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n-QUASI-ISOTOPY: I. QUESTIONS OF NILPOTENCE

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ABSTRACT

It is well-known that no knot can be cancelled in a connected sum with another knot, whereas every link can be cancelled up to link homotopy in a (componentwise) connected sum with another link. In this paper we address the question whether the noncancellation property of knots holds for (piecewise-linear) links up to some stronger analogue of link homotopy, which still does not distinguish between sufficiently close C^0-approximations of a topological link. We introduce a sequence of such increasingly stronger equivalence relations under the name of k-quasi-isotopy, $k \in \mathbb{N}$; all of them are weaker than isotopy (in the sense of Milnor). We prove that every link can be cancelled up to peripheral structure preserving isomorphism of any quotient of the fundamental group, *functorially* invariant under k-quasi-isotopy; functoriality means that the isomorphism between the quotients for links related by any allowable crossing change fits in the commutative diagram with the fundamental group of the complement to the intermediate singular link. The proof invokes Baer's theorem on the join of subnormal locally nilpotent subgroups. On the other hand, the integral generalized ($\mathrm{lk} \neq 0$) Sato–Levine invariant $\tilde{\beta}$ is invariant under 1-quasi-isotopy, but is not determined by any quotient of the fundamental group (endowed with the peripheral structure), functorially invariant under 1-quasi-isotopy — in contrast to Waldhausen's theorem.

As a byproduct, we use $\tilde{\beta}$ to determine the image of the Kirk–Koschorke invariant $\tilde{\sigma}$ of fibered link maps.

Keywords: PL isotopy; TOP isotopy; link homotopy; F-isotopy; I-equivalence; concordance; generalized Sato–Levine invariant; Kirk invariant; Jin suspension; lower central series; nilpotent group; subnormal cyclic subgroup; Milnor link group.

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1. Introduction

1.1. *The definition*

Consider PL links $L,\ L' : mS^1 \hookrightarrow S^3$, where mS^1 denotes $S^1_1 \sqcup \cdots \sqcup S^1_m$, and let $f : mS^1 \to S^3$ be any singular link in a generic PL homotopy between them. Then f is called a 0-*quasi-embedding*, or simply a *link map*, if its (unique) double point is a self-intersection of some component $f(S^1_i)$ (rather than an intersection between distinct components). If this is the case, $f\big(S^1_i\big)$ is a wedge of two loops ℓ, ℓ' $\big($following [31], we call them the *lobes* of $f\big(S^1_i\big)\big)$, and we say that f is a 1-*quasi-embedding*, if at least one of them, say ℓ, is null-homotopic in the complement to the non-singular components, i.e. in $X := S^3 \backslash f(mS^1 \backslash S^1_i)$. If this also is the case, we say that f is a 2-*quasi-embedding*, if a generic PL null-homotopy $F : D^2 \to X$ of $\ell = F(\partial D^2)$ can be chosen so that for some arc $I \subset \ell'$ containing $\ell' \cap F(D^2)$, the polyhedron $F(D^2) \cup I$ is null-homotopic in X.

In general, a PL map $f : mS^1 \to S^3$ with precisely one double point x will be called a k-*quasi-embedding*, $k \in \mathbb{N}$, if, in addition to the singleton $P_0 = \{x\}$, there exist compact subpolyhedra $P_1, \ldots, P_k \subset S^3$ and arcs $J_0, \ldots, J_k \subset mS^1$ such that $f^{-1}(P_j) \subset J_j$ for each $j \le k$, $P_j \cup f(J_j) \subset P_{j+1}$ for each $j < k$, and the latter inclusion is null-homotopic for each $j < k$. As a matter of convenience, we shall also assume that $\partial J_0 = f^{-1}(x)$.

A reader familiar with embeddings of higher-dimensional manifolds may recognize this construction as k steps of the Penrose–Whitehead–Zeeman–Irwin trick [50, 26, 57], adjusted to codimension two; see also [56, Comment 6.3]. Note that the original construction of Casson handles [9] can be thought of as an instance of infinitely many steps of this (adjusted) trick.

Two links $L,\ L' : mS^1 \to S^3$ are said to be k-*quasi-isotopic*, if they can be joined by a generic PL homotopy, all whose singular links are k-quasi-embeddings. Thus 0-quasi-isotopy is nothing but *link homotopy* (i.e. homotopy through link maps), whereas *PL isotopy* (i.e. PL homotopy through one-to-one maps) implies k-quasi-isotopy for all $k \in \mathbb{N}$. (Note that a PL isotopy may insert/shrink local knots.) It is not hard to see (cf. "n-*Quasi-isotopy II*") that the $(k+1)$th Milnor link and the $(k+1)$th Whitehead link are k-quasi-isotopic to the unlink.

It will be convenient to have two modifications of the definitions. *Strong* (*weak*) k-*quasi-embedding*, and consequently *strong* (*weak*) k-*quasi-isotopy*, are defined as above, except that "compact polyhedra" is replaced with "PL 3-balls" (respectively, "null-homotopic" is replaced with "induces trivial homomorphisms on reduced integral homology"). Strong k-quasi-isotopy may be thought of as arising from a slightly different interpretation of the PWZI trick, where the engulfing lemma is taken for granted, rather than regarded as a part of the construction.

1.2. *A motivation*

The concept of k-quasi-isotopy arose in attempts of the first author to solve the following problem (cf. [41, Questions I and III]). Let M^1 be a compact 1-manifold (it does not seem to really matter which one). A (continuous) map $f : M \to \mathbb{R}^3$ is said to be *realizable by TOP isotopy*, if there exists a homotopy $h_t : M \to \mathbb{R}^3$, $t \in [0,1]$, such that $h_1 = f$ and each h_{t_0} with $t_0 < 1$ is one-to-one.

Problem 1.1. *Does there exist a map $M^1 \to \mathbb{R}^3$, non-realizable by TOP isotopy?*

As pointed out in [41], it would be especially plausible to find such a map among locally flat TOP immersions (i.e. maps that tamely embed a neighborhood of every point). For each $n \geq 3$ there exists a locally flat TOP immersion $S^n \to \mathbb{R}^{2n}$, which is ε-approximable by an embedding for each $\varepsilon > 0$, but is not realizable by TOP isotopy [43] (see also [41, Theorem 1.12]).

A possible candidate $f : I \sqcup I \to \mathbb{R}^3$ (with one double point) for being non-realizable by TOP isotopy is depicted in Fig. 1. This locally flat TOP immersion can be regarded as a "connected sum" of infinitely many (left handed) Whitehead links (resembling the Wilder arc, which is an infinite "connected sum" of trefoils). Note also that f is a composition of a PL map and a TOP embedding; it was shown in [41] that in codimension ≥ 3, such a composition is realizable by a TOP isotopy if and only if it is ε-approximable by an embedding for each $\varepsilon > 0$.

Proposition 1.2. *The map f of Fig.* 1 *is not realizable by PL isotopy.*

The proof is not hard, but reveals the link-theoretic nature of Problem 1.1. We first introduce some notation to be used in this proof and throughout Sec. 2.

A two-component *string link* is a PL embedding $L : (I_+ \sqcup I_-, \partial) \hookrightarrow (I \times \mathbb{R}^2, \partial)$ such that $L(i, \pm) = (i, \pm p)$ for $i = 0, 1$ and some fixed $p \in \mathbb{R}^2 \backslash \{0\}$. String links are considered up to ambient isotopy, and their connected sum $L_1 \# L_2$ is defined by stacking together the two copies of (I_+, I_-, I).

If L is a two-component string link, let $\operatorname{denom}(L) : S^1 \hookrightarrow S^3$ be the "denominator" closure of L, obtained by adjoining to L the two arcs $\partial I \times \{\lambda p \mid \lambda \in [-1, 1]\}$ in $\partial I \times \mathbb{R}^2$, and let $\operatorname{numer}(L) : S^1 \sqcup S^1 \hookrightarrow S^3$ be the "numerator" closure of L, defined to be the composition of $L/\partial : I_+/\partial \sqcup I_-/\partial \hookrightarrow I/\partial \times \mathbb{R}^2$ and the inclusion $S^1 \times \mathbb{R}^2 \subset S^1 \times D^2 \cup D^2 \times S^1 = \partial(D^2 \times D^2)$.

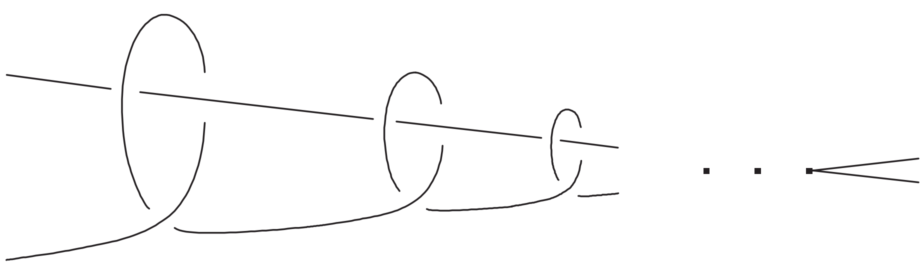

Fig. 1.

Finally, we call the string link W that appears three times in Fig. 1 the *Whitehead string link*, since its linking number zero closure (cf. Sec. 2) is the Whitehead link $\mathcal{W}$.

Proof. Suppose that $h_t : I \sqcup I \to \mathbb{R}^3$, $t \in [0,1]$, is a homotopy such that $h_1 = f$, each h_{t_0} with $t_0 < 1$ is one-to-one, and which restricts to a PL homotopy for $t \in [0,1)$. We may assume that $f(I \sqcup I)$ is contained in the interior of $I \times \mathbb{R}^2 \subset \mathbb{R}^3$. Let U be a regular neighborhood of the 4 endpoints $f(\partial I \sqcup \partial I)$ in $I \times \mathbb{R}^2$ such that $f^{-1}(U)$ is a regular neighborhood of $\partial I \sqcup \partial I$ in $I \sqcup I$, and each of the 4 balls of U meets $\partial I \times \mathbb{R}^2$ in a disk. Without loss of generality, $h_{t_0}(I \sqcup I)$ is contained in the interior of $I \times \mathbb{R}^2$ and $h_{t_0}(\partial I \sqcup \partial I)$ is contained in U for each t_0. Then h_t for $t \in [0,1)$ extends, via an identification of I with a subarc of $I_\pm$, to a PL isotopy $H_t : (I_+ \sqcup I_-, \partial) \hookrightarrow (I \times \mathbb{R}^2, \partial)$, keeping the boundary fixed and such that $H_t(I_+ \backslash I \sqcup I_- \backslash I) \subset U$.

Each $H_{1-\varepsilon}$ is a string link, which for a sufficiently small $\varepsilon > 0$ can be represented as a connected sum $W\#\cdots\#W\#L'$ of arbitrarily many copies of the Whitehead string link W and some residual string link L'. Therefore, in order to get a contradiction it suffices to find an invariant $\mathcal{I}$ of PL isotopy of 2-component string links such that

(i) $\mathcal{I}(L)$ is a nonnegative integer for every L;
(ii) $\mathcal{I}(W) > 0$;
(iii) $\mathcal{I}(L_1\#L_2) \geq \mathcal{I}(L_1) + \mathcal{I}(L_2)$ for any L_1 and L_2.

If L is a two-component string link, let $\mathrm{loc}(L)$ denote the knot product of the local knots of L or, more precisely, of all one-component factors in the prime factorization [21] of the non-split 3-component link $\mathrm{numer}(L) \cup M$, where M is a meridian of the solid torus $S^1 \times D^2$ from the definition of $\mathrm{numer}(L)$ above. A collection of disjoint balls B_i in $S^3 \backslash M$ meeting $\mathrm{numer}(L)$ in arcs I_i such that (B_i, I_i) represent the one-component prime factors of $\mathrm{numer}(L) \cup M$ can be chosen so that each B_i is disjoint from a given 2-disk D bounded by M and meeting $\mathrm{numer}(L)$ in two points (this is proved similarly to the uniqueness of knot factorization, see [37]). It follows that $\mathrm{loc}(L)$ is a factor of $\mathrm{denom}(L)$, i.e. there exists a knot $\mathrm{glob}(L)$ such that $\mathrm{denom}(L) = \mathrm{loc}(L)\#\mathrm{glob}(L)$.

We define $\mathcal{I}(L)$ to be the genus of $\mathrm{glob}(L)$. Then $\mathcal{I}(L)$ is invariant under PL isotopy since $\mathrm{glob}(L)$ is. Since $\mathrm{loc}(W)$ is the unknot and $\mathrm{denom}(W)$ is the trefoil, $\mathcal{I}(W) = 1$. Finally, $\mathrm{loc}(L_1\#L_2) = \mathrm{loc}(L_1)\#\mathrm{loc}(L_2)$ for any L_1, L_2 by a version of the above assertion where the 2-disk D is replaced by two such 2-disks with disjoint interiors (the proof is analogous). Therefore, since knot genus is additive under connected sum, $\mathcal{I}(L_1\#L_2) = \mathcal{I}(L_1) + \mathcal{I}(L_2)$. $\square$

Remark. One could try to prove Proposition 1.1 only using closed links. As in the above proof, it is easy to extend h_t for $t \in [0,1)$ to a PL isotopy $H_t : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$ such that the link $H_{1-\varepsilon}$ for a sufficiently small $\varepsilon > 0$ is ambient isotopic a connected sum[a] $\mathcal{W}\#\cdots\#\mathcal{W}\#\mathcal{L}$ of arbitrarily many copies of the Whitehead link $\mathcal{W}$ and some residual link $\mathcal{L}$. However, it turns out that there exists no invariant of 2-component links satisfying the analogues of conditions (i)–(iii); indeed, a certain connected sum of the Whitehead link with its reflection (along the bands, "orthogonal" to the "mirror") is ambient isotopic to the unlink.

Problem 1.1 would be solved if we could show that the invariant $\mathcal{I}$ from the proof of Proposition 1.2 stabilizes for sufficiently close (in the C^0 topology) embedded PL approximations of any given topological string link; for this would imply invariance of $\mathcal{I}$ under TOP isotopy with parameter in the (compact) interval $[0,1]$. The following example shows that this is not the case, moreover, $\mathcal{I}$ assumes arbitrarily large values on arbitrarily close PL approximations of a certain wild string link.

Example. Consider the wild link $\mathcal{L}(2S^1) = C \cup \mathcal{K}$, depicted in Fig. 2, and let K be a PL knot that is the image of an ε-approximation of the parametrization of $\mathcal{K}$ for some small $\varepsilon > 0$. As a knot, K can be gradually untangled from left to right into a connected sum of trefoils $(B^3_i, B^3_i \cap K)$, $1 \leq i \leq N$, and possibly some residual prime knots $(B^3_i, B^3_i \cap K)$, $N+1 \leq i \leq M$, where $N = N(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and the balls B^3_i are pairwise disjoint. None of these N trefoils is a local knot (i.e. a one-component factor in the prime decomposition) of the link $L := C \cup K$, since the free homotopy class of C, regarded as a conjugacy class in the amalgamated free product $\pi_1(S^3 \backslash K) \cong \pi_1(B^3_1 \backslash K) \underset{\mathbb{Z}}{*} \cdots \underset{\mathbb{Z}}{*} \pi_1(B^3_M \backslash K)$, can be seen to remain nontrivial $\pi_1(S^3 \backslash K) \to \pi_1(B^3_i \backslash K)$, $i \leq N$, given by abelianization

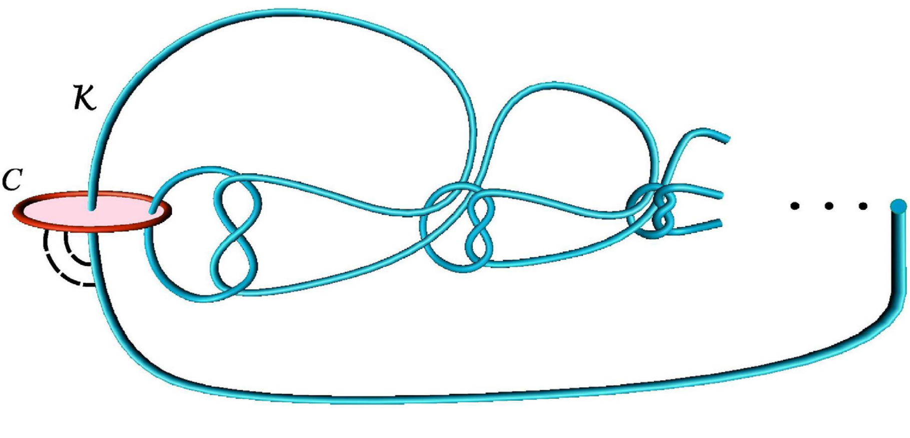

Fig. 2.

[a]Following [11, Sec. 5] and [22, Sec. 2.7], we denote by $L_1\#L_2$ any link of the form numer $(S_1\#S_2)$, where L_1 = numer (S_1) and L_2 = numer (S_2). This multi-valued operation of componentwise connected sum should not be confused with the multi-valued operation of connected sum along selected components as in the prime decomposition theorem of [21].

under each homomorphism of all the factors except for the ith one.[b] This homomorphism does not depend on the choice of the prime decomposition $(B^3_1, \dots, B^3_M)$ of K since the proof of the uniqueness of knot factorization (see [7]) shows that the collection of solid tori $(\overline{B^3_1 \setminus N(K)}, \dots, \overline{B^3_M \setminus N(K)})$, where $N(K)$ is a small regular neighborhood of K, is unique up to a self-homeomorphism of (S^3, K).

Now let $\mathcal{K}'$ and K' be obtained from $\mathcal{K}$ and K by the clockwise twist by 2π of the visible side of the disk shown in Fig. 2 (the disk is bounded by C and meets K in two points); set $\mathcal{L}' = \mathcal{C} \cup \mathcal{K}'$ and $L' = C \cup K'$. The twist yields a homeomorphism between the complements of L and L' sending meridians of K onto those of K', therefore $\mathrm{loc}(L') = \mathrm{loc}(L)$. If $L' = \mathrm{numer}(S)$ for some string link S, the knot $\mathrm{denom}(S)$ is a band connected sum of the components of L'; moreover, S is uniquely determined by L' and the band. The band indicated by dashed arcs in Fig. 2 yields $\mathrm{denom}(S) = K$. Since $\mathrm{loc}(S)$ is a factor of $\mathrm{loc}(L')$, the knot $\mathrm{glob}(S)$ has at least N trefoils among its prime factors, and so $\mathcal{I}(S) \geq N$. Thus $\mathcal{I}$ can be arbitrarily large for arbitrarily close PL approximations of the wild string link $\mathcal{S}$ that is determined by $\mathcal{L}'$ and the band.

1.3. *The main problem*

In general, it seems to be not easy to find a nonzero invariant of topological string links, satisfying the analogues of conditions (i) and (iii) from the proof of Proposition 1.2 (compare [41, Example 1.5] and [33]). Since we are not interested in exploring invariants of wild links per se, we will take a different approach in this paper, staying in the class of PL links by the price of imposing an additional restriction on the desired invariant, which ensures its stabilization on sufficiently close approximations of any topological string link. This approach is based on

Theorem 1.3. *For each $k \in \mathbb{N}$ and any TOP isotopy (i.e. homotopy through one-to-one maps) $h_t : mS^1 \hookrightarrow \mathbb{R}^3$, $t \in [0,1]$, there exists an $\varepsilon > 0$ such that any generic PL homotopy, ε-close to h_t, is a strong k-quasi-isotopy.*

Proof. For the sake of clarity we only give an explicit proof for $k = 1$.

Let $\delta > 0$ be the minimal distance between $h_t(S^1_i)$ and $h_t(S^1_j)$, $i \neq j$, $t \in I$. Let $C_i \subset (S^1_i)^2 \times I$ denote the set of all triples (p, q, t) such that p and q are not antipodal in S^1_i, and the h_t-image of the shortest arc $[p, q] \subset S^1_i$ lies in the open $\frac{\delta}{5}$-neighborhood of the middle point m_{pq} of the line segment $[h_t(p), h_t(q)] \subset \mathbb{R}^3$. Then C_i contains a neighborhood of $\Delta_{S^1_i} \times I$, and it follows that the union C of all C_i's contains $\cup_t (h^2_t)^{-1}(N)$ for some neighborhood N of $\Delta_{\mathbb{R}^3}$. Hence the minimal distance γ between $h_t(p)$ and $h_t(q)$ over all triples (p, q, t) in the (compact) complement to C is non-zero. Set $\varepsilon = \min\left(\frac{\gamma}{3}, \frac{\delta}{5}\right)$.

Let l_t be a generic PL homotopy, ε-close to h_t. Then l_t has at most one double point $x := l_t(p) = l_t(q)$ for each $t \in I$. In this case $h_t(p)$ and $h_t(q)$ are $\frac{2\gamma}{3}$-close.

[b] Thus $\mathcal{L}$ is a counterexample to the assertion in the last sentence of [41, Example 1.3]. The mistake in the proof of this assertion was pointed out in [42].

So (p,q,t) lies in C_i for some i. Hence $h_t([p,q])$ lies in the $\frac{\delta}{5}$-neighborhood of m_{pq}. It follows that the loop $\ell := l_t([p,q])$ lies in the $\frac{3}{5}\delta$-neighborhood of x. We claim that ℓ is null-homotopic in $S^3\backslash l_t(mS^1\backslash S^1_i)$. Indeed, if the radial null-homotopy of ℓ towards x met $l_t(S^1_j)$, then a point of ℓ and a point of $l_t\big(S^1_j\big)$ would lie on the same radius of the $\frac{3}{5}\delta$-neighborhood of x, consequently a point of $h_t([p,q])$ and a point of $h_t\big(S^1_j\big)$ would be δ-close, contradicting our choice of δ. $\square$

The same argument works to prove the analogous assertion for string links. (The definition of k-quasi-isotopy for string links is the same, except that "at least one of the two loops" must be changed to "the loop" in the reformulations for $k = 1, 2$.)

Corollary 1.4. (a) *For each $k \in \mathbb{N}$, strong k-quasi-isotopy classes of all sufficiently close PL approximations to a topological link* (*or topological string link*) *coincide.*

(b) *TOP isotopic PL links* (*or PL string links*) *are strongly k-quasi-isotopic for all $k \in \mathbb{N}$.*

Thus, positive solution to Problem 1.1 is implied by positive answer to either part of the following question of independent interest.

Problem 1.5. *Does there exist*, *for some k, $m \in \mathbb{N}$, a nonnegative integer valued invariant $\mathcal{I}$ of k-quasi-isotopy of m-component PL string links such that*

(a) *$\mathcal{I}(L\#L') \geq \mathcal{I}(L)+\mathcal{I}(L')$ for any L, L', and $\mathcal{I}(L_0) > 0$ for some L_0, preferably with unknotted components*?

(b) (non-commutative version) *$\mathcal{I}(L\#L') \geq \mathcal{I}(L)$ for any L, L', and there exist L_0, $L_1, \dots$, preferably with unknotted components*, *such that $\mathcal{I}(L_0\#\cdots\#L_n) \geq n$ for each n*?

(a$'$), (b$'$) *Same questions for PL links* (*with any choice of bands in $L\#L'$ and some choice of bands in $L_0\#\cdots\#L_n$*).

Additional interest in Problem 1.5 arises from its interpretation as a proper formalization of the intuitive question, whether the "phenomenon of linking (modulo knotting)", regarded solely as a feature of interaction between distinct components, admits the possibility of "accumulation of complexity" (like it happens in the case of the "phenomenon of knotting").

For $k = 0$ the answer to Problem 1.5 is well-known to be negative since concordance implies link homotopy [16, 18]. Indeed, the connected sum of any PL link L with its reflection ρL (i.e. the composition of L with reflections in both domain and range) along the bands, "orthogonal" to the "mirror", is a slice link. For completeness, the proof of the analogous assertion for string links is included in the proof of Theorem 2.2 below.

1.4. *Main results*

A straightforward way to find an invariant as required in Problem 1.5 is to extract it from some quotient of the fundamental group $\pi(L) := \pi_1(S^3\backslash L(mS^1))$, invariant under k-quasi-isotopy. In view of Stallings' theorem on lower central series [55, 8, 10], this is impossible if the quotient is nilpotent. Indeed, it follows from the Stallings theorem that any nilpotent quotient is invariant under link concordance, whereas any PL link (or string link) is cancelled by its reflection up to concordance. This suggests that the first step towards the above problems might be to look for non-nilpotent quotients of $\pi(L)$, invariant under k-quasi-isotopy of L.

The following encouraging fact exploits the power of the generalized Sato–Levine invariant $\tilde{\beta}$, whose remarkability emerged independently in the works of Traldi [59, Sec. 10] and Polyak and Viro [49] (see also [48, 3]); Kirk and Livingston [31, 39]; Akhmetiev [1]; [2]; Nakanishi and Ohyama [46]; and, implicitly (see proof of Theorem 2.6 below), Koschorke [34, discussion of Fact 2.9].

Corollary 1.6. *No nilpotent quotient of the fundamental group* (*even if endowed with the peripheral structure*[c]) *can be a complete invariant of* 1*-quasi-isotopy of* 2*-component links.*

Similarly for string links; *specifically, the connected sum $W\#rW$ of the Whitehead string link and its reflection is not* 1*-quasi-isotopic to the trivial string link.*

However, we were unable to find any non-nilpotent quotient of the fundamental group, invariant under k-quasi-isotopy for some k. The difficulty that we encountered can be summarized as follows.

Let R be an equivalence relation on links, obtained from ambient isotopy by allowing certain types of transversal self-intersections of components, and $\mathcal{F}$ be a function assigning to every PL link L some group $\mathcal{F}(L)$ together with an epimorphism $p_L : \pi(L) \to \mathcal{F}(L)$. Then we say that $\mathcal{F}$ is *functorially invariant* under R with respect to p_L if it is invariant under ambient isotopy, and for any links L_+, L_- obtained one from another by an allowed crossing change, with L_s being the intermediate singular link, there is an isomorphism between the groups $\mathcal{F}(L_+)$ and $\mathcal{F}(L_-)$ making the following diagram commutative:

$$\begin{tikzcd}
& \pi_1(S^3\backslash\text{regular neighborhood of } L_s(mS^1)) \arrow[dl, "i_*"'] \arrow[dr, "i_*"] & \\
\pi(L_+) \arrow[d, "p_{L_+}"'] & & \pi(L_-) \arrow[d, "p_{L_-}"] \\
\mathcal{F}(L_+) \arrow[rr, "\simeq"] & & \mathcal{F}(L_-)
\end{tikzcd}$$

[c]We recall that the *peripheral structure* in the fundamental group of an m-component link L is a collection of m pairs (m_i, l_i), defined up to simultaneous conjugation of m_i and l_i by an element of $\pi(L)$, where m_i is a meridian to the ith component of L and l_i is the corresponding longitude. The *meridian* and the *longitude* (or parallel) corresponding to a path p that joins the basepoints of the link complement and of the boundary of a small regular neighborhood T_i of the ith component, are the images of the distinguished generators of $\pi_1(\partial T_i) \simeq H_1(\partial T_i) \simeq H_1(\overline{S^3\backslash T_i}) \oplus H_1(T_i)$ under the homomorphism $i_* : \pi_1(\partial T_i) \to \pi(L)$ determined by p.

Corollary 1.7. *Every quotient of the fundamental group of a link* (*or string link*), *functorially invariant under* k*-quasi-isotopy for some* k, *is nilpotent.*

By a theorem of Waldhausen (see [29]) the ambient isotopy class of L is completely determined by the fundamental group $\pi(L)$ together with the peripheral structure $P(L)$. Therefore, given an equivalence relation R on links, it must be possible, in principle, to extract any invariant of links up to R from the pair (π, P). Nevertheless, the two results above combine to produce

Corollary 1.8. *Let* $\mathcal{G}_k(L)$ *denote the finest quotient of* $\pi(L)$, *functorially invariant under* k*-quasi-isotopy,*[d] *and* $\mathcal{P}_k(L)$ *the finest peripheral structure in* $\mathcal{G}_k(L)$, *invariant under* k*-quasi-isotopy.*[e] *For each* $k > 0$, *the pair* $(\mathcal{G}_k, \mathcal{P}_k)$ *is not a complete invariant of* k*-quasi-isotopy, even for* 2*-component links* (*and string links*).

In fact, the gap between k-quasi-isotopy and what this pair can tell about it must be extremely wide, apparently including distinction of most slice and boundary links if k is large enough. Certainly, concordant links need not be 1-quasi-isotopic (see Sec. 2), whereas the boundary of two Whitehead-linked Möbuis bands [47, Fig. 7] is a $\mathbb{Z}/2$-boundary link with lk $= 0$, which is 1-quasi-isotopically non-trivial since it has a nonzero Sato–Levine invariant (cf. Sec. 2).

1.5. *Discussion and remarks*

Surprisingly, for $k = 0$ it seems to be a hard problem whether $(\mathcal{G}_0, \mathcal{P}_0)$ is a complete invariant of link homotopy (= 0-quasi-isotopy). It is not hard to see (see Sec. 3 or [36, proof of Theorem 4]) that $\mathcal{G}_0(L)$ coincides with Milnor's link group $\mathcal{G}(L)$ (i.e. the quotient of $\pi(L)$ obtained by forcing each meridian to commute with all of its conjugates [45]), and $\mathcal{P}_0(L)$ with the peripheral structure $\mathcal{P}(L)$ where the set Λ_i corresponding to the class $\bar{m}_i$ of a meridian $m_i \in \pi(L)$ is the right coset $N(m_i)\bar{l}_i$ of the subgroup[f] $N(m_i) = \langle [g^{-1}, g^{m_i}] \mid g \in \mathcal{G}(L) \rangle$ containing the class $\bar{l}_i$ of the longitude corresponding to m_i. In fact, Levine explicitly considered only a coarser peripheral structure $\mathcal{P}'(L)$ in [36], with the right coset of $N(m_i)$ being enlarged to the coset of the normal closure N_i of $N(m_i)$; but this $\mathcal{P}'$ was still finer than Milnor's original $\mathcal{P}''$ where instead of N_i the normal closure of m_i in $\mathcal{G}$ was used [45].

It has been known for some time that Milnor's pair $(\mathcal{G}, \mathcal{P}'')$ is a complete invariant of link homotopy in the cases of homotopically Brunnian and $\leq$ 3-component

[d]To see that $\mathcal{G}_k(L)$ is well-defined, note that it is the quotient of $\pi(L)$ over the product of the kernels of $i_* : \pi(L) \to \pi_1(S^3 \times I \backslash H(mS^1 \times I))$ for all k-quasi-isotopies H starting with L.

[e]More precisely, $\mathcal{P}_k(L)$ denotes the collection of m pairs $(\bar{m}_i, \Lambda_i)$, each defined up to conjugation, where $\bar{m}_i \in \mathcal{G}_k(L)$ is the coset of a meridian $m_i \in \pi(L)$ to the ith component of L, and $\Lambda_i \subset \mathcal{G}_k(L)$ is the set of cosets $\bar{l}_{i\alpha} \in \mathcal{G}_k(L_\alpha) \simeq \mathcal{G}_k(L)$ of the longitudes $l_{i\alpha}$ corresponding to some representatives $m_{i\alpha}$ of $\bar{m}_i$ in the fundamental groups of all links L_α, k-quasi-isotopic to L (that $m_{i\alpha}$ are meridians follows from functoriality).

[f]We use the notation $[a, b] = a^{-1}b^{-1}ab$, $a^b = b^{-1}ab$ throughout the paper, but in the notation $[a, b] = aba^{-1}b^{-1}$, $a^b = bab^{-1}$ of [36] the same formula happens to define the same group (by substituting g^{-1} for g).

links [45], as well as of links obtained by adjoining a component in an arbitrary way to a homotopically Brunnian link [36]; yet another case was found in [25, Theorem 3.1]. This is not the case for 4-component links [24], where nevertheless Levine's peripheral structure $\mathcal{P}'$ yields a complete classification [36]. On the other hand, Levine's program [36] for inductive determination of the link homotopy class of a link from its $(\mathcal{G}, \mathcal{P}')$ via a description of the automorphisms of the Milnor group of a principal sublink, induced by its self-link-homotopies, was found in [24, 25] to be not as certain to succeed as it might have been expected, because for certain 5-component link not all automorphisms preserving the peripheral structure $\mathcal{P}'$ are induced by self-link-homotopies [24]. Another viewpoint of these difficulties is that they arise from non-uniqueness of representation of link as the closure of a string link [19]; indeed, string links are completely classified by their (integer) $\bar{\mu}$-invariants and so by the string link analogue of $(\mathcal{G}, \mathcal{P}'')$.

We conclude this section with general remarks on the notion of k-quasi-isotopy.

Remarks. (i) The term "k-quasi-isotopy" is motivated by the following considerations. A compact polyhedron X is said to be *quasi-embeddable* (cf. [54]) into a PL manifold Q, if for each $\varepsilon > 0$ there exists a map $X \to Q$ with all point-inverses of diameter at most ε (such a map is called an *ε-map*). It is natural to call two embeddings $X \hookrightarrow Q$ *quasi-isotopic* if for each $\varepsilon > 0$ they can be joined by a homotopy in the class of ε-maps. (For links of spheres in a sphere in codimension ≥ 2 this is clearly equivalent to being link homotopic.) These correspond to the case $k = 0$ of the following definition.

(ii) In order to clarify our motivations (and for future reference) we define k-quasi-isotopy ($k \in \mathbb{N}$) in the most general situation, that is for PL embeddings of a compact polyhedron X into a polyhedron Y. Let $f : X \to Y$ be an arbitrary PL map, and consider the singular set $S_f = \text{closure}(\{x \in X \mid f^{-1}(f(x)) \neq \{x\}\})$. Then f is called a *strong k-quasi-embedding* if there exist a commutative PL diagram

$$\begin{tikzcd}
S_f \arrow[d] \arrow[r, phantom, "\subset"] & J_0 \arrow[d, dash] \arrow[dr] \arrow[rr, phantom, "\subset"] & & J_1 \arrow[d, dash] \arrow[dr] \arrow[rr, phantom, "\subset"] & & \cdots & J_{k-1} \arrow[d, dash] \arrow[dr] \arrow[rr, phantom, "\subset"] & & J_k \arrow[d, dash] \arrow[r, phantom, "\subset"] & X \arrow[d, "f"] \\
P_0 \arrow[d, "\varphi_0"] & \subset \arrow[d, "\psi_0"] & P_1 \arrow[d, "\varphi_1"] & \subset \arrow[d, "\psi_1"] & P_2 \arrow[d, "\varphi_2"] & \cdots & \subset \arrow[d, "\psi_{k-1}"] & P_k \arrow[d, "\varphi_k"] & \subset \arrow[d, "\psi_k"] & Y \arrow[d] \\
Q_0 \arrow[r, two heads] & \tilde{Q}_0 \arrow[r, two heads] & Q_1 \arrow[r, two heads] & \tilde{Q}_1 \arrow[r, two heads] & Q_2 \arrow[r, two heads] & \cdots \arrow[r, two heads] & \tilde{Q}_{k-1} \arrow[r, two heads] & Q_k \arrow[r, two heads] & \tilde{Q}_k \arrow[r] & pt
\end{tikzcd}$$

and a triangulation $K_0 \twoheadrightarrow \tilde{K}_0 \twoheadrightarrow K_1 \twoheadrightarrow \tilde{K}_1 \twoheadrightarrow \cdots \twoheadrightarrow K_k \twoheadrightarrow \tilde{K}_k$ of the bottom line, satisfying the following conditions for each $i = 0, \dots, k$:

(a) $f^{-1}(P_i) \subset J_i$;
(b) $\varphi_i^{-1}(C)$ is collapsible for each cone C of the dual cone complex K_i^*;
(c) $\psi_i^{-1}(C)$ is collapsible for each cone C of the dual cone complex $\tilde{K}_i^*$.

Note that $\varphi_0 : P_0 \to Q_0$ can always be chosen to be id : $f(S_f) \to f(S_f)$; and that if X and Y are PL manifolds, "collapsible" may be replaced by "a codimension zero PL ball" by the theory of regular neighborhoods. (Recall that the empty set is not collapsible.) The definition of a *k-quasi-embedding* is similar, with (b) and (c) replaced by the following conditions, where $J_{-1} = P_{-1} = \emptyset$:

(b$'$) the inclusion $P_{i-1} \cup f(J_{i-1}) \subset P_i$ extends to a map $H_i : M_i \to P_i$ of the mapping cylinder M_i of the composition $P_{i-1} \cup f(J_{i-1}) \subset P_i \xrightarrow{\varphi_i} Q_i$ so that $(\varphi_i H_i)^{-1}(C) = R_i^{-1}(C)$ for each $C \in K_i^*$, where $R_i : M_i \to Q_i$ is the projection.

(c$'$) the inclusion $J_{i-1} \cup f^{-1}(P_i) \subset J_i$ extends to a map $h_i : N_i \to J_i$ of the mapping cylinder N_i of the composition $J_{i-1} \cup f^{-1}(P_i) \subset J_i \xrightarrow{\psi_i} \tilde{Q}_i$ so that $(\psi_i h_i)^{-1}(C) = r_i^{-1}(C)$ for each $C \in \tilde{K}_i^*$, where $r_i : N_i \to \tilde{Q}_i$ is the projection.

X is said to *k-quasi-embed* in Y if there exists a k-quasi-embedding $f : X \to Y$. Embeddings $g, h : X \hookrightarrow Y$ (or more generally k-quasi-embeddings) are called *k-quasi-concordant* if they can be joined by a k-quasi-embedding $F : X \times I \to Y \times I$. Finally, a *$k$-quasi-isotopy* is a level preserving k-quasi-concordance F for which pt can be replaced by I in the above diagram (that is, the composition $S_F \subset X \times I \to I$ factors through P_0 and all the Q_i's).

(iii) The above definitions are modelled after the controlled version of the Penrose–Whitehead–Zeeman–Irwin trick as it appears in the Homma–Bryant proof [6] of the Chernavskij–Miller theorem (compare [41, Sec. 3]) that topological embeddings between PL manifolds in codimension ≥ 3 are approximable by PL ones. In more detail, let us say that a (strong) k-quasi-embedding $f : X \to Y$ is *ε-controlled* if $\varphi_k^{-1}(C)$ and $f(\psi_k^{-1}(D))$ have diameters $< \varepsilon$ for each $C \in K_k^*$ and $D \in \tilde{K}_k^*$.

In these terms, Bryant's arguments show that, firstly, if $\xi : X^n \to Y^m$ is a topological embedding between PL manifolds with $m - n \geq 3$, for each $k \in \mathbb{N}$ there exists an $\varepsilon = \varepsilon(k, \xi) > 0$ such that any generic PL ε-approximation f of ξ is a $c\varepsilon$-controlled strong k-quasi-embedding for some constant $c = c(m, n)$. Secondly, the very same arguments of [6] prove that there exists a constant $k(m, n)$ such that every ε-controlled $k(m, n)$-quasi-embedding $X \to Y$ is ε-close to a PL embedding.

In particular, $k(m, n)$-quasi-embeddability of X in Y implies PL embeddability, and similarly $k(m, n)$-quasi-concordance of PL embeddings $X \hookrightarrow Y$ implies PL concordance, which in turn implies (cf. [41]) ambient PL isotopy.

The higher-dimensional version of k-quasi-isotopy also seems to be of some interest in connection with configuration spaces of $(k+2)$-tuples of points (compare [41, Conjecture 1.9]). However, we will not pursue it any further in this paper.

(iv) There are some reasons to regard the Homma–Bryant technique as somewhat complementary or alternative to the technique of gropes. Indeed, the latter was first employed by Shtan'ko to prove a version of the Chernavskij–Miller theorem for compacta (cf. [15]), and according to [15, last line on p. 96] it works to give yet another proof of the Chernavskij–Miller theorem (but note that, though Shtan'ko's Theorem implies the Chernavskij–Miller theorem via a theorem of Bryant–Seebeck,

this does not yield the desired proof, since the latter rests on the Chernavskij–Miller theorem itself). Later, both gropes and Casson handles have played an important role in 4-manifold topology. Now, both viewpoints — gropes and k-quasi-isotopy — yield geometric approaches to finite type invariants, respectively, of knots [13] and of links "modulo knots" (see a further paper by the first author).
(v) In looking for an equivalence relation on classical PL links such that any two PL links sufficiently close to a given topological link are equivalent, one is rather naturally led to k-quasi-isotopy. Recall that in codimension ≥ 3, it was proved by Edwards [14] that any two PL embeddings of a compact polyhedron X into a PL manifold Y sufficiently close to a given topological embedding are joined by a small PL ambient isotopy. So we could obtain the desired equivalence relation by examining general position properties, essential for Edwards' theorem. Edwards' proof rests mainly on radial engulfing and on his "Slicing lemma", proved by a version of the Penrose–Whitehead–Zeeman–Irwin trick where the usual application of engulfing is excluded. An alternative proof of Edwards' theorem in the case where X is a PL manifold can be given along the lines of [58, proof of Theorem 24] using the Homma–Bryant technique in place of the Penrose–Whitehead–Zeeman–Irwin trick. (This is a more direct argument than the straightforward combination of the relative version of the Chernavskij–Miller theorem and the "small concordance implies small ambient isotopy" theorem of [41].) In either case, we arrive at some variant of k-quasi-isotopy.

2. Invariants of 1-Quasi-Isotopy

2.1. *First example*

A counterpart to the 0-quasi-isotopically nontrivial Hopf link $\mathcal{H}$ (Fig. 3(a)), the (left handed) Whitehead link $\mathcal{W}$ (Fig. 3(b)) is not weakly 1-quasi-isotopic to the unlink. Indeed, the crossing change formula [28] for the Sato–Levine invariant $\beta = \bar{\mu}(1122)$ of 2-component links with zero linking number implies its invariance under weak 1-quasi-isotopy:

$$\beta(L_+) - \beta(L_-) = -n^2, \tag{2.1}$$

where L_+, $L_- : S^1 \sqcup S^1 \hookrightarrow S^3$ differ by a single positive[g] self-intersection of one component, leading to two lobes whose linking numbers with the other component are n and $-n$. The formula (2.1) along with the requirement that $\beta(\text{unlink}) = 0$ clearly suffices to evaluate β on any null-homotopic 2-component link, so $\beta(\mathcal{W}) = 1$ and the assertion follows.

Next, one could ask, whether the three versions of 1-quasi-isotopy are distinct. The untwisted left handed Whitehead double of $\mathcal{W}$, denoted $\mathcal{W}_2$ (Fig. 3(c)), is

[g]We call the process of self-intersection *positive* if, given a choice of orientation of the component that crosses itself and a choice of ordering of the two singular points p, q, the triple consisting of the positive tangent vector at $L_-(p)$, of that at $L_-(q)$, and of the vector $L_-(p) - L_-(q)$, agrees with a fixed orientation of the ambient space (this is clearly independent of the two choices).

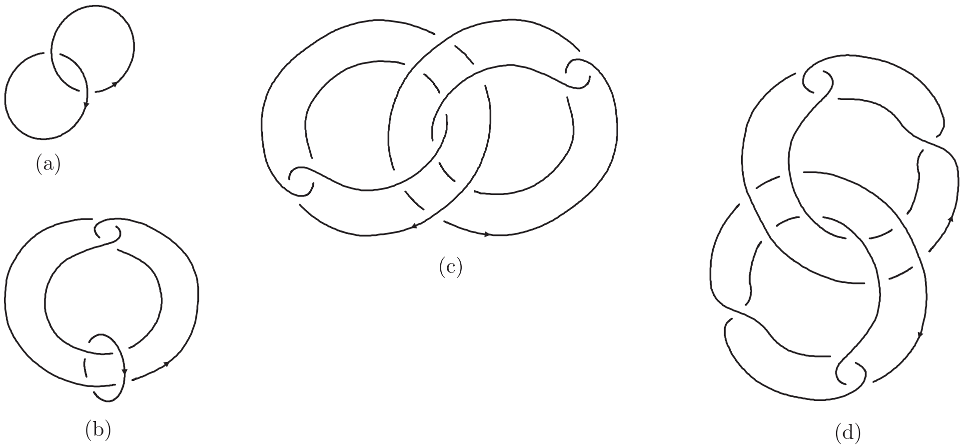

Fig. 3.

clearly 1-quasi-isotopically trivial but *seemingly* not strongly 1-quasi-isotopically trivial since the Whitehead curve in the solid torus cannot be engulfed (i.e. $\mathcal{W}$ is not a split link). Whereas a link of two trefoils $\mathcal{W}_2'$ obtained by a twisted Whitehead doubling of each component of the Hopf link (Fig. 3(d)) can be seen to weakly 1-quasi-isotop onto the unlink, but *conjecturally* is nontrivial up to 1-quasi-isotopy.

Problem 2.1. *Find invariants distinguishing the three versions of* 1*-quasi-isotopy.*

2.2. *The generalized Sato–Levine invariant*

Apart from these matters, let us see the difference between (either version of) 1-quasi-isotopy and (either version of) concordance. Consider the well-known link $\mathcal{M}$ depicted in Fig. 4(a). We call it the *Mazur link* (compare [40]) since it was employed, *inter alia*, in the construction of Mazur's contractible 4-manifold. The Mazur link is obviously F-isotopic[h] to the Hopf link and hence topologically I-equivalent to it, by virtue of Giffen's shift-spinning construction (see [22, Theorem 1.9]). It is known that $\mathcal{M}$ is not PL I-equivalent to $\mathcal{H}$ [52], but nevertheless the *fake Mazur link* (cf. [40, p. 470]) $\mathcal{M}'$, shown in Fig. 4(b), is even PL concordant (i.e. locally flat PL I-equivalent) to $\mathcal{H}$. This is clear from Fig. 4(c), where the ambient isotopy class of $\mathcal{M}'$ is represented in a different way.

On the other hand, both $\mathcal{M}$ and $\mathcal{M}'$ are not weakly 1-quasi-isotopic to $\mathcal{H}$, as detected by the *generalized Sato–Levine invariant* $\tilde{\beta}$ (compare [31, p. 1351] and [47, Fig. 6]). By definition, $\tilde{\beta}(L) = a_3(L) - a_1(L)(a_2(K_1) + a_2(K_2))$, where K_1 and K_2 are the components of the link L, and a_i denotes the coefficient at z^i of the Conway

[h]Two links are called *F-isotopic* if one can be obtained from another by a sequence of substitutions of a PL-embedded component K_i with an arbitrary knot K_i' lying in its regular neighborhood $V(K_i)$ and homotopic to K_i in $V(K_i)$, and of reverse operations.

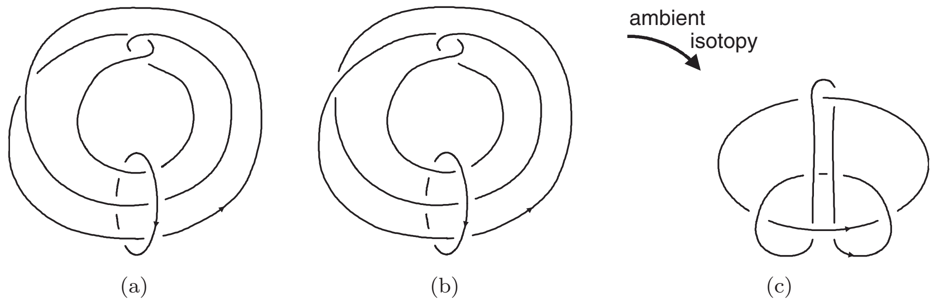

Fig. 4.

link polynomial. It is easy to verify (cf. [39]), using the skein relation for the Conway polynomial, that, given two links $L_+, L_- : S^1 \sqcup S^1 \hookrightarrow S^3$, related by a single positive crossing change on one component so that the two lobes' linking numbers with the other component are n and $l - n$, where $l = \mathrm{lk}(L_+) = \mathrm{lk}(L_-)$, we have

$$\tilde{\beta}(L_+) - \tilde{\beta}(L_-) = n(l - n). \tag{2.2}$$

Hence $\tilde{\beta}(\mathcal{H}) \neq \tilde{\beta}(\mathcal{M}) = \tilde{\beta}(\mathcal{M}')$, meanwhile (2.2) immediately implies invariance of $\tilde{\beta}$ under weak 1-quasi-isotopy. This proves the case of links in the following

Theorem 2.2. *PL concordance does not imply weak* 1-*quasi-isotopy, for both links and string links.*

Proceeding to the case of string links, let us first introduce some notation to be used in the rest of this section.

If $L : I_+ \sqcup I_- \hookrightarrow \mathbb{R}^2 \times I_0$ is a string link, let ρL denote the *reflection* $R \circ L \circ r$ of L, where r and R are the self-homeomorphisms of $I_+ \sqcup I_-$ and $\mathbb{R}^2 \times I_0$ given by the reflections of I_+, I_- and I_0 in their midpoints.

Next, let H_n denote the string link that can be described as the braid $(\sigma_{12})^{2n}$ or the integer tangle $2n$ (see [17]). The link $H := H_1$ will be called the *Hopf string link* since its numerator closure (see Sec. 1) $\mathrm{numer}(H)$ is the Hopf link $\mathcal{H}$. The *linking number l closure* of a 2-component string link L is defined to be the link $\mathrm{numer}_l(L) := \mathrm{numer}(L \# H_{l-n})$, where $n = \mathrm{lk}(\mathrm{numer}(L))$.

Finally, recall from Sec. 1 that W denotes the Whitehead string link, shown three times in Fig. 1; it can also be described as the rational tangle $\frac{1}{1+\frac{1}{2}}$.

Proof of the case of string links in Theorem 2.2. The link $\mathrm{numer}_1(W \# \rho W)$ is the fake Mazur link (Fig. 4(c)), which is not 1-quasi-isotopic to the Hopf link. Therefore the string link W is not cancelled by ρW up to 1-quasi-isotopy.

However, $W \# \rho W$ is concordant to the trivial string link. Indeed, for any string link $L : I_+ \sqcup I_- \hookrightarrow \mathbb{R}^2 \times I_0$, the embedding $L \times \mathbf{1}_I : (I_+ \sqcup I_-) \times I \to \mathbb{R}^2 \times I_0 \times I$ is a collection of slicing disks for the restriction of $L \times \mathbf{1}_I$ to $(I_+ \sqcup I_-) \times \partial I \cup (\{1\} \sqcup \{1\}) \times I$, which is a string link ambient isotopic to $L \# \rho L$. $\square$

Remarks. (i) An axiomatic definition of $\tilde\beta$ can be given by the formula (2.2) along with the requirement that $\tilde\beta(\mathcal{H}_n) = 0$ for each $n \in \mathbb{Z}$, where $\mathcal{H}_n = \mathrm{numer}(H_n)$. This definition was used in [31], where $\tilde\beta$ was related to the Casson–Walker invariant of rational homology 3-spheres, in [2], where a direct proof of its existence was given, without references to skein theory, and in [39]. An invariant defined by Gauss diagrams in [49] (the misprint in the definition was corrected in [48] and [3]) turned out to coincide with $\tilde\beta$ [3] (as defined above and in [31]; the axiomatic definition of $\tilde\beta$ in [3] uses a slightly different choice of $\mathcal{H}_n$, discussed in [31, p. 1337]). Yet another choice of H_n has to be made for the version of $\tilde\beta$ in [59, Sec. 10]. Nakanishi and Ohyama recently proved that 2-component links have the same lk and $\tilde\beta$ iff they are related by a sequence of the so-called self-Δ-moves [46]. Relevance of $\tilde\beta$ for magnetohydrodynamics is discussed in [1].

(ii) It follows from [12, Sec. 4] (see also [11, Sec. 9]) that $\tilde\beta$ is an integral lifting of Milnor's invariant $\bar\mu(1122)$. To add some clarity to the subject (cf. [31, Addenda]), we give a direct proof of this fact in Appendix A.

By the Stallings theorem, the quotients of the fundamental groups of topologically I-equivalent links (or string links) by any term of the lower central series are isomorphic [55, 8, 10], and the isomorphism preserves the image of the peripheral structure in the quotient (see [8, Remark 1]). Thus we have

Corollary 2.3. *No nilpotent quotient of the fundamental group, even if equipped with the peripheral structure, can classify two-component links, or string links, up to* 1*-quasi-isotopy.*

2.3. *The η-function and Jin suspension*

Let us recall the definition of the η-function of Kojima and Yamasaki [33]. Given a linking number zero link $L(2S^1) = K_+ \sqcup K_- \subset S^3$, let $\tilde K_+$ be a lift of K_+ into the infinite cyclic cover X_- of $S^3 \setminus K_-$, and let $\tilde K_+^*$ be the nearby lift of the zero pushoff of K_+ (which may be a nonzero pushoff of $\tilde K_+$ if K_+ happens to be unknotted). Let $f(t)$ be any Laurent polynomial annihilating the class of $\tilde K_+$ in the $\mathbb{Z}[t^{\pm1}]$-module $H_1(X_-)$,[i] for example the Alexander polynomial of K_-. Then the 1-cycle $f(t)\tilde K_+$ bounds a chain ζ in X_-. Define

$$\eta_L^+(t) = \frac{1}{f(t)} \sum_{n=-\infty}^{\infty} (\zeta \cdot t^n \tilde K_+^*) t^n,$$

[i]The Wang sequence for a map $S^3 \setminus K_- \to S^1$ representing a generator of $H^1(S^3 \setminus K_-)$, which is the same thing as the long sequence associated to the sequence $0 \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}] \xrightarrow{t=1} \mathbb{Z} \to 0$ of coefficient modules for $H_*(S^3 \setminus K_-; \cdots)$, shows that multiplication by $t-1$ is an isomorphism on $H_1(X_-)$. Since $H_1(X_-)$ is finitely generated over $\mathbb{Z}[\mathbb{Z}]$, which is Noetherian, $H_1(X_-)$ is Noetherian, therefore infinite divisibility by $t-1$ yields, for each $m \in H_1(X_-)$, a Laurent polynomial p such that $(t-1)^{-1}m = pm$. It follows that $H_1(X_-)$ is torsion over $\mathbb{Z}[\mathbb{Z}]$.

where $(\,\cdot\,)$ stands for the intersection pairing in X_-, and only finitely many of the summands may be nonzero; thus $\eta^+_L(t) \in \mathbb{Q}(t)$, the field of fractions of $\mathbb{Z}[t^{\pm 1}]$. Then η^+_L does not depend on the choice of ζ since $H_2(X_-) = 0$,[j] and consequently on the choice of $f(t)$, for η^+_L clearly remains unchanged if $f(t)$ and ζ are simultaneously multiplied by some polynomial $g(t)$. By interchanging K_+ and K_-, one can similarly define η^-_L. Since the projection of ζ to $S^3 \setminus K_-$ has zero intersection number with K^*_+, we have $\eta^+_L(1) = 0$; therefore $\eta^+_L(t) = (1-\epsilon)\big[f(t)^{-1}\sum_{n\neq 0}(\zeta\cdot t^n\tilde{K}^*_+)t^n\big]$, where $\epsilon\,\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z} \subset \mathbb{Z}[\mathbb{Z}]$ is the augmentation, $\epsilon(t) = 1$.

Remark. If F is a Seifert surface for K_-, disjoint from K_+, one can similarly define a pairing $\langle\cdot\rangle$ on disjoint cycles in a lift of $S^3 \setminus F$ into X_- such that $\eta^+_L = \langle \tilde{K}_+ \cdot \tilde{K}^*_+\rangle$ and which descends to the Blanchfield pairing (see [22]) on $H_1(X_-)$, well-defined up to addition of (finite) polynomials. The pairing $\langle\cdot\rangle$ was used in [11, Sec. 7] to show that the rational function $\eta^+_L(t)$ can be expanded as a rational power series $\sum_{k=1}^{\infty}\beta^k_+ z^k$ in the variable $z = -(t+t^{-1}-2)$, where the coefficients β^k_+ are known as Cochran's derived invariants (in particular, β^1_+ is the Sato–Levine invariant β).

If K_- happens to be unknotted, η^+_L is by definition the Laurent polynomial $\sum_{n\geq 1}\operatorname{lk}(\tilde{K}_+, t^n\tilde{K}_+)(t^n + t^{-n} - 2)$, where lk is the usual linking number in $\mathbb{R}^3$. More generally, by a well-known unpublished observation of Jin, the η-function satisfies the following crossing change formula for a positive self-intersection of K_+:

$$\eta^+_{L'} - \eta^+_L = t^s + t^{-s} - 2, \tag{2.3}$$

where s is the linking number between K_- and one of the lobes of the singular component. Indeed, we may assume that the trace of the homotopy between K_+ and K'_+ is a small disk D such that $\partial D = K'_+ - K_+$ as chains, and which meets the common arc of K_+ and K'_+ transversally in an interior point p. Then $f(t)\tilde{K}'_+$ bounds the chain $\zeta' = \zeta + f(t)\tilde{D}$, where $\tilde{D}$ is the appropriate lift of D, and we may assume that ζ' meets the union of all translates of the lift $\tilde{D}^*$ of the pushoff of $\tilde{D}$ along the 1-chain $f(t)\tilde{J}^*$ where $\tilde{J}^*$ is the lift of the pushoff of an arc $J\imath D$ joining p with a point of ∂D. We have

$$\eta^+_{L'}(t) - \eta^+_L(t) = \sum_{n\neq 0}(\tilde{D}\cdot t^n\tilde{K}^*_+)(t^n - 1) + (1-\epsilon)\left[\frac{1}{f(t)}\sum_{n\neq 0}(\zeta'\cdot t^n\partial\tilde{D}^*)t^n\right],$$

where the first summand of the right hand side equals $t^s + t^{-s} - 2$ and the second $(1-\epsilon)[1] = 0$. See [27] for a different proof and [32] for generalizations of the formula (2.3).

Since the right-hand side of (2.3) turns to zero when $s = 0$, we get

Proposition 2.4. *η^+_L is invariant under weak 1-quasi-isotopy with support in K_+.*

[j] The same argument as in the preceding footnote shows that $H_2(X_-)$ is $\mathbb{Z}[\mathbb{Z}]$-torsion. But $S^3 \setminus K_-$ is homotopically 2-dimensinoal, so $H_2(X_-) = \ker[C_2(X_-) \to C_1(X_-)]$ is free over $\mathbb{Z}[\mathbb{Z}]$.

A link $S^1 \sqcup S^1 \hookrightarrow S^3$ is called *semi-contractible* [30, 34] if each component is null-homotopic in the complement to the other. In the class of semi-contractible links there is yet another way to think of Cochran's invariants. The *Jin suspension* [30] of a semi-contractible link L is a link map $S^2 \sqcup S^2 \to S^4$, formed by the tracks of some null-homotopies of the components of L in each other's complement. By the Sphere theorem of Papakyriakopoulos, the result is well-defined up to link homotopy (cf. [34]). This construction, applied by Fenn and Rolfsen to the Whitehead link $\mathcal{W}$, produced the first example of a link map $S^2 \sqcup S^2 \to S^4$, nontrivial up to link homotopy.

To define Kirk's invariant of a link map $\Lambda : S^2_+ \sqcup S^2_- \to S^4$, we assume by general position that f only has transversal double points. Let $z = \Lambda(x) = \Lambda(y)$ be one, with $x, y \in S^2_+$. Let p be a path in S^2_+ joining x to y, then $\Lambda(p)$ is a closed loop in $S^4 \backslash \Lambda(S^2_-)$. Denote by n_z the absolute value of the linking number between this loop and $\Lambda(S^2_-)$, and by ε_z the sign, relating the orientations of the two sheets crossing at z to that of S^4. Then $\sigma_+(t) := \sum_z \varepsilon_z(t^{n_z} - 1)$, together with the analogously defined $\sigma_-(t)$, form Kirk's invariant $\sigma(\Lambda)$ of link homotopy of Λ [30]. By (2.3), the pair (η^+_L, η^-_L) for a semi-contractible link L contains the same information as the Kirk invariant of the Jin suspension of L. So Proposition 2.4 implies

Proposition 2.5. *Let L and L' be semi-contractible links that can be joined by two weak* 1*-quasi-isotopies such that the ith carries the ith component by a locally-flat PL isotopy. Then the Jin suspensions of L and L' have identical Kirk's invariant.*

Remark. According to Teichner, it remains an open problem (as of Spring 2003) whether two link maps $S^2 \sqcup S^2 \to S^4$ with the same Kirk invariant may be not link homotopic (compare [5]).

2.4. *Koschorke's refined Kirk invariant*

We conclude this section with a result on link maps, imposing a constraint on existence of invariants of 1-quasi-isotopy of string links. (The constraint is discussed in Remark (i) at the end of the section.)

Under a *fibered disk link map* $\Lambda : (I_+ \sqcup I_-) \times I \to (\mathbb{R}^2 \times I_0) \times I$ we understand a self-link-homotopy of the trivial string link. (In terms of the map Λ, we assume that $\Lambda(I_\pm \times \{t\}) \subset \mathbb{R}^2 \times I_0 \times \{t\}$ for each $t \in I$, and that $\Lambda(\pm, s, t) = (\pm p, s, t)$ whenever $(s, t) \in \partial(I \times I)$, where $p \in \mathbb{R}^2 \backslash \{0\}$ is a fixed point.) The *trivial* fibered disk link map is given by $\Lambda_0(\pm, s, t) = (\pm p, s, t)$ for all s and t. To every fibered disk link map Λ one can associate a spherical link map $\mathrm{Numer}(\Lambda) : S^2_+ \sqcup S^2_- \to S^4$, defined to be the composition of $\Lambda/\partial : (I_+ \times I)/\partial \sqcup (I_- \times I)/\partial \to \mathbb{R}^2 \times ((I_0 \times I)/\partial)$ and the inclusion $\mathbb{R}^2 \times S^2 \subset D^2 \times S^2 \cup S^1 \times D^3 = \partial(D^2 \times D^3)$.

For fibered disk link maps there is a refined version of the Kirk invariant, well-defined up to *fibered* link homotopy [34]. Given a Λ as above, by general position we may assume that it only has transversal double points. Let $z = \Lambda(x, t) = \Lambda(y, t)$ be one, with $x, y \in I_+$. There is a unique path p in I_+ joining x to y. Then $\Lambda(p, t)$

is a closed loop in $\mathbb{R}^2 \times I_0 \times I \backslash \Lambda(I_- \times I)$, which has a well-defined linking number with $\Lambda(I_- \times I)$. More precisely, let l_z denote the algebraic number of intersections between $\Lambda(I_- \times \{t\})$ and a generic disk, spanned by $\Lambda(p)$ in $\mathbb{R}^2 \times I_0 \times \{t\}$. We emphasize that not only the absolute value, but also the sign of l_z is well-defined, due to the natural ordering of x and y in I_+. Denote by ε_z the sign, relating the orientations of the two sheets crossing at z to that of the ambient 4-space. We define $\tilde{\sigma}_+$ to be the Laurent polynomial $\sum_z \varepsilon_z (t^{l_z} - 1)$, where z runs over the double points of the $+$-component $\Lambda|_{I_+ \times I}$, and $\tilde{\sigma}_-$ by interchanging I_+ and I_-.

To see that $\tilde{\sigma} := (\tilde{\sigma}_+, \tilde{\sigma}_-)$ is unchanged under a fibered link homotopy between fibered disk link maps Λ and Λ', approximate Λ' arbitrarily closely by a Λ'', fibered *regularly* homotopic with Λ (cf. [53, 4.4]). Clearly, $\tilde{\sigma}(\Lambda'') = \tilde{\sigma}(\Lambda')$, and since l_z is a continuous function on the 1-manifold of double points of the fibered regular homotopy, $\tilde{\sigma}(\Lambda'') = \tilde{\sigma}(\Lambda)$. Of course, $\sigma(\mathrm{Numer}(\Lambda)) = (\varphi \oplus \varphi)\tilde{\sigma}(\Lambda)$, where $\varphi : \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t]$ denotes the $\mathbb{Z}$-homomorphism given by $t^n \mapsto t^{|n|}$.

Here are some examples. Semi-contractible string links and their Jin suspensions, denoted $\mathfrak{J}(L)$, are defined in the obvious fashion. Let $h_n^{\pm} : S_+^1 \sqcup S_-^1 \to S_+^1 \sqcup S_-^1$ be a map that has degree n on $S_{\pm}^1$ and degree 1 on $S_{\mp}^1$. Let us fix some 2-component string link $W_{\pm}^n$, whose linking number zero closure $\mathrm{numer}_0(W_{\pm}^n)$ is isotopic to a C^0-approximation of the composition of $h_n^{\pm}$ and the Whitehead link $\mathcal{W}$. In particular, W_+^1 and W_-^1 may be chosen to coincide with the Whitehead string link W (shown three times in Fig. 1). It is not hard to see (cf. [34]) that

$$\tilde{\sigma}(\mathfrak{J}(W_+^n)) = \left(\frac{n^2}{2}(t + t^{-1} - 2) + \frac{n}{2}(t - t^{-1}), 1 - t^n \right),$$

and it follows that

$$\tilde{\sigma}(\mathfrak{J}(W_+^n \# \rho W_+^n)) = (n(t - t^{-1}), t^{-n} - t^n).$$

Theorem 2.6. (a) [30] $\mathrm{im}\,\sigma = \ker \delta$, *where* $\delta : \mathbb{Z}[t] \oplus \mathbb{Z}[t] \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ *is given by*

$$(f, g) \mapsto (f|_{t=1},\ g|_{t=1},\ f' + g' + f'' + g''|_{t=1}).$$

(b) $\mathrm{im}\,\tilde{\sigma} = \ker \tilde{\delta}$, *where* $\tilde{\delta} : \mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ *is given by*

$$(f, g) \mapsto (f|_{t=1},\ g|_{t=1},\ f' + g'|_{t=1},\ f'' + g''|_{t=1}).$$

Proof of (b). By the definition of $\tilde{\sigma}$, it satisfies $\tilde{\sigma}_+|_{t=1} = \tilde{\sigma}_-|_{t=1} = 0$. Let h_t be a self-link-homotopy of the trivial string link. Then $\mathrm{numer}_0(h_t)$ is a self-link-homotopy of the unlink, and $\mathrm{numer}_1(h_t)$ is a self-link-homotopy of the Hopf link. Since the generalized Sato–Levine invariant is a well-defined invariant of links, it does not change under each homotopy. On the other hand, it jumps by $\sum_z \varepsilon_z l_z (0 - l_z) = -\tilde{\sigma}_0' - \tilde{\sigma}_0''|_{t=1}$ in the former case, and by $\sum_z \varepsilon_z l_z (1 - l_z) = -\tilde{\sigma}_0''|_{t=1}$ in the latter, where $\tilde{\sigma}_0 = \tilde{\sigma}_+ + \tilde{\sigma}_-$. Thus $\mathrm{im}\,\tilde{\sigma} \subset \ker \tilde{\delta}$.

The reverse inclusion is due to Koschorke [34], and follows easily from the above mentioned facts. Indeed, $\sigma(\mathrm{Numer}(\mathfrak{J}(W_{\pm}^n))) = (\varphi \oplus \varphi)\tilde{\sigma}(\mathfrak{J}(W_{\pm}^n))$ generate $\ker \delta$.

On the other hand, $\tilde\sigma(\mathfrak{J}(W^n_\pm\#\rho W^n_\pm))$ generate the kernel of the $\mathbb{Z}$-homomorphism $\mathbb{Z}[t^{\pm1}]\oplus\mathbb{Z}[t^{\pm1}]\to\mathbb{Z}[t]\oplus\mathbb{Z}[t]\oplus\mathbb{Z}$, given by $(f,g)\mapsto(\varphi(f),\varphi(g),f'+g'|_{t=1})$. $\square$

The first part of the above argument yields

Proposition 2.7. *If two semi-contractible* 2*-component links* (*string links*) *are weakly* 1*-quasi-isotopic, their Jin suspensions have identical* $\sigma'_\pm(1)+\sigma''_\pm(1)$ (*respectively,* $\tilde\sigma'_\pm(1)$ *and* $\tilde\sigma''_\pm(1)$).

The following seems to be implicit in [34].

Theorem 2.8. *Fibered concordance does not imply fibered link homotopy.*

Proof. Consider the fibered disk link map $\mathfrak{J}(W\#\rho W)$. By a standard construction (cf. the proof of Theorem 2.2) it is fiberwise concordant with the trivial fibered disk link map. However, they are not fiberwise link homotopic, since $\tilde\sigma(\mathfrak{J}(W\#\rho W))=(t-t^{-1},t^{-1}-t)\neq(0,0)$. $\square$

This yields a counterexample (for disk link maps) to the 1-parametric version of Lin's theorem [38] (see also [16, 18]) that singular concordance implies link homotopy in the classical dimension, and to the fibered version of Teichner's theorem [5, Theorem 5].

Remarks. (i) The definition of $\tilde\sigma$ applies equally well to an arbitrary link homotopy of string links, rather than a self-link-homotopy of the trivial string link; we denote the extended invariant by $\bar{\tilde\sigma}=(\bar{\tilde\sigma}_+,\bar{\tilde\sigma}_-)$. Following the proof of "$\subset$" in Theorem 2.6(b), the invariants $\beta(\mathrm{numer}_0(L))$ and $\tilde\beta(\mathrm{numer}_1(L))$ of a string link L can now be viewed as determined by the extended invariant $\bar{\tilde\sigma}_0:=\bar{\tilde\sigma}_++\bar{\tilde\sigma}_-$ of any link homotopy H from L to the trivial string link, namely

$$\beta(\mathrm{numer}_0(L))=-\bar{\tilde\sigma}'_0(H)-\bar{\tilde\sigma}''_0(H)|_{t=1}\quad\text{and}\quad\tilde\beta(\mathrm{numer}_1(L))=-\bar{\tilde\sigma}''_0(H)|_{t=1}.$$

On the other hand, the "$\supset$" part of Theorem 2.6(b) implies that any invariant of a string link L that is a $\mathbb{Z}$-linear function of $\bar{\tilde\sigma}$ of a link homotopy from L to the trivial string link, is a linear combination of $\beta\circ\mathrm{numer}_0$ and $\tilde\beta\circ\mathrm{numer}_1$.

(ii) The mod 2 reduction of $\bar{\tilde\sigma}'_+|_{t=1}$, i.e. $\sum_z\varepsilon_z l_z\,(\mathrm{mod}\,2)$, where z runs over the double points of the $+$-component, is nothing but Hudson's obstruction [23] to embedding the $+$-component by homotopy in the complement of the $-$-component. Note that the mod 2 reduction of $\bar{\tilde\sigma}''_+|_{t=1}$ is identically zero.

(iii) The string links W_+^{-1} and W_-^{-1} may be chosen to coincide with the string link W^{-1} that can be described as the rational tangle $\frac{1}{2+\frac{1}{2}}$. Its reflection $\bar W:=\rho W^{-1}$ may be called the right handed Whitehead string link, since its linking number zero closure $\mathrm{numer}_0(\bar W)$ is the right handed Whitehead link $\bar{\mathcal{W}}$ (i.e. $\mathcal{W}$ composed with a reflection of S^3 and precomposed with a reflection of one component). Interestingly, $W\#\bar W$ and $\bar W\#W$ have zero invariants $\beta\circ\mathrm{numer}_0$ and $\tilde\beta\circ\mathrm{numer}_1$ as well as $\eta^+\circ\mathrm{numer}_0$ and $\eta^-\circ\mathrm{numer}_0$, but seem unlikely to be weakly null-1-quasi-isotopic.

3. The Fundamental Group

3.1. *Preliminaries*

Let us fix some notation. Unless otherwise mentioned, we work in the PL category. For any link $L : mS^1 \hookrightarrow S^3$ we write $\pi(L)$ for its fundamental group $\pi_1(S^3 \backslash L(mS^1))$. If G is a group, the conjugate $h^{-1}gh$ of $g \in G$ by $h \in G$ will be denoted by h^g, and the normal closure of a subgroup H in G by H^G. We also write $g^{\pm h}$ for $(g^h)^{\pm 1}$.

For the following lemma, suppose that links L_+, $L_- : mS^1 \hookrightarrow S^3$ are related by a crossing change on the ith component, and $L_s : mS^1 \to S^3$ is the intermediate singular link, with lobes ℓ and ℓ'. We may assume that the images of L_+, L_- and L_s are contained in some $\mathbb{R}^3 \subset S^3$ and coincide outside the cube $Q := [-1,1]^3$, meeting $L_s(mS^1)$ in $[-1,1] \times \{0\} \times \{0\} \cup \{0\} \times \{0\} \times [-1,1]$. Attach 1-handles $H, H' \cong D^2 \times I$ to the interiors of four faces $F_\pm$, $F'_\pm$ of Q so that $Q \cup H$, $Q \cup H'$ and $N := Q \cup H \cup H'$ are regular neighborhoods of ℓ, ℓ' and $L_s(S^1_i)$ in $S^3 \backslash L_s(mS^1 \backslash S^1_i)$.

By a *meridian* of the lobe ℓ we mean the image in $\pi(L_s) := \pi_1(S^3 \backslash N)$ of a generator α of $\pi_1(H \cap \partial N) \simeq \mathbb{Z}$ under the inclusion induced homomorphism, which is determined by a path ρ joining the basepoints of these spaces. The *longitude* of ℓ, corresponding to ρ, is the image in $\pi(L_s)$ of a generator β of $\pi_1((H \cup F_+ \cup F_-) \cap \partial N)$ that can be represented by a loop crossing the edge $F_+ \cap F_-$ of Q (geometrically) exactly once, and having zero linking number with ℓ.

Lemma 3.1. *Suppose that links L_+ and L_- differ by a single crossing change on the ith component so that the intermediate singular link L_s is a k-quasi-embedding. Let $\ell = J_0$ be the lobe of L_s as in the definition of k-quasi-embedding. Let μ, $\lambda \in \pi(L_s)$ be a meridian of ℓ and the corresponding longitude, and μ_+, λ_+ be their images in $\pi(L_+)$. Then*

$$\lambda_+ \in \langle \mu_+ \rangle_k^{\pi(L_+)},$$

where for a subgroup H of a group G we set $H_0^G = G$ and inductively $H_{k+1}^G = H^{H_k^G}$.

It may be easier to guess the following argument than to read it.

Proof. The proof of the general case is best exposed by analogy with the cases $k = 1, 2$, considered below.

Without loss of generality, $Q \cup H \subset P_1$, where P_1 is as in the definition of k-quasi-embedding. Let $L_s(p) = L_s(q)$ be the double point of L_s, and y the basepoint of $H \cap \partial N$. Let ρ be a path joining the basepoint of $S^3 \backslash N$ to y, and $o : S^1 \to H \cap \partial N$, $l : S^1 \to (H \cup F_+ \cup F_-) \cap \partial N$ be some representatives of α, β above. Pick a generic PL null-homotopy $F : D^2 \to P_1$ of the loop l, transversal to $L_s(S^1_i)$ and such that $F(\operatorname{Int} D^2)$ is disjoint from Q. Suppose, for definiteness, that l is clockwise with respect to a fixed orientation of D^2. Note that $y = F(y)$.

If N is small enough, each intersection point $L_s(p_j) = F(q_j)$ corresponds to a small circle $C_j \subset F^{-1}(N)$, circling around q_j. Pick any $y_j \in C_j$, and let $I_j \subset D^2$ be an arc joining y to y_j and disjoint from all C_u's, except at y_j. Let us parameterize $F(C_j)$ by a clockwise loop l_j with ends at $F(y_j)$, and parameterize $F(I_j)$ by a path r_j directed from $F(y)$ to $F(y_j)$. Then the loop l is homotopic in $F(D^2)\backslash \operatorname{Int} N$ to $\bar{r}_1 l_1 r_1 \cdots \bar{r}_n l_n r_n$, so λ_+ is a product of the meridians $[\bar{\rho}\bar{r}_j l_j r_j \rho]$ to the ith component. But any two meridians to the same component are conjugate, thus $\lambda_+ \in \langle \mu_+ \rangle^{\pi(L_+)}$. This completes the proof in the case $k = 1$.

For $k \geq 2$ we need an additional ingredient for our construction. Pick a parallel push-off h of $L_+(S^1_i)$ into ∂N such that $h(p) = y$. Each y_j could have been chosen so that $F(y_j) = h(p_j)$.

Now let I'_j denote the subarc of J_1 joining p to p_j, and let $\tilde{r}_j$ be a path parameterizing $h(I'_j)$ in the direction from $h(p) = F(y)$ to $h(p_j) = F(y_j)$. Pick a generic PL null-homotopy $F_j : D^2_j \to P_2$ of the loop $\bar{r}_j \tilde{r}_j$, transversal to $L_s(S^1_i)$ and such that $F_j(\operatorname{Int} D^2_j)$ is disjoint from Q. For convenience, we assume that $\bar{r}_j \tilde{r}_j$ is counterclockwise with respect to a fixed orientation of D^2_j. Note that $F_j(y) = y$ and $F_j(y_j) = y_j$.

If N is small enough, each intersection point $L_s(p_{ju}) = F_j(q_{ju})$ corresponds to a small circle $C_{ju} \subset F_j^{-1}(N)$, circling around q_{ju}. Pick any $y_{ju} \in C_{ju}$, and let $I_{ju} \subset D^2_j$ be an arc joining y to y_{ju} and disjoint from all C_{ju}'s, except at y_{ju}. Let us parameterize $F_j(C_{ju})$ by a clockwise loop l_{ju} with ends at $F_j(y_{ju})$, and parameterize $F_j(I_{ju})$ by a path r_{ju} directed from $F_j(y)$ to $F_j(y_{ju})$.

Then the loop $\bar{r}_j \tilde{r}_j \tilde{l}_j$ is null-homotopic in $F_j(D^2_j)\backslash \operatorname{Int} N$, where $\tilde{l}_j$ denotes $\bar{r}_{j1} l_{j1} r_{j1} \cdots \bar{r}_{jn_j} l_{jn_j} r_{jn_j}$. It follows that the loop $\bar{r}_j l_j r_j$ is homotopic in $F_j(D^2_j)\backslash \operatorname{Int} N$ to $\bar{\tilde{l}}_j \bar{\tilde{r}}_j l_j \tilde{r}_j \tilde{l}_j$. But the loop $\bar{\tilde{r}}_j l_j \tilde{r}_j$ is homotopic in $N\backslash L_+(S^1_i)$ (though possibly not in $N\backslash L_s(S^1_i)$) to o, so $\bar{\rho}\bar{r}_j l_j r_j \rho$ is homotopic in $S^3\backslash L_+(mS^1)$ to $(\bar{\rho}\bar{\tilde{l}}_j \rho)\bar{\rho} o \rho(\bar{\rho}\tilde{l}_j \rho)$, which represents $\mu_+^{[\tilde{l}_j]}$. Thus λ_+ is a product of conjugates of μ_+ by products of the meridians $[\bar{\rho}\bar{r}_{ju} l_{ju} r_{ju} \rho] \in \langle \mu_+ \rangle^{\pi(L_+)}$. ☐

Remark. To continue the proof for $k \geq 3$, we should have chosen each y_{ju} so that $F_j(y_{ju}) = h(p_{ju})$.

3.2. *Higher analogues of Milnor's group*

Let L be an m-component link and $M \subset \pi(L)$ be the set of all meridians of L, which is closed under inverse and conjugation, though is not a subgroup. Let μ_k denote the subgroup of $\pi(L)$ generated by the set

$$\left\{ [m, m^g] \mid m \in M, g \in \langle m \rangle_k^{\pi(L)} \right\},$$

where $[x, y] = x^{-1} y^{-1} x y$. This subgroup is normal since $[m, m^g]^h = [m^h, (m^h)^{g^h}]$. Moreover, since any two meridians to one component are either conjugate or inverse-conjugate, while the commutator identity $[x, y^{-1}] = [y, x]^{y^{-1}}$ implies the equality $[m^{-1},(m^{-1})^g] = [m,m^g]^{(m^g m)^{-1}}$, our μ_k coincides with the normal closure of

$$\big\{[m_i,m_i^g] \mid i=1,\dots,m;\ g\in\langle m_i\rangle_k^{\pi(L)}\big\},$$

where $\{m_1,\dots,m_m\}$ is a fixed collection of meridians. Recall that $\pi(L)/\mu_0$ is Milnor's group $\mathcal{G}(L)$, invariant under homotopy [45, Theorem 2], [36, Theorem 4]. In the same spirit we consider the quotient $\mathcal{G}_k(L)=\pi(L)/\mu_k$ (this agrees with the notation of Sec. 1, as we will see from Theorems 3.2 and 3.5).[k]

Theorem 3.2. *If two links L, $L' : mS^1 \hookrightarrow S^3$ are k-quasi-isotopic then the groups $\mathcal{G}_k(L)$, $\mathcal{G}_k(L')$ are isomorphic.*

Moreover, the isomorphism takes any element m representing a meridian of $L(S^1_i)$ onto an element representing a meridian of $L'(S^1_i)$, whereas the element representing the corresponding longitude of $L'(S^1_i)$ is the image of the product of an element ξ of

$$N_k(m)=\big\langle [g^{-1},g^m] \mid g\in\langle m\rangle_k^{\mathcal{G}_k(L)}\big\rangle$$

and the element representing the corresponding longitude of $L(S^1_i)$.

Furthermore, $\mathcal{G}_k(L)$ is functorially invariant under k-quasi-isotopy of L with respect to the quotient map $\pi(L)\to\pi(L)/\mu_k=\mathcal{G}_k(L)$.

Proof. It suffices to consider the case where L' is obtained from L by a k-quasi-isotopy with a single self-intersection of the ith component. We can assume that there is a plane projection, where this self-intersection corresponds to replacement of an underpass by the overpass. Say, an arc $L(I_-)$ underpasses an arc $L(I_+)$, and the arc $L'(I_-)$ overpasses the arc $L'(I_+)$, where I_+ and I_- are disjoint arcs in S^1_i. Furthermore, we can assume that the k-quasi-isotopy has its support in $I_+\cup I_-$.

Suppose that $\partial I_- = \{p_1,p_2\}$ and $\partial I_+ = \{p_3,p_4\}$ so that the points p_1, p_2, p_3, p_4 lie in the cyclic order on S^1_i. For $j = 1, 2, 3, 4$ let m_j be the meridian, starting and ending in the basepoint (which we assume to lie above the plane) and underpassing $L'\big(S^1_i\big)$ once below the point $L'(p_j)$ in the direction inherited from the main underpass. The groups $\pi(L)$ and $\pi(L')$ can be presented by the same generators, including (the classes of) m_j's, and by the same relations, including $m_1m_2 = m_4m_3$, with one exception: the relation $m_1 = m_3$ in $\pi(L)$ is replaced with the relation $m_2 = m_4$ in $\pi(L')$ (cf. [45], [9, proof of Lemma 1 in Lecture 1], [36, p. 386]).

Let L_s denote the singular link arising in the k-quasi-isotopy between L and L', and let λ denote the longitude of the lobe $J_0=\ell$ of $L_s(S^1_i)$. Then $m_2 = m_3^\lambda$ both in $\pi(L')$ and in $\pi(L)$. By Lemma 3.1, $\lambda\in\langle m_2\rangle_k^{\pi(L)}$ and analogously for $\pi(L')$.

[k]It also follows from the proofs of Theorems 3.2 and 3.5 that the set Λ_i from the definition of the peripheral structure $\mathcal{P}_k$ in Sec. 1 coincides with the right coset $N_k(m_i)\bar{l}_i$ of the subgroup $N_k(m_i)$ (defined in the statement of Theorem 3.2) containing the class $\bar{l}_i$ of the longitude l_i corresponding to a representative m_i of $\bar{m}_i$.

Therefore the commutator $[m_2, m_3]$ lies in $\mu_k\pi(L)$, and consequently the relation $m_3m_2 = m_2m_3$ holds in the quotient $\mathcal{G}_k(L)$. Analogously, the same relation holds in $\mathcal{G}_k(L')$. Now, this latter relation and either two of the relations $m_1 = m_3$, $m_2 = m_4$, $m_1m_2 = m_4m_3$ imply the third one, which means that all of these relations do hold both in $\mathcal{G}_k(L)$ and in $\mathcal{G}_k(L')$. Thus these groups are isomorphic and the first assertion is proved. The assertion on the longitudes now follows from Lemma 3.3 below, since, substituting $g = \lambda^{m^{-1}}$ we obtain $\xi = [g^m, g^{-1}]$. $\square$

Lemma 3.3. [36, proof of Theorem 4 (note that there $[x, y]$ means $xyx^{-1}y^{-1}$)] *If L and L' are related by a homotopy with a single self-intersection of the ith component, then, in the above notation, $\xi = [\lambda, [\lambda^{-1}, m^{-1}]]$.*

Let us call a singular link with one double point satisfying the conclusion of Lemma 3.1 a *virtual k-quasi-embedding*. While all our major results on k-quasi-isotopy clearly hold for virtual k-quasi-isotopy (defined in the obvious way), coincidence of the two notions is unclear for $k \geq 2$.

Problem 3.4. *Does virtual k-quasi-isotopy imply k-quasi-isotopy, for all k?*

3.3. *Maximality and nilpotence of $G_k(L)$*

In view of Theorem 3.2, the following implies, in particular, that no difference between k-quasi-isotopy, virtual k-quasi-isotopy and strong k-quasi-isotopy can be seen from functorially invariant (in the sense of Sec. 1) quotients of $\pi(L)$.

Theorem 3.5. *If $\mathcal{F}(L)$ is a group, functorially invariant under strong k-quasi-isotopy with respect to an epimorphism $q_L : \pi(L) \to \mathcal{F}(L)$, then q_L factors through the quotient map $\pi(L) \to \pi(L)/\mu_k = \mathcal{G}_k(L)$ for every link L.*

Proof. In other words, we want to show that each relation imposed on $\pi(L)$ in order to obtain $\mathcal{G}_k(L)$ must also be satisfied in any quotient $\mathcal{F}(L)$, functorially invariant under strong k-quasi-isotopy.

We begin by noting that the requirement of functorial invariance can be restated as follows. In the notation of the definition of functorial invariance, let p be a path joining the double point of the singular link L_s to the basepoint, and R be a small regular neighborhood of p. We may assume that each $L_\pm$ is sufficiently close to L_s, so that $L_\pm(mS^1)$ meets R in a couple of arcs properly embedded in R. Now functorial invariance of $\mathcal{F}(L)$ is equivalent to the requirement that, starting with an arbitrary allowable singular link L_s and resolving it in the two ways, we have for the obtained links L_+, L_- that each of the compositions $\pi_1(R\backslash L_\pm(mS^1)) \xrightarrow{i_*} \pi(L_\pm) \to \mathcal{F}(L_\pm)$ is the abelianization $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ (see [36, Sec. 4]; compare [45, Fig. 2]).

Next, notice that there are two loops contained in R and representing meridians m, m^g of L_+ conjugated by the element g represented by the loop l starting at the basepoint, going along $\bar{p}$ (i.e. the path p in the reverse direction), then along either lobe q of the singular component of L_s (which we assume disjoint from L_+ and L_-), and finally back along p. Conversely, given any link L, any path p joining a point x near a component K of L to the basepoint, and any $g \in \pi(L)$, we can represent g as a loop $l = \bar{p}qp$ for some loop q starting at x, and isotop L by pushing a finger along q to obtain a link L_+, and then push further through a singular link L_s to obtain a link L_- as in the above situation.[1] Let $m = m(p)$ denote the meridian represented by the loop $\bar{p}op$ where o is a small loop around K starting at x.

Then it suffices to prove that for any link L and any path p as above, an arbitrary element $g \in \langle m\rangle_k^{\pi(L)}$, where $m = m(p)$, can be represented by a loop $l = \bar{p}qp$ such that the corresponding singular link L_s (obtained using L and q as above) is allowable with respect to k-quasi-isotopy. The latter will follow once we show that $l \cup o \subset B_1 \subset \cdots \subset B_k \subset S^3\backslash(L(mS^1)\backslash K)$ for some balls $B_1, \ldots, B_k$ such that $L(mS^1) \cap B_i$ is contained in an arc contained in B_{i+1} (indeed, then without loss of generality the whole finger will lie in B_1).

Since $g \in \langle m\rangle_k^{\pi(L)}$, we can write g as $g = m^{\iota_1 g_1} \cdots m^{\iota_n g_n}$ where each $g_i \in \langle m\rangle_{k-1}^{\pi(L)}$ and each $\iota_i = \pm 1$. Analogously, each $g_i = m^{\iota_{i1} g_{i1}} \cdots m^{\iota_{i,n_i} g_{i,n_i}}$ where $g_{ij} \in \langle m\rangle_{k-2}^{\pi(L)}$ and $\iota_{ij} = \pm 1$, and so on. We will refer to these as $g_\mathcal{I}$'s and $\iota_\mathcal{I}$'s where $\mathcal{I}$ runs over a set S_g of multi-indices of length $\leq k$, including the multi-index of length 0. Let $S_{g,i}$ be the subset of S_g consisting of all multi-indices of length i, let $\varphi: S_{g,i} \to S_{g,i-1}$ be the map forgetting the last index, and let $n_\mathcal{I} = |\varphi^{-1}(\mathcal{I})|$.

The path p possesses a small regular neighborhood $T \subset S^3\backslash(L(mS^1)\backslash K)$ containing the loop o and such that $T \cap L$ is an arc. Let us fix a PL homeomorphism $T \simeq [-1,1] \times I^2$, where $I = [0,1]$, such that $T \cap L = \frac{1}{2} \times \frac{1}{2} \times I$, and $[-1,0] \times I^2$ is a small regular neighborhood of the basepoint. Let us represent the conjugators $g_\mathcal{J}$, $\mathcal{J} \in S_{g,k}$, of the last stage by pairwise disjoint embedded paths $l_\mathcal{J}$ starting in $[-\frac{1}{k+1}, 0) \times I \times 1$, ending in $[-1, -\frac{1}{k+1}) \times I \times 1$ and disjoint from T elsewhere. Also let us represent m by $|S_g|$ distinct paths $r_\mathcal{I}$, $\mathcal{I} \in S_g$, setting $r_\mathcal{I} = ([0, \frac{3}{4}] \times \{\frac{1}{4}, \frac{3}{4}\} \cup \{\frac{3}{4}\} \times [\frac{1}{4}, \frac{3}{4}]) \times f(\mathcal{I})$ with orientation given by $\iota_\mathcal{I}$, where $f: S_g \to [0,1]$ is an injective map such that $f(S_{g,i}) \subset (\frac{i}{k+1}, \frac{i+1}{k+1}]$ for each i. Notice that the arc $r_\mathcal{I}$ is properly contained in the ball $D_\mathcal{I} = I^2 \times U$ where U is a closed neighborhood of $f(\mathcal{I})$ in $I\backslash f(S_g\backslash\{\mathcal{I}\})$.

Assume that for some $i \in \{1, \ldots, k\}$, each of the elements $g_\mathcal{I}$, $\mathcal{I} \in S_{g,i}$ is represented by an embedded path $l_\mathcal{I}$ with the initial endpoint in $[-1 + \frac{i}{k+1}, 0) \times I \times \{\frac{i+1}{k+1}\}$ and the terminal endpoint in $[-1, -1 + \frac{i}{k+1}) \times I \times \{\frac{i+1}{k+1}\}$, disjoint from $[-1,1] \times I \times [0, \frac{i+1}{k+1}]$ except in the endpoints, and disjoint from the other paths $l_\mathcal{J}$, $\mathcal{J} \in S_{g,i}$. Let us use this to construct for each $\mathcal{I} \in S_{g,i-1}$ an embedded path representing $g_\mathcal{I}$. We have $g_\mathcal{I} = m^{\iota_1 g_{\mathcal{I},1}} \cdots m^{\iota_{n_\mathcal{I}} g_{\mathcal{I},n_\mathcal{I}}}$, where each $\iota_i = \pm 1$, and each of the conjugators $g_\mathcal{J}$, $J \in \varphi^{-1}(I)$, is represented by an embedded path $l_\mathcal{J}$ by the assumption. Let us connect the initial endpoint of each $l_\mathcal{J}$ to the point $0 \times \frac{1}{2} \times f(\mathcal{J})$

[1]Notice that this construction depends on the choice of representative l of g; for instance, tying a local knot on the loop q is likely to change the isotopy class of L_-.

by a path $l^+_{\mathcal{J}}$ contained in $[-1+\frac{i}{k+1}, 0]\times I\times(\frac{i}{k+1}, \frac{i+1}{k+1}]$. Now let us consider two close push-offs $l'_{\mathcal{J}}$ and $l''_{\mathcal{J}}$ of the path $l_{\mathcal{J}}\cup l^+_{\mathcal{J}}$, contained in a small regular neighborhood $L_{\mathcal{J}}$ of $l_{\mathcal{J}}\cup l^+_{\mathcal{J}}$ in the exterior of $([-1, -1+\frac{i}{k+1}]\cup[0, 1])\times I\times[0, \frac{i+1}{k+1}]$. We may assume that the initial endpoints of $l'_{\mathcal{J}}$ and $l''_{\mathcal{J}}$ coincide with the two endpoints of $r_{\mathcal{J}}$. Finally, connect the terminal endpoint of each $l'_{\mathcal{I},i}$ to the terminal endpoint of each $l''_{\mathcal{I},i+1}$ by some path $c_{\mathcal{I},i}$ contained in $[-1, -1+\frac{i}{k+1})\times I\times(\frac{i}{k+1}, \frac{i+1}{k+1}]$, and connect the terminal endpoint of $l'_{\mathcal{I},1}$ (respectively $l''_{\mathcal{I},n_{\mathcal{I}}}$) to some point in $[-1+\frac{i-1}{k+1}, -1+\frac{i}{k+1})\times I\times\{\frac{i}{k+1}\}$ (respectively to some point in $[-1, -1+\frac{i-1}{k+1})\times I\times\{\frac{i}{k+1}\}$) by some path $c_{\mathcal{I},0}$ (respectively $c_{\mathcal{I},n_{\mathcal{I}}}$) contained in $[-1, -1+\frac{i}{k+1})\times I\times[\frac{i}{k+1}, \frac{i+1}{k+1}]$. Define

$$l_I = \bar{c}_{\mathcal{I},0}(\bar{l}'_{\mathcal{I},1}r_{\mathcal{I},1}l''_{\mathcal{I},1}c_{\mathcal{I},1})\cdots(\bar{l}'_{\mathcal{I},n_{\mathcal{I}}}r_{\mathcal{I},n_{\mathcal{I}}}l''_{\mathcal{I},n_{\mathcal{I}}}c_{\mathcal{I},n_{\mathcal{I}}}).$$

In particular, this defines the required path l, and so it remains to construct the balls $B_1, \dots, B_k$. Set

$$\begin{aligned} B_i = {} & [-1, 1]\times I\times\left[0, \frac{i}{k+1}\right] \\ & \cup\left[-1, -1+\frac{i}{k+1}\right]\times I\times\left[\frac{i}{k+1}, \frac{i+1}{k+1}\right]\cup\bigcup_{\mathcal{I}\in S_{g,i}}(L_{\mathcal{I}}\cup D_{\mathcal{I}}). \end{aligned}$$

It is easy to see that $l\subset B_1\subset\cdots\subset B_k\subset S^3\backslash(L(mS^1)\backslash K)$ and that $L(mS^1)\cap B_i$ is contained in an arc contained in B_{i+1} for each i. Also, without loss of generality, $o\subset B_1$. $\square$

Theorem 3.6. *For any k and L the group $\mathcal{G}_k(L)$ is nilpotent.*

Remark. In the case where $k = 1$ and $\pi(L)$ can be generated by two meridians, Theorem 3.6 is due to Mikhailov, who used a thorough analysis of the Hall basis to verify it in this case. Since the fundamental group of the Mazur link can be generated by two meridians, this was enough to imply that $\mathcal{G}_1(L)$, as defined in this section (Theorem 3.5 was not available at that time), is not a complete invariant of 1-quasi-isotopy [44]. However, since the fundamental group of the string link $W\#\rho W$ cannot be generated by two meridians, Mikhailov's result added little with respect to Problem 1.5. In fact, misled by similarity between $\mathcal{G}_k(L)$ and groups generated by their $(k+2)$-Engel elements (which will be discussed in "*n-Quasi-isotopy III*"), the first author and Mikhailov were trying to prove that $\mathcal{G}_1(W\#rW)$ was not nilpotent for a considerable time; cf. [44].

Proof. We recall that $\mathcal{G}_k(L) = \pi(L)/\mu_k$ where

$$\mu_k = \langle[m, m^g]\mid m\in M, g\in\langle m\rangle_k^{\pi(L)}\rangle.$$

Using the commutator identities $[a, bc] = [a, c][a, b]^c$ and $[a, b^{-1}] = [a, b]^{-b^{-1}}$ together with the fact that μ_k is normal in $\pi(L)$, we see that μ_k contains any commutator of type $[m, m^{\iota_1 g_1}\cdots m^{\iota_r g_r}]$ where $m\in M$, r is a positive integer, $\iota_i = \pm1$ and

$g_i \in \langle m\rangle_k^{\pi(L)}$. Hence μ_k is the subgroup generated by $\{[m,g] \mid m \in M, g \in \langle m\rangle_{k+1}^{\pi(L)}\}$. Analogously any commutator $[m^n, g]$, where $m \in M$, $n \in \mathbb{Z}$ and $g \in \langle m\rangle_{k+1}^{\pi(L)}$, can be expressed in the elements of μ_k, thus

$$\mu_k = \left\langle \bigcup_{m\in M} \left[\langle m\rangle, \langle m\rangle_{k+1}^{\pi(L)}\right] \right\rangle. \tag{3.1}$$

Then in the quotient over μ_k, each element $\bar{m}$ of the image $\bar{M}$ of M under the quotient map is contained in the center of the subgroup $\langle\bar{m}\rangle_{k+1}^{\mathcal{G}_k}$. In particular, the cyclic subgroup $\langle\bar{m}\rangle$ is normal in $\langle\bar{m}\rangle_{k+1}^{\mathcal{G}_k}$, and thus subnormal[m] (of defect $\le k+2$) in $\mathcal{G}_k$, namely, we have that $\langle\bar{m}\rangle \lhd \langle m\rangle_{k+1}^{\mathcal{G}_k} \lhd \langle m\rangle_k^{\mathcal{G}_k} \lhd \cdots \lhd \langle m\rangle_0^{\mathcal{G}_k} = \mathcal{G}_k$.

So our group $\mathcal{G}_k(L)$ is generated by a finite number of its subnormal cyclic subgroups, and it remains to apply the following group-theoretic fact. $\square$

Theorem (Baer). [4], [35, Theorem 1.6.2] *A group generated by finitely many of its subnormal finitely generated nilpotent subgroups is nilpotent.*

For convenience of the reader, we sketch a quick proof of the Baer theorem modulo the following textbook level result.

Theorem (Hirsch–Plotkin). [51] *Let H_1 and H_2 be normal locally nilpotent*[n] *subgroups of a group G. Then H_1H_2 is a normal locally nilpotent subgroup of G.*

Clearly, the union of ascending chain of locally nilpotent subgroups is locally nilpotent, so using the Zorn lemma (for our purpose, the countable case suffices) one obtains that in any group G there is a unique normal locally nilpotent subgroup (called the Hirsch–Plotkin radical of G) containing all normal locally nilpotent subgroups of G. Using an induction on the defect and applying, at each step, the Hirsch–Plotkin theorem (which immediately generalizes to the case of any (ordinal) number of subgroups), one proves (cf. [51]) that the Hirsch–Plotkin radical contains all the subnormal locally nilpotent subgroups. In particular, we obtain that any group generated by its subnormal locally nilpotent subgroups is locally nilpotent.

Remark. In connection with Milnor's original definition of $\mathcal{G}(L) = \mathcal{G}_0(L)$ [45], we notice that μ_k is the subgroup generated by the commutator subgroups of the subgroups $\langle m\rangle_{k+1}^{\pi(L)}$ where m runs over all meridians. Indeed, by the definition, $\langle m\rangle_{k+1}^{\pi(L)}$ is generated by the conjugates of m under the action of $\langle m\rangle_k^{\pi(L)}$, and if m^g and m^h are two of them, $[m^g, m^h] = [m, m^{hg^{-1}}]^g \in \mu_k$. Conversely, formula (3.1) implies that μ_k is contained in the subgroup generated by these commutators.

[m] A subgroup H of a group G is said to be *subnormal* in G if there exists a finite chain of subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_d = G$ such that each H_i is normal in H_{i+1}. The minimal such d is called the *defect* of H.

[n] A group is called *locally nilpotent* if all of its finitely generated subgroups are nilpotent. The same statement with "locally nilpotent" replaced by "nilpotent" (Fitting's theorem [51]) would not fit in our argument since a group generated by its two subnormal nilpotent subgroups may be non-nilpotent [35, p. 22], in contrast to the Baer theorem.

Corollary 3.7. *Every quotient of the fundamental group of a link* (*or string link*), *functorially invariant under strong k-quasi-isotopy for some k*, *is nilpotent.*

The case of links follows from Theorems 3.5 and 3.6; the proof in the case of string links is analogous.

Appendix A. A Proof that $\tilde{\beta} \equiv \bar{\mu}(1122)$ (mod lk)

We give a direct proof that the Milnor invariant $\bar{\mu}(1122) \in \mathbb{Z}/\mathrm{lk}$ satisfies the crossing change formula (2.2). It is easy to check that $\bar{\mu}(1122)$ vanishes on the links T_n from [31], hence coincides with the residue class of $\tilde{\beta}$ mod lk (which also follows from [12]).

We recall the definition of $\bar{\mu}$-invariants following Levine's approach [36]. Given a link $L' = L_1 \sqcup \cdots \sqcup L_n \subset S^3$, we assume inductively that the $\bar{\mu}$-invariants of $L = L_1 \sqcup \cdots \sqcup L_{n-1}$ are already defined. To define those of L', we only need to know the homotopy class of L_n in $\pi(L) = \pi_1(S^3 \backslash L)$ (up to conjugation). Furthermore, we proceed to the quotient $\mathcal{G}(L) = \pi(L)/\mu_0$ (see Sec. 3). A presentation for $\mathcal{G}(L)$ is (cf. [36])

$$\langle m_1, m_2, \ldots, m_{n-1} \mid [m_i, l_i] = 1, R_j = 1 \rangle,$$

where $m_1, \ldots, m_{n-1}$ is a collection of meridians, one to each component, $l_1, \ldots, l_{n-1}$ are the corresponding longitudes (regarded as words in m_i's), and R_j, $j = 1, 2, \ldots$ are all commutators in m_i's with repeats, i.e. with some m_i occurring at least twice.

Now any element $\alpha \in \mathcal{G}(T)$, where T is the trivial link and therefore $\pi(T)$ is the free group F_{n-1} with free generators $m_1, \ldots, m_{n-1}$, has unique decomposition into product of powers of *basic commutators without repeats* (see [36] for definition and proof; for a general treatment of basic commutators see [20]), which is finite since $\mathcal{G}(L)$ is nilpotent by [45]. For our case $n = 4$ the decomposition looks like

$$\alpha = x^{e_1} y^{e_2} z^{e_3} [y, x]^{e_4} [z, y]^{e_5} [z, x]^{e_6} [[y, x], z]^{e_7} [[z, x], y]^{e_8}, \tag{A.1}$$

where m_1, m_2, m_3 are substituted for convenience by x, y, z respectively. The integers e_i depend on the choice of basis in F_{n-1}, and hence should be considered only up to certain transformations (all of them are generated by conjugations $m_i \mapsto (m_i)^g$, see [36] for details). Next, a choice of meridians yields a homomorphism $\pi(T) = F_{n-1} \to \pi(L)$ which descends to $\varphi : \mathcal{G}(T) \to \mathcal{G}(L)$, so for arbitrary L and $\alpha \in \mathcal{G}(L)$ we are led to the representation (A.1). However, in general l_i are not trivial and so the relations $[m_i, l_i] = 1$ arising from the peripheral tori are not vacuous, hence the representation (A.1) is not unique with respect to a fixed basis of m_i. The commutator numbers $e_i = (e_i)_j$ for $\alpha = \alpha_j = [m_j, l_j]$ give the precise indeterminacy in this case. Finally, if we set α to be the class of L_n in $\mathcal{G}(L)$, the obtained commutator numbers e_i up to the prescribed indeterminacy are, together with the invariants of L, the complete set of homotopy invariants of L' [36], including all Milnor's invariants with their original indeterminacy. In particular, in the formula (A.1) above, e_1, e_2, e_3 are the linking numbers of L_4 with L_1, L_2, L_3 respectively, the residue classes of e_4, e_5, e_6 are the triple $\bar{\mu}$-invariants corresponding to (L_i, L_j, L_4), where (i, j) run over the two-element subsets of $1, 2, 3$, and the residue classes of e_7, e_8 are the two independent quadruple invariants $\bar{\mu}(1234)$, $\bar{\mu}(1324)$, respectively.

Now let $\bar{L}_\pm = K_1^\pm \sqcup K_3^\pm$ be two-component links of linking number l, related by a single positive crossing change so that the formula (2.2) applies. Let $K_2^\pm$ and $K_4^\pm$ be zero push-offs of $K_1^\pm$ and $K_3^\pm$, respectively, and let us consider the links $L_\pm := \bar{L}_\pm \sqcup K_2^\pm$ and $L'_\pm := L_\pm \sqcup K_4^\pm$.

Let us see the difference between the commutator numbers for the elements $\alpha_- = [K_4^-] \in \mathcal{G}(L_-)$ and $\alpha_+ = [K_4^+] \in \mathcal{G}(L_+)$. Since L_- (Fig. 5(a)) and L_+ (Fig. 5(b)) are link homotopic (by the obvious homotopy with a single self-intersection on $K_3^\pm$), the two groups $\mathcal{G}(L_-)$, $\mathcal{G}(L_+)$ are isomorphic by an isomorphism h taking meridians onto meridians and longitudes algebraically somewhat close to longitudes [45, Theorem 2], [36, Theorem 4] or Theorem 3.2. In our case the only longitude one has to worry about is α_-, and its behavior can be determined. Let τ_1, $\tau_2 \in \mathcal{G}(L_+)$ denote the classes of the two lobes' longitudes, joined to the basepoint as shown in Fig. 5(b), so that the linking number of each τ_i with K_3^+ is zero. By Lemma 3.3, $h(\alpha_-) = \xi\alpha_+$, where $\xi = [\tau_1, [\tau_1^{-1}, z^{-1}]]$. We will systematically use the fact that the fourth central term $\gamma_4\mathcal{G}(L_+) = 1$ since L_+ has three components [45] allowing us to use simplified commutator identities, e.g. it follows that $\xi = [[z, \tau_1], \tau_1]$. In fact, as we noted above, ξ is not well defined as a word in meridians, and can be rewritten using the relations $[m_i, l_i] = 1$ arbitrarily. In particular, since $\alpha_+ = z\tau_1 z^{-1}\tau_2$, the relation $[z^{-1}, \alpha_+] = 1$, or, equivalently, $[\tau_2^{-1}, z] = [z^{-1}, \tau_1]$, implies that ξ can be rewritten e.g. as $\xi' = [[\tau_2, z], \tau_1]$.

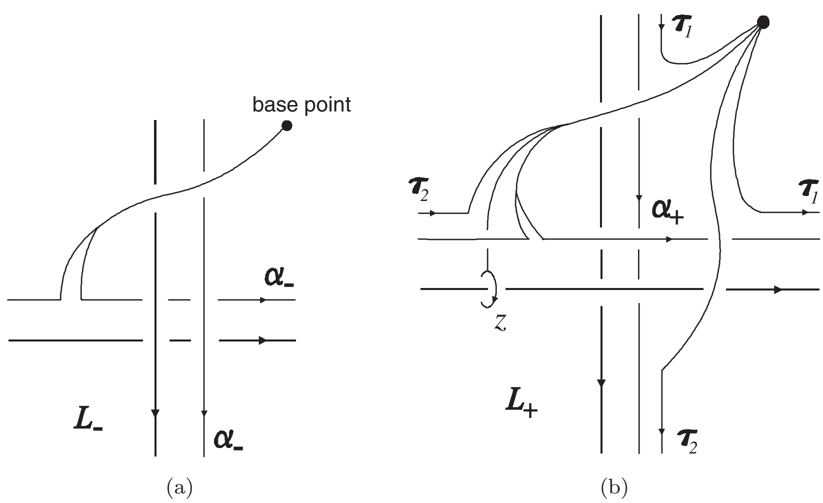

Fig. 5.

Now suppose that the linking numbers of the lobes with the first component of $\bar{L}_+$ are n and $l-n$. Then the representations (A.1) of τ_i's are as follows:

$$\tau_1 = x^n y^n \cdots; \quad \tau_2 = x^{l-n} y^{l-n} \cdots$$

The higher terms are omitted since they will not affect the result, as all commutators with repeats or of weight four are trivial. Making the substitutions for τ_i's, we obtain (using the Hall–Witt identity $[[x,y],z][[y,z],x][[z,x],y] = 1 \bmod \gamma_4$)

$$\begin{aligned}\xi' &= [[x^{l-n}y^{l-n}, z], x^n y^n] \\ &= [[y,z],x]^{n(l-n)}[[x,z],y]^{n(l-n)} = [[y,x],z]^{n(l-n)}[[z,x],y]]^{-2n(l-n)}\end{aligned}$$

which is not surprising, since $\bar{\mu}(1212) = -2\bar{\mu}(1122)$ [45], and proves the crossing change formula (2.2) for $\bar{\mu}(1122)$. The analogous calculation for ξ:

$$\xi = [[z, x^n y^n], x^n y^n] = [[y,x],z]^{-n^2}[[z,x],y]^{2n^2}$$

shows that the integer e_7 is not well-defined even for the two-component link.

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