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BRIEF COMMUNICATIONS

The Jaworowski Method in the Problem of the Preservation of Extensor Properties by the Orbit Functor

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Dedicated to Professor Yu. M. Smirnov
on the occasion of his eightieth birthday

1. INTRODUCTION

Recent progress in the study of the topological structure of the Banach–Mazur compact space $Q(n)$ [1, 2] is largely based on the following result concerning the preservation of extensor properties by the orbit functor.

Theorem 1.1. *If G is a compact Lie group and $\mathbb{X}$ is a metric* G-A[N]E *space, then the orbit space of X is an* A[N]E *space.*

The application of this theorem to the groups $O(n)$ and the space $L(n)$ of convex bodies, whose unit balls are the outer Löwner ellipsoids, gives, in particular, an answer to an old question about the extensor properties of $Q(n)$. The initial proof of Theorem 1.1 was fairly cumbersome [3]. Later, a shorter proof based on the Curtis theorem and applicable to arbitrary compact groups was obtained [4] (see also [5], where locally compact Lie groups are considered). However, the problem of the preservation of equivariant extensor properties by the orbit functor was tackled much earlier by Jaworowski, who studied the symmetric product functor SP^n_G [6]. He proved the following theorem.

Theorem 1.2. *If X is a compact metric* ANE *space and G is the group of permutations of n elements, then the symmetric product $\mathrm{SP}^n_G X$ associated to G is an* A[N]E.

In the language of the equivariant theory of extensors, this result can be written as follows:

$$X \in \mathrm{ANE} \implies X^n \in \mathrm{G\text{-}ANE} \implies \mathrm{SP}^n_G X = X^n/G \in \mathrm{ANE}.$$

Thereby Jaworowski proved Theorem 1.1 for any finite subgroup $G < S_n$ and any compact metric G-ANE space X^n. Jaworowski's paper is concerned with a special topic and contains a gap [7, 8] (which can easily be fixed, though); however, the potential of the approach is very high. Thus, from the standpoint of the modern theory of equivariant extensors, there is no difficulty in making Jaworowski's argument rigorous enough and obtaining a very elegant proof of Theorem 1.1. It is easy to see from the proof given below that Theorem 1.1 extends to an arbitrary class of spaces for which the Whitehead–Borsuk–Hanner theorem is valid. We used to believe that such classes

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included the class of stratifiable spaces [9], but Professor R. Cauty kindly informed us that his paper [10] had a gap, and the validity of its main result was an open question.

2. PRELIMINARIES

Below, we give the basic notions of the theory of G-spaces [11, 12]. By an action of a compact group G on a space $\mathbb{X}$, we understand a continuous map μ from the product $G \times \mathbb{X}$ to $\mathbb{X}$ with the following properties:

(1) $\mu(g, \mu(h, x)) = \mu(g \cdot h, x)$;
(2) $\mu(e, x) = x$ for all $x \in \mathbb{X}$ and $g, h \in G$ (here e is the identity of the group G).

Instead of $\mu(g, x)$, we usually write $g \cdot x$ or simply gx. A space $\mathbb{X}$ with an action of a group G is said to be a *G-space*. A map $f\colon \mathbb{X} \to \mathbb{Y}$ of two G-spaces is called a *G-map*, or an *equivariant map*, if $f(g \cdot x) = g \cdot f(x)$ for all $x \in \mathbb{X}$ and $g \in G$.

The *orbit* $G(x)$ of a point $x \in \mathbb{X}$ is the subset $\{g \cdot x \mid g \in G\} = G \cdot x$; it is always closed. The natural map $p\colon \mathbb{X} \to X$, defined by $x \mapsto G(x)$, of the space $\mathbb{X}$ to the quotient space $X \rightleftharpoons \mathbb{X}/G$ is said to be the *orbit projection*. The quotient space X endowed with the quotient topology generated by p is called the *orbit space*.

For any subset $A \subset \mathbb{X}$, its *saturation* by orbits is $p^{-1}p(A) = G \cdot A$. If the saturation coincides with the initial subset A, then A is said to be *invariant*, or a *G-subset*.

For each point $x \in \mathbb{X}$, the subset $G_x = \{g \in G \mid g \cdot x = x\}$ is a closed subgroup in the group G; it is called the *stabilizer of the point* x. For a closed subgroup $H < G$, the set $\mathbb{X}^H = \{x \in \mathbb{X} \mid H \cdot x = x\}$ is called the *set of H-fixed points*. Recall the definition of the *cross product*: if $\mathbb{Y}$ is an H-space and $H < G$, then $G \times_H \mathbb{Y}$ is the orbit space of $G \times \mathbb{Y}$ with action $h \cdot (g, y) \rightleftharpoons (g \cdot h^{-1}, h \cdot y)$; i.e., an arbitrary element of the cross product can be represented in the form

$$H \cdot (g, y) = \{(g \cdot h^{-1}, h \cdot y) \mid h \in H\}.$$

Note that

(3) $G \times_H \mathbb{Y}$ is a G-space with action $g_1 \cdot (H \cdot (g, y)) = H \cdot (g_1 \cdot g, y)$.

Let us introduce a number of notions related to the extension of G-maps. A space $\mathbb{X}$ with an action of a compact group G is an *equivariant absolute neighborhood extensor for metric spaces* (symbolically, $\mathbb{X} \in$ G-ANE) if any G-map $\varphi\colon \mathbb{A} \to \mathbb{X}$ defined on a closed G-subset $\mathbb{A} \subset \mathbb{Z}$ of a metric G-space $\mathbb{Z}$ (such a map is called a *partial G-map*) can be G-continued over some G-neighborhood $\mathbb{U} \subset \mathbb{Z}$ of the set $\mathbb{A}$, i.e., there exists a $\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X}$ such that $\widehat{\varphi}\restriction_{\mathbb{A}} = \varphi$. If it is always possible to G-extend φ over $\mathbb{U} = \mathbb{Z}$, then $\mathbb{X}$ is called an *equivariant absolute extensor* (symbolically, $\mathbb{X} \in$ G-AE). If the action of the group G is trivial (i.e., the spaces are considered without actions), then the notion introduced above coincides with the notion of absolute (neighborhood) extensors (A[N]Es) for metric spaces. Our further consideration essentially uses the following two facts.

Theorem 2.1 (Whitehead–Borsuk–Hanner [13, 14]). *If*

(4) $Y \in$ A[N]E *is a closed subset of a metric space* X;
(5) $X \setminus Y \in$ A[N]E;
(6) Y *is a strong deformation* (*strong deformation neighborhood*) *retract of* X (*i.e., there exists a homotopy* $H_t\colon U \to X$ *of a space* $U \rightleftharpoons X$ (*of some neighborhood* U *of* Y) *such that* $H_0 = \mathrm{Id}$, $H_t \restriction_Y = \mathrm{Id}$, *and* H_1 *is a retraction of* U *into* Y),

then $X \in$ A[N]E.

The application of the inductive argument in the proof of Theorem 1.1 is based on the slice theorem.

Definition 2.2. We say that a G-subset $\mathbb{A} \subset \mathbb{X}$ is K-*tubular*, where $K < G$ is a closed subgroup, if there exists a K-invariant subset $\mathbb{S} \subset \mathbb{A}$ such that

(7) $G \cdot \mathbb{S} = \mathbb{A}$;
(8) the natural map $\theta\colon G \times_K \mathbb{S} \to \mathbb{A}$ defined by $\theta(K \cdot (g, s)) = g \cdot s$ is a G-homeomorphism.

A K-invariant subset $\mathbb{S} \subset \mathbb{A}$ is said to be a K-*slice* of the set $\mathbb{A}$.

Note that a K-slice $\mathbb{S}$ of a tubular set $\mathbb{A} = G \times_K \mathbb{S}$ satisfies the following conditions:

(9) $\mathbb{S}$ is a K-invariant closed subset of $\mathbb{A}$;
(10) $g \cdot \mathbb{S} \cap \mathbb{S} \neq \varnothing \iff g \in K$.

The following assertion is easy to verify.

Proposition 2.3. *Any subset* $\mathbb{S} \subset \mathbb{A}$ *satisfying conditions* (8), (9), *and* (10) *is a* K-*slice of the* K-*tubular set* $\mathbb{A}$, *and* $\mathbb{A} = G \times_K \mathbb{S}$.

Theorem 2.4 (on slices of G-spaces [11]). *If a compact Lie group* G *acts on a completely regular* G-*space* $\mathbb{X}$, *then each point* $x \in \mathbb{X}$ *has a* G_x-*tubular neighborhood* $\mathbb{U} = G \times_{G_x} \mathbb{S}$.

The slice theorem implies that any orbit in $\mathbb{X}$ is an equivariant neighborhood extensor for the class of completely regular G-spaces.

It is easy to show that the dimension $\operatorname{ind} G \rightleftharpoons \dim G + \mathcal{K}_G$, where $\mathcal{K}_G$ is the number of connected components in the group G, satisfies the following conditions:

(11) $\operatorname{ind} G \geq 2$; $\operatorname{ind} G = 2$ if and only if G is the trivial group;
(12) if $H \lneq G$, then $\operatorname{ind} H < \operatorname{ind} G$.

For $x \notin \mathbb{X}^G$, the subgroup G_x is proper, and hence $\operatorname{ind} G_x < \operatorname{ind} G$; therefore, in many situations, the slice theorem makes it possible to argue by induction and reduce studying the equivariant extensor properties of $\mathbb{X} \setminus \mathbb{X}^G$ to studying the equivariant extensor properties of spaces with actions of groups G_x having smaller indices.

In [15], the relation between the equivariant extensor properties of tubular neighborhoods and those of their slices is established.

Theorem 2.5. *If a compact Lie group* G *acts on a completely regular* G-*space* $\mathbb{X}$ *and a tubular neighborhood* $\mathbb{U} = G \times_{G_x} \mathbb{S} \subset \mathbb{X}$ *is a* G-ANE, *then the slice* $\mathbb{S}$ *at the point* x *is a* G_x-ANE.

This theorem is a corollary of the following two assertions, which are also proved in [15, p. 500].

Theorem 2.6. *If* $X \in$ G-A[N]E, *then* $X \in$ H-A[N]E *for any subgroup* $H < G$.

Theorem 2.7. *If* H *is a subgroup of* G *and* $G \times_H \mathbb{S} \in$ G-A[N]E, *then* $\mathbb{S} \in$ H-A[N]E.

3. PROOF OF THE THEOREM

We shall consider in detail only the implication $\mathbb{X} \in$ G-ANE $\Longrightarrow X \in$ ANE, believing that this will make the proof of the global assertion obvious to the reader.

If the group G is trivial, then, clearly, $X \in$ ANE. Arguing by induction, we assume that the assertion is valid for all groups with $\operatorname{ind} G < n$ and prove it for G with $\operatorname{ind} G = n$. First, note that the set $\mathbb{X}^G$ of G-fixed points is a strong deformation neighborhood G-retract of $\mathbb{X}$; therefore, it is a G-ANE. Hence

(1) $(X^G)/G = X^G$ is a strong deformation neighborhood retract of $X = \mathbb{X}/G$;
(2) $X^G \in$ ANE.

Now, if we prove that

(3) $Y = X \setminus X^G \in$ ANE,

then the Whitehead–Borsuk–Hanner theorem will imply the required inclusion $X \in \mathrm{ANE}$.

To prove (3), it suffices to show that

(4) any point $y = p(x) \in Y$, where $x \notin \mathbb{X}^G$, has a neighborhood $O(y) \in \mathrm{ANE}$.

By Theorem 2.4, the point $x \in \mathbb{X}$ has a G_x-tubular neighborhood $\mathbb{U} = G \times_{G_x} \mathbb{S}$. Being an invariant neighborhood of a G-ANE-space, $\mathbb{U}$ belongs to the class G-ANE. By Theorem 2.6 $\mathbb{U}(x) \in \mathrm{G}_x$-ANE. Since G_x is a proper subgroup of G, we have $\mathbb{U}(x)/G_x \in \mathrm{G}_x$-ANE by the induction hypothesis ($\mathrm{ind}(G_x) < n$).

But the orbit space $U(x) \rightleftharpoons \mathbb{U}(x)/G$ is a retract of $\mathbb{U}(x)/G_x \in \mathrm{ANE}$. To see this, consider the map $h\colon \mathbb{S}/G_x \to U(x)$ defined by $h(G_x(z)) = G(z)$ for $z \in \mathbb{S}$, which is obviously a homeomorphism. Since $\mathbb{S}$ is closed in $\mathbb{U}(x)$, $\mathbb{S}/G_x$ is closed in $\mathbb{U}(x)/G_x$. A direct verification shows that the map $H\colon \mathbb{U}(x)/G_x \to U(x)$ defined by $H(G_x \cdot z) = G(z)$ for $z \in \mathbb{U}(x)$ is a continuous extension of the homeomorphism h, i.e., $H \restriction_{\mathbb{S}/G_x} = h$. Hence $h^{-1} \circ H$ is a retraction of $\mathbb{U}/G_x$ onto the space $\mathbb{S}/G_x$, which is therefore an ANE. Consequently, $U(x) \cong \mathbb{S}/G_x$ is also an ANE.

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