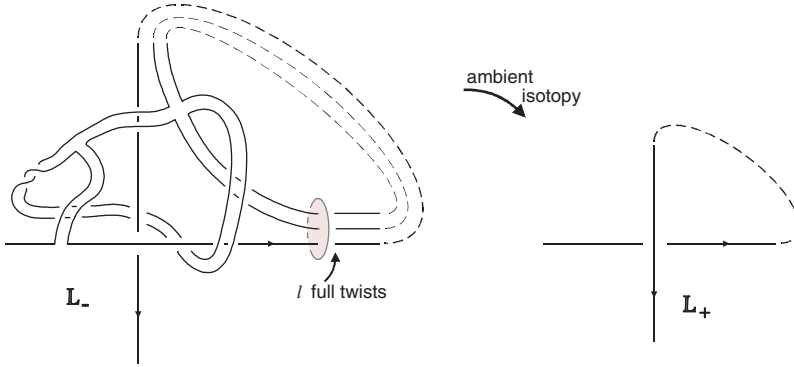
Fig. 5.

T in the complement of L_-, as shown in Fig. 5 (the ribbon, forming a half of the punctured torus, is twisted l times around the right lobe in order to cancel the linking number l of the two lobes). The resulting link $L_-\#\partial T$ is easily seen to ambient isotop onto L_+; meanwhile the natural generators of $\pi_1(T)$ represent the conjugate classes of τ and $\mu^{-1}\mu^\tau = [\mu^{-1}, \tau^{-1}]$, where μ is the meridian and τ the twisted longitude of the right lobe. By Proposition 3.1(b), τ lies in $\gamma_{k+1}\pi(L_-)$, thus the resulting surface V is indeed a $(k+1)$-cobordism.

Replacing each component V_i of V by n_i pairwise unlinked pushoffs, we get a $(k+1)$-cobordism between $D_I(L_0)$ and $D_I(L_1)$. The pushoffs of each punctured torus T as above can be taken in the same level $S^3 \times \{t\}$ where these codimension one submanifolds become naturally ordered. Let us, however, shift these n_i pushoffs in $S^3 \times \{t\}$ vertically to different levels $S^3 \times \{t+\varepsilon\}, \ldots, S^3 \times \{t+n_i\varepsilon\}$ in the order just specified. Thus the $(k+1)$-cobordism splits into a sequence of isotopies and iterated additions of boundaries of punctured tori:

$$D_I(L_-) \mapsto D_I(L_-)\underset{i_1}{\#}\partial T_1 \mapsto \cdots \mapsto D_I(L_-)\underset{i_1}{\#}\partial T_1 \underset{i_2}{\#}\cdots\underset{i_i}{\#}\partial T_{n_i} = D_I(L_+).$$

(The subscript of $\#$ indicates the component being amended.) It remains to show that every two consecutive links $L(j) := D_I(L_-)\#_{i_1}\cdots\#_{i_j}T_{i_j}$ and $L(j+1)$ in such a string are link homotopic. Now $L(j)$ and $L(j+1)$ only differ in their (i_{j+1})th components $K(j)$ and $K(j+1) = K(j)\#\partial T_{i_{j+1}}$, and by [27] it suffices to show that these represent the same conjugate class of Milnor's group $\pi(L_j')/\mu_0\pi(L_j')$ of $L_j' := L(j)\backslash K(j) = L(j+1)\backslash K(j+1)$, in other words, that the conjugate class of $\partial T_{i_{j+1}}$ is contained in the normal subgroup $\mu_0\pi(L_j')$.

In the case $j = 0$ we can simply quote the above observation that the generators σ, τ of $\pi_1(T)$ lie in $\gamma_{k+1}\pi(L_-)$ and $\gamma_{k+2}\pi(L_-)$, respectively. Hence the class of ∂T_{i_1} lies in $\gamma_{2k+3}\pi(L_0') \subset \gamma_m\pi(L_0') \subset \mu_0\pi(L_0')$, where the latter containment holds by [27] since $L_0' = L_-\backslash K(0)$ has $m-1$ components. On the other hand, in the case $j = n_i - 1$ it suffices to notice that one generator σ of $\pi_1(T)$ bounds an embedded disk in

the complement to L_+; hence $\partial T_{i_{n_i}}$ is null-homotopic in the complement to $L'_{n_i-1} = L_+\backslash K(n_i)$. Finally, when $0 < j < n_i - 1$, it takes just a little more patience to check that the generators $\tau_{i_{j+1}}$, $\sigma_{i_{j+1}}$ of $T_{i_{j+1}}$ are homotopic in the complement to $L(j)$ respectively to τ and $(\mu_{i_{j+1}}\cdots\mu_{i_{n_i}})^{-1}(\mu_{i_{j+1}}\cdots\mu_{i_{n_i}})^{\tau} = [(\mu_{i_{j+1}}\cdots\mu_{i_{n_i}})^{-1}, \tau^{-1}]$, where τ denotes the longitude of the right lobe and μ_{i_l} the meridian of the (i_l)th component of $L(j)$. (To see this, one may assume that $n_i = 2$, since all non-amended components can be grouped together and all amended components can also be grouped together, using that the punctured tori T_{i_j} are ordered in agreement with position of their projections to $S^3 \times \{t\}$.) Thus $\tau_{i_{j+1}} \in \gamma_{k+1}\pi(L(j))$ and $\sigma_{i_{j+1}} \in \gamma_{k+2}\pi(L(j))$, whence the class of $\partial T_{i_{j+1}}$ lies in $\gamma_{2k+3}\pi(L'_j) \subset \gamma_m\pi(L'_j) \subset \mu_0\pi(L'_j)$. $\qquad\square$

Remark. The restriction $2k + 3$ is sharp, since Milnor's link $\mathcal{M}_{k+1}$ is strongly k-quasi-isotopic to the unlink (see Sec. 1) but has nontrivial $\bar{\mu}(\underbrace{11\ldots11}_{2k+2}22)$ (cf. [28]).

3.4. *Conway polynomial and Cochran's invariants*

We recall that the Conway polynomial of an m-component link is of the form $z^{m-1}(c_0 + c_1z^2 + \cdots + c_nz^{2n})$ (see, for instance, [4]).

Corollary 3.5. *Let c_k denote the coefficient of the Conway polynomial at z^{m-1+2k}. The residue class of c_k modulo $\gcd(c_0, \ldots, c_{k-1})$ is invariant under k-quasi-isotopy.*
  Set $\lambda = \lceil\frac{(l-1)(m-1)}{2}\rceil$. The residue class of $c_{\lambda+k}$ modulo $\gcd$ of $c_\lambda, \ldots, c_{\lambda+k-1}$ and all $\bar{\mu}$-invariants of length $\leq l$ is invariant under $(\lfloor\frac{\lambda}{m-1}\rfloor + k)$-quasi-isotopy.

  Here $\lceil x\rceil = n$ if $x \in [n - \frac{1}{2}, n + \frac{1}{2})$, and $\lfloor x\rfloor = n$ if $x \in (n - \frac{1}{2}, n + \frac{1}{2}]$ for $n \in \mathbb{Z}$. The second part contains the first as well as Cochran's observation that c_1 is a homotopy invariant if $m \geq 3$ and the linking numbers vanish [4, Lemma 5.2].

Proof. Clearly, k-quasi-isotopic links are closures of k-quasi-isotopic string links. (String links and their closures are defined, for instance, in [18], and we assume their k-quasi-isotopy to be fixed on the boundary.) It is entirely analogous to the proof of Theorem 3.4 to show that μ-invariants of string links of length $\leq 2k+3$, as well as their modifications in [18] are invariant under k-quasi-isotopy. (The novelty in the definition of μ-invariants in [18] is that instead of the longitude λ_i one uses the product $\lambda_i\mu_i^{-l_i}$, where μ_i is the corresponding meridian, and l_i is the sum of the linking numbers of the ith component with the other components; in particular, this leads to $\mu(i,i) = -l_i$.) Let $a_i(S)$ denote the coefficient at z^i of the power series $\Gamma_S(z)$, defined for a string link S in the statement of [18, Theorem 1]. (Note that the determinant in the definition of Γ_S refers to any diagonal $(m-1) \times (m-1)$ minor, rather than to the full matrix.) The definition of Γ_S shows that $a_{m-1+2k}(S)$ is determined by the μ-invariants of S of length $\leq 2k + 2$, and hence is invariant under k-quasi-isotopy by Theorem 3.4. On the other hand, [18, Theorem 1] (see

also [15, 20, 39] and [35, Theorem 8.2 and the third line on p. 254]) implies that the closure L of S satisfies $c_k(L) \equiv a_{m-1+2k}(S)$ modulo the greatest common divisor of $a_i(S)$ with $i < m - 1 + 2k$. Since the c_k's are the only possibly nonzero coefficients of the Conway polynomial, the same congruence holds modulo $\gcd(c_0, \ldots, c_{k-1})$.

The second part follows by the same argument, taking into account that, by the definition of Γ_S, the residue class of $a_{l(m-1)+n}$ modulo gcd of all μ-invariants of S of length $\leq l$ (equivalently, of all $\bar{\mu}$-invariants of L of length $\leq l$) is determined by the μ-invariants of S of length $\leq l + n + 1$.   □

From Theorem 3.3 and [6, 2.1] we immediately obtain

Corollary 3.6. *Cochran's derived invariants β^i, $i \leq k$, of each two-component sublink with vanishing linking number are invariant under k-quasi-isotopy.*

Using Cochran's original geometric definition of his invariants [3] (see also [39]) it is particularly easy to see that $\beta^k(\mathcal{M}_k) = 1$ (whereas $\beta^k(\text{unlink}) = 0$). Thus $\mathcal{M}_k$ is not k-quasi-isotopic to the unlink. On the other hand, the statements of Corollary 3.6 and Theorem 3.3 are sharp, that is, β^{k+1} is not invariant under k-quasi-isotopy, which therefore does not imply $(k+2)$-cobordism. In fact, it is known [5, 34] (see also [30]) that β^i is an integral lifting of $\pm\bar{\mu}(\underbrace{11\ldots11}_{2i \text{ times}}22)$.

Cochran showed that the power series $\sum_{k=1}^{\infty} \beta^k z^k$ is rational, and is equivalent to the rational function $\eta_L^+(t)$ of Kojima–Yamasaki [17] (see definition in "*n-Quasi-isotopy I*") by the change of variable $z = 2 - t - t^{-1}$ [3, Sec. 7] (see also [39]). Kojima and Yamasaki were able to prove that the η-function is invariant under topological I-equivalence (i.e. the non-locally-flat version of concordance) [17]. However, the following comments are found in [17, Introduction]: "In the study of the η-function, we became aware of impossibility to define it for wild links. The reason of this is essentially due to the fact that the knot module of some wile knot is not $\mathbb{Z}[t^{\pm 1}]$-torsion." Indeed, in the definition of η_L^+, where $L = K_+ \cup K_-$, one has to divide by a nonzero Laurent polynomial annihilating the element $[\tilde{K}_+]$ of the knot module of K_-. It is shown in [17] that no such Laurent polynomial exists for a twisted version $\widetilde{\mathcal{M}}_\infty$ of Milnor's wild link $\mathcal{M}_\infty$ (which can be approximated by a twisted version $\widetilde{\mathcal{M}}_n$ of the links $\mathcal{M}_n$, with one half-twist along each disk as in Fig. 1).

On the other hand, Corollary 3.6 and [25, Corollary 1.4(a)] imply

Corollary 3.7. *Given any $i \in \mathbb{N}$ and any topological link $\mathfrak{L}$, there exists an $\varepsilon > 0$ such that all PL links, C^0 ε-close to $\mathfrak{L}$, have the same β^i.*

This has already been known, by Milnor's work on $\bar{\mu}$-invariants [28], for the residue class of β^k modulo $\gcd(\beta^1, \ldots, \beta^{k-1})$. Corollary 3.7 yields a natural extension of each β^i to all topological links, which by [25, Corollary 1.4(b)] or by compactness of I is invariant under topological isotopy, i.e. homotopy through embeddings.

In particular, since $\widetilde{\mathcal{M}}_\infty$ is isotopic to the unlink (see [28]), Cochran's reparametrization $\sum \beta^i z^i$ of η_L^+ of an arbitrary PL ε-approximation of $\widetilde{\mathcal{M}}_\infty$ converges to zero (as a formal power series) as $\varepsilon \to 0$.

Acknowledgments

References

[1] W. A. Blankinship and R. H. Fox, Remarks on certain pathological open subsets of 3-space and their fundamental groups, *Proc. Amer. Math. Soc.* **1** (1950) 618–624.

[2] Chan Van Hao, Nilgroups of finite rank, *Sibir. Mat. Zh.* **5**(2) (1964) 459–464 (in Russian).

[3] T. D. Cochran, Geometric invariants of link cobordism, *Comment. Math. Helv.* **60** (1985) 291–311.

[4] T. D. Cochran, Concordance invariance of coefficients of Conway's link polynomial, *Invent. Math.* **82** (1985) 527–541.

[5] T. D. Cochran, Derivatives of links: Milnor's concordance invariants and Massey products, *Mem. Amer. Math. Soc.* **84**(274) (1990).

[6] T. D. Cochran, k-cobordism for links in S^3, *Trans. Amer. Math. Soc.* **327** (1991) 641–654.

[7] T. D. Cochran and K. E. Orr, Stability of lower central series of compact 3-manifold groups, *Topology* **37** (1998) 497–526.

[8] O. L. Costich, P. H. Doyle and D. E. Galewski, A characterization of punctured open 3-cells, *Proc. Amer. Math. Soc.* **28** (1971) 295–298.

[9] M. Eisermann, A surgery proof of Bing's characterization of S^3, *J. Knot Theory Ramifications* **13** (2004) 307–309.

[10] R. H. Fox and E. Artin, Some wild cells and spheres in three-dimensional space, *Ann. of Math.* **49** (1948) 979–990.

[11] B. Freedman and M. H. Freedman, Kneser–Haken finiteness for bounded 3-manifolds, locally free groups, and cyclic covers, *Topology* **37** (1998) 133–147.

[12] C. McA. Gordon, *The Theory of Normal Surfaces*, Lecture notes typeset by R. P. Kent, IV, http://www.ma.utexas.edu/users/rkent/.

[13] W. Haken, On homotopy 3-spheres, *Illinois J. Math.* **10** (1966) 159–178.

[14] W. Jaco, *Lectures on Three-Manifold Topology*, CBMS Regional Conference Series in Mathematics, Vol. 43 (American Mathematical Society, Providence, RI, 1980).

[15] P. Kirk, C. Livingston and Z. Wang, The Gassner representation for string links, *Commun. Cont. Math.* **3** (2001) 87–136; preprint, math.GT/9806035.

[16] K. Kobayashi, Boundary links and h-split links, in *Low-Dimensional Topology (Funchal, 1998)*, Contemporary Mathematics, Vol. 233 (1999), pp. 173–186.

[17] S. Kojima, M. Yamasaki, Some new invariants of links, *Invent. Math.* **54** (1979) 213–228.

[18] J. Levine, A factorization of Conway's polynomial, *Comment. Math. Helv.* **74** (1999) 1–27.

[19] X.-S. Lin, Null k-cobordant links in S^3, *Comment. Math. Helv.* **66** (1991) 333–339.

[20] G. Masbaum and A. Vaintrob, Milnor numbers, spanning trees, and the Alexandar–Conway polynomial, *Adv. Math.* **180** (2003) 765–797; preprint, math.GT/0111102.

[21] S. Matveev, *Algorithmic Topology and Classification of 3-Manifolds* (Springer, Berlin, Heidelberg, New York, 2003).

[22] D. R. McMillan, Compact, acyclic subsets of three-manifolds, *Michigan Math. J.* **16** (1969) 129–136.

[23] D. R. McMillan, Acyclicity in 3-manifolds, *Bull. Amer. Math. Soc.* **76** (1970) 942–964.

[24] D. R. McMillan and T. L. Thickstun, Open three-manifolds and the Poincaré conjecture, *Topology* **19** (1980) 313–320.

[25] S. A. Melikhov and D. Repovš, n-Quasi-isotopy: I. Questions of nilpotence, *J. Knot Theory Ramifications* (in this issue).

[26] R. Mikhailov, Transfinite lower central series of groups: Parafree properties and topological applications, *Tr. Mat. Inst. Steklova* **239** (2002) 251–267; English translation *Proc. Steklov Math. Inst.* **239** (2002) 236–252.

[27] J. Milnor, Link groups, *Ann. of Math.* **59** (1954) 177–195.

[28] J. Milnor, Isotopy of links, in *Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz*, eds. R. H. Fox, D. Spencer and J. W. Tucker (Princeton University Press, 1957), pp. 208–306.

[29] K. E. Orr, Homotopy invariants of links, *Invent. Math.* **95** (1989) 379–394.

[30] K. E. Orr, Link concordance invariants and Massey products, *Topology* **30** (1991) 699–710.

[31] N. Sato, Cobordism of semi-boundary links, *Topol. Appl.* **18** (1984) 225–231.

[32] N. Smythe, n-linking and n-splitting, *Amer. J. Math* **92** (1970) 272–282.

[33] N. Smythe, Handlebodies in 3-manifolds, *Proc. Amer. Math. Soc.* **26** (1970) 534–538.

[34] D. Stein, Massey products in the cohomology of groups with applications to link theory, *Trans. Amer. Math. Soc.* **318** (1990) 301–325.

[35] L. Traldi, Conway's potential function and its Taylor series, *Kobe J. Math.* **5** (1988) 233–264.

[36] J. H. C. Whitehead, A certain open manifold whose group is unity, *Quart. J. Math.* **6** (1935) 268–279.

[37] D. Rolfsen, Some counterexamples in link theory, *Canad. J. Math.* **26** (1974) 978–984.

[38] N. Smythe, Topological invariants of isotopy of links, I, *Amer. J. Math.* **92** (1970) 86–98.

[39] T. Tsukamoto and A. Yasuhara, A factorization of the Conway polynomial and covering linkage invariants, preprint, math.GT/0405158.