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ON INCOMPLETENESS OF THE DELETED PRODUCT OBSTRUCTION FOR EMBEDDABILITY

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ABSTRACT. Let $\tilde{N} = N \times N \setminus (\Delta N)$, where ΔN denotes the diagonal. The purpose of this paper is to construct counterexamples to the deleted product criterion for embeddability into $\mathbb{R}^m$ for certain dimensions. Two counterexamples are constructed: (1) an example of a 3–dimensional manifold N with boundary which is not embeddable in $\mathbb{R}^3$, but for which there exists an equivariant mapping $\varphi\colon \Sigma\tilde{N} \to \Sigma S^2$ and (2) an example of a closed smooth $4k$–dimensional manifold N which does not smoothly embed into $\mathbb{R}^{6k-1}$, but for which there exists an equivariant mapping $\tilde{N} \to S^{6k-2}$.

1. Introduction

Recall that *the deleted product* of a space N is $\tilde{N} = N \times N \setminus (\Delta N)$, where ΔN is the diagonal. If $f\colon N \hookrightarrow \mathbb{R}^m$ is an embedding, then define the mapping $\tilde{f}\colon \tilde{N} \to S^{m-1}$ by the formula

$$\tilde{f}(x,y) = \frac{f(x)-f(y)}{||f(x)-f(y)||} \tag{1.1}$$

The mapping $\tilde{f}$ is equivariant with respect to the action of $\mathbb{Z}_2$:

- on $\tilde{N}$, acting as the symmetry $(x,y) \to (y,x)$, and
- on the sphere S^{m-1}, taking a point into its antipode.

Consider the following assertion (for PL or DIFF categories):

(*) *For a smooth n-manifold or an n-polyhedron N, if there exists an equivariant map $\tilde{N} \to S^{m-1}$, then N piecewise-linearly or smoothly embeds into $\mathbb{R}^m$.*

The existence of an equivariant map $\tilde{N} \to S^{m-1}$ can be checked for many cases ([2, beginning of §2], [4], [5, 1.7.1], [1, 7.1], [17]). Thus if the assertion (*) is true, the embedding problem is reduced to manageable (although not trivial) algebraic problems. Therefore a problem appeared in the 1960's to find conditions under which the asserion (*) above is true.

The assertion (*) is true for:

- $m = 2, n = 1$ (see [9], [24]);
- $m \geq \frac{3(n+1)}{2}$ (see [4], [23]); and
- $m \geq n+3$, a PL $(3n-2m+2)$-connected closed n-manifold N and in the PL category (see [21], [22]).

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However, in general, the assertion (*) above is false when the dimension m is too low. More precisely, it is known to be false in the following cases:

- in the PL case for $\max(4,n) \le m < \frac{3(n+1)}{2}$ (see [7], [3], [12], [19], [20]);
- in the smooth case for $m = n+3 \in \{11, 12, 13, 19\}$ and certain homotopy n-sphere N (see [6], [10]).

The only open cases for the PL version of the assertion (*) are $m = 3$ and $n = 2, 3$. The counterexamples to the assertion (*) for $m = n \ge 4$ and $m = n+1 \ge 4$ ([12], [7]) cannot be directly extended to $m = 3$ because they use the m-dimensional Mazur contractible manifolds, whose analogues do not exist for $m = 3$.

The first main result of this paper gives a weaker counterexample for $m = n = 3$ (up to a desuspension):

THEOREM (1.2). *There exists a manifold N^3 with boundary (a certain punctured homology 3-sphere) such that N^3 is not embeddable into $\mathbb{R}^3$, but there exists an equivariant mapping*

$$\varphi\colon \Sigma\tilde{N}^3 \to \Sigma S^2.$$

Theorem (1.2) also holds if one replaces N^3 by its special spine P^2. Note that it would be easier to construct a desuspension of a map $\Sigma\tilde{P}^2 \to S^3$ than that of $\Sigma\tilde{N}^3 \to S^3$.

Our second main result shows that *the smooth case of the assertion (*) above is false for $m < 3n/2$:*

THEOREM (1.3). *For each $k > 1$ there exists a closed smooth $4k$-manifold M^{4k} which PL embeds into $\mathbb{R}^{4k+2}$ (hence there exists an equivariant map $\tilde{M}^{4k} \to S^{4k+1} \subset S^{6k-2}$), but does not smoothly embed into $\mathbb{R}^{6k-1}$.*

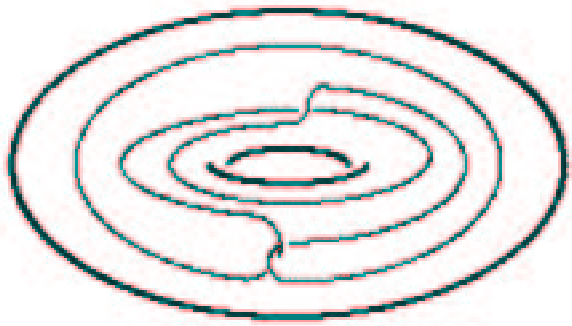
Theorem (1.3) is also true for $k = 1$, but this case is not interesting because $4\cdot 1 + 2 > 6 \cdot 1 - 1$.

2. Proof of Theorem (1.2)

Proof of Theorem (1.2) is based on the following classical result [13]:

LEMMA (2.1). (Mazur) *There exists a contractible 4-manifold M, embeddable into $\mathbb{R}^4$ and such that ∂M^4 is a nontrivial homology 3–sphere.*

Construction of M. The manifold M is obtained by gluing a 4–cell I^4 to a solid 4–torus $S^1 \times I^3$ along a solid 3–torus $\partial I^2 \times I^2$, identified with a tubular neighborhood of $S^1 \times 0$ in the boundary $\partial(S^1 \times I^3) = S^1 \times S^2$ of a knotted circle depicted in Figure 1. A meridian of $\partial I^2 \times I^2$ is identified with *any* longitude of the knot in $S^2 \times S^1$. □



Proof of Theorem (1.2). Take the manifold M^{4k} given by Lemma (2.1). Let B^3 be a 3–cell lying in ∂M^4. Define the manifold N^3 with boundary by $N^3 = \partial M^4 \setminus B^3$. The manifold N^3 cannot be embedded into S^3 since the embeddability of N^3 into S^3 would imply that $S^3 \setminus \int N^3 = B^3$.

In order to construct the required equivariant map φ, we begin by an embedding

$$f\colon N^3 \hookrightarrow \partial M^4 \subset M^4 \subset \mathbb{R}^4.$$

Since M^4 is contractible, there exists a homotopy $f_t\colon N^3 \to M^4$, $t \in I$, such that $f_0 = f$ and f_1 is a constant mapping. Using a collar, the homotopy $f_t\colon N^3 \to M^4$ can be modified in such a way that $f(x) \neq f_t(y)$, for all $x \neq y \in N^3$, $t \in I$.

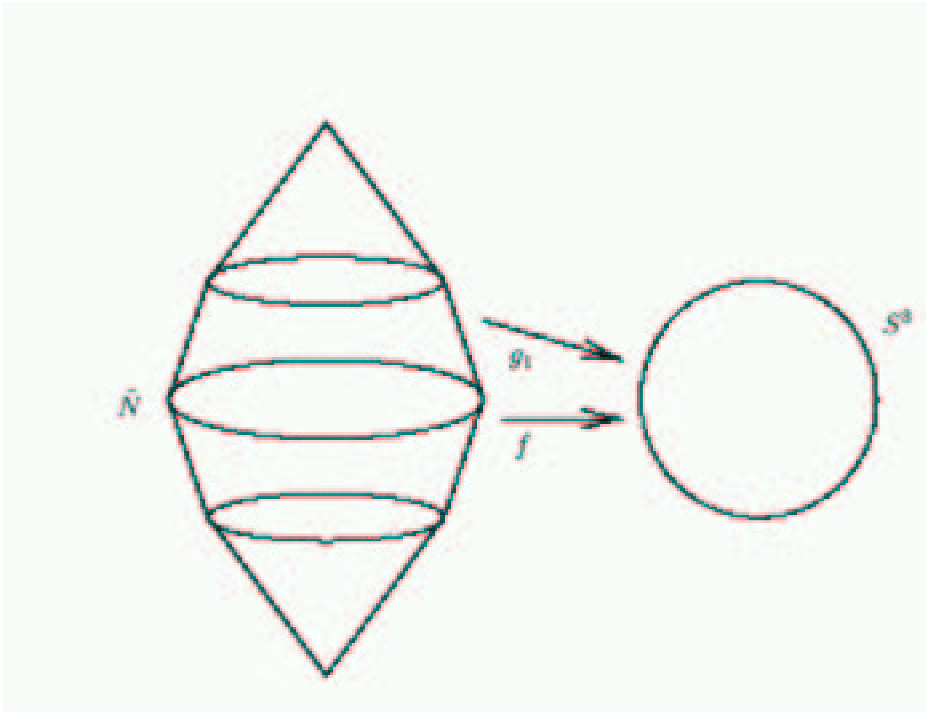
Define a homotopy $g_t\colon \tilde{N}^3 \to S^3$ by the formula:

$$g_t(x,y) = \frac{f(x) - f_t(y)}{||f(x) - f_t(y)||}, \quad x \neq y \in N^3, \quad t \in I.$$

Then the mapping $g_0\colon \tilde{N}^3 \to S^3$ coincides with the equivariant mapping $\tilde{f}\colon \tilde{N}^3 \to S^3$, defined in the formula (1.1). The mapping $g_1\colon \tilde{N}^3 \to S^3$ is given by the formula

$$g_1(x,y) = \frac{f(x) - \mathrm{pt}}{||f(x) - \mathrm{pt}||}, \quad x, y \in N^3$$

hence it depends only to the variable x. Since N^3 is a 3-manifold with boundary, it collapses to a 2-dimensional polyhedron. Therefore g_1 is nullhomotopic. By composing both homotopies, we obtain a homotopy between $\tilde{f}$ and a constant mapping, which enables us to construct the required equivariant mapping $\varphi\colon \Sigma\tilde{N}^3 \to S^3 = \Sigma S^2$ (cf. Figure 2). □



3. Proof of Theorem (1.3)

Proof of Theorem (1.3) is based on the following classical result – for completeness we have included its proof, because the paper [14] is not easily accessible (cf. [11, Corollary 1.17], [8], [14, Lemma 7.4]):

LEMMA (3.1). (Kervaire-Milnor) *For each k there exists a closed smooth almost parallelizable $4k$-manifold M^{4k} with $p_k(M^{4k}, \mathbb{R}) \neq 0$.*

Proof. Consider the J-homomorphism $J\colon \pi_{4k-1}(SO) \to \pi^S_{4k-1}$. Take a non-zero element $\alpha \in \ker J$ and an integer $N > 4k$. Take a framing of the normal bundle of the standard sphere $S^{4k-1} \subset S^{N+4k-1}$, corresonding to α.

By Pontryagin construction, the standard sphere with this framing represents an element in π^S_{4k-1}. Clearly, this element is $J\alpha$. Since $J\alpha = 0$, by Pontryagin construction the framed submanifold $S^{4k-1} \subset S^{N+4k-1}$ is the boundary of a framed manifold $M_0^{4k} \subset D^{N+4k-1}$.

Let M^{4k} be the union of M_0^{4k} and D^{4k} along their boundaries. Since M_0^{4k} has a trivial stable normal bundle, it follows that M^{4k} is almost parallelizable. By [14, Lemmata 1 and 2] (see also [11, §1]), $p_k(M^{4k}, \mathbb{R})$ equals to a multiple of α and hence is non-zero. □

Proof of Theorem (1.3). Take the manifold M^{4k} given by Lemma (3.1). We can modify it by surgery (cf. [11]) to obtain a $(2k-1)$-connected manifold with the same properties. This new manifold will be denoted by the same letter M^{4k}.

By the duality theorem for the real Pontryagin classes ([24], cf. also [15]), $\bar{p}_k(M^{4k}, \mathbb{R}) \neq 0$. Hence M^{4k} does not smoothly embed (it even does not immerse) into $\mathbb{R}^{6k-1}$ ([16], [24]).

Let M_0^{4k} be the complement in M^{4k} of an open $4k$-ball. Then M_0^{4k} is parallelizable and hence there is an immersion $f\colon M_0^{4k} \to \mathbb{R}^{4k+1}$. Since $k > 1$ and M_0^{4k} is $(2k-1)$-connected, it follows that it possesses a $2k$–dimensional spine $P^{2k} \subset M_0^{4k}$.

By general position the restriction of f to this spine is an embedding. Hence the restriction of f onto a neighborhood of this spine P^{2k} is an embedding. This neighborhood is homeomorphic to M_0^{4k}. So there is an embedding $g\colon M_0^{4k} \to \mathbb{R}^{4k+1}$. By extending the embedding $g|_{\partial M_0^{4k}}$ as a cone in $\mathbb{R}^{4k+2}$, we obtain a PL embedding of M^{4k} into $\mathbb{R}^{4k+2}$. ☐

Note that the smooth embedding $g|_{\partial M_0^{4k}}\colon S^{4k-1} \to \mathbb{R}^{4k+1}$ is non-trivial as immersion, even after composing with the inclusion $\mathbb{R}^{4k+1} \subset \mathbb{R}^{6k-1}$ (cf. [5]).

Note that embeddability of M^{4k} into $\mathbb{R}^{4k+1}$ also follows from [18, Corollary A3]. In the proof of this corollary one should apply the cone construction as above, rather than the Penrose-Whitehead-Zeeman trick (as was suggested in [18]), because the Penrose-Whitehead-Zeeman trick works only in codimension ≥ 3.

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