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J. Math. Anal. Appl. 285 (2003) 62–73

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On controlled extensions of functions

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Received 19 November 2002

Submitted by B.S. Thomson

Abstract

For any numerical function $E:\mathbb{R}^2 \to \mathbb{R}$ we give sufficient conditions for resolving the controlled extension problem for a closed subset A of a normal space X. Namely, if the functions $f:A \to \mathbb{R}$, $g:A \to \mathbb{R}$ and $h:X \to \mathbb{R}$ satisfy the equality $E(f(a), g(a)) = h(a)$, for every $a \in A$, then we are interested to find the extensions $\hat{f}$ and $\hat{g}$ of f and g, respectively, such that $E(\hat{f}(x), \hat{g}(x)) = h(x)$, for every $x \in X$. We generalize earlier results concerning $E(u, v) = u \cdot v$ by using the techniques of selections of paraconvex-valued LSC mappings and soft single-valued mappings.

Keywords: Multivalued mapping; Continuous selection; Controlled extension; Normal space; Soft mapping

0. Introduction

For a nonnegative continuous function $h:X \to \mathbb{R}$ on a normal space X and for any two nonnegative continuous functions $f:A \to \mathbb{R}$ and $g:A \to \mathbb{R}$ on a closed subset $A \subset X$ such that $f(a) \cdot g(a) = h(a)$, for every $a \in A$, Shchepin [12] proved the existence of their nonnegative continuous extensions over X, $\hat{f}$ and $\hat{g}$ say, with $\hat{f}(x) \cdot \hat{g}(x) = h(x)$, for every $x \in X$.

Frantz [4] proved the following extension theorem for functions of nonconstant sign:

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0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00292-0

Theorem 0.1. *For any closed subset A of a compact metric space X and any continuous functions $f : A \to \mathbb{R}$, $g : A \to [0, \infty)$ and $h : X \to \mathbb{R}$ such that $g^{-1}(0) \subset f^{-1}(0)$ and $f \cdot g = h|A$, there exist continuous extensions $\hat{f} : X \to \mathbb{R}$ and $\hat{g} : X \to [0, \infty)$ of f and g such that $\hat{f} \cdot \hat{g} = h$.*

See also [4] for examples showing the essentiality of the hypotheses $g \geqslant 0$ and $g^{-1}(0) \subset f^{-1}(0)$. One can easily find such examples on the unit circle. Barov and Dijkstra [1] have generalized Theorem 0.1 to arbitrary normal domains, giving a short proof via a direct analytical expression for the desired extensions.

Having in mind these results we introduce the following definition.

Definition 0.2. Let $E : \mathbb{R}^2 \to \mathbb{R}$ and $h : X \to \mathbb{R}$ be any continuous functions. For any subset $A \subset X$ let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions which satisfy the equality

$$E\big(f(a), g(a)\big) = h(a), \quad a \in A.$$

Then the functions $\hat{f} : X \to \mathbb{R}$ and $\hat{g} : X \to \mathbb{R}$ are called an (E, h)-*controlled extension* of f and g, respectively, if $\hat{f}$ extends f, $\hat{g}$ extends g and

$$E\big(\hat{f}(x), \hat{g}(x)\big) = h(x), \quad x \in X.$$

For a simple example, let $E(u, v) = a \cdot v + b \cdot u$. Then one can extend g to $\hat{g}$ in an arbitrary manner, applying the Tietze–Urysohn theorem, and then directly set $\hat{f} = (h - b\hat{g})/a$. More generally, if the equation $E(u, v) = c$, $c \in \mathbb{R}$, admits an explicit representation $u = \psi(v, c)$ by a continuous function ψ, then we can simply put $\hat{f}(x) = \psi(\hat{g}(x), h(x))$, using an arbitrary extension $\hat{g}$ of g.

Clearly, one can rephrase the results above as existence theorems for (E, h)-controlled extensions for the multiplication function $E(u, v) = u \cdot v$. Note that Definition 0.2 is a version of Frantz's definition [4]. But, as he wrote, "...there are many other cases, however, for which the answers are not clear." The aim of the present paper is to show some ways to fill this gap.

In Section 1 below we formulate our results and introduce necessary technical notions. Section 2 presents the proofs: we show general properties of a mapping $E : \mathbb{R}^2 \to \mathbb{R}$ which are sufficient for substitution of the multiplication mapping $(u, v) \mapsto u \cdot v$ in the result. We propose two variants. One of them, in abstract terms concerning convexity-like properties of the level sets $E^{-1}(t)$, $t \in \mathbb{R}$ (cf. Theorem 1.2) and the other one deals with concrete analytical properties of the function E (cf. Theorem 1.5).

Note that in Definition 0.2 there is no mention of boundary restrictions of the type $g^{-1}(0) \subset f^{-1}(0)$. We consider this more definitely in Section 3. There we also reformulate the controlled extension problem in terms of *soft* mappings (in the sense of Shchepin [11]) or as a suitable selection problem. Such a general point of view gives various ways for solving this problem.

In conclusion, we recall that a single-valued mapping $f : X \to Y$ is said to be a *selection* of a given multivalued mapping $F : X \to Y$ if $f(x) \in F(x)$ for each $x \in X$. Furthermore, the *lower semicontinuity* of a multivalued mapping $F : X \to Y$ between topological spaces means that for each $x \in X$ and $y \in F(x)$, and each open neighborhood $U(y)$ there exists

an open neighborhood $V(x)$ such that $F(x') \cap U(y) \neq \emptyset$ whenever $x' \in V(x)$. For general facts and references on the selection theory see [5,6,9].

1. Preliminaries

We need some terminology concerning sets with controlled degree of nonconvexity. Let P be a nonempty closed subset of a normed space B. The number $\delta(P, D) = \sup\{\operatorname{dist}(q, P)/r \mid q \in \operatorname{conv}(P \cap D)\}$ is a natural upper estimate for a relative measure of nonconvexity of the intersection of the set P with the open ball D of radius r. *The function of nonconvexity* $\alpha_P(\cdot)$ of the set P associates to each number $r > 0$ the supremum of the set $\{\delta(P, D)\}$ over all open balls of the radius r. Clearly, the identity $\alpha_P(\cdot) \equiv 0$ is equivalent to the convexity of the set P. If $\alpha_P(r) \leqslant \alpha(r)$ for all positive r, then the set P is said to be *α-paraconvex*. The following selection theorem was proved in [8]:

Theorem 1.1. *Let $\alpha : (0, \infty) \to [0, 1)$ be any increasing continuous function and $\Phi : X \to B$ a lower semicontinuous mapping from a paracompact space X into a Banach space B with α-paraconvex values $\Phi(x)$, for every $x \in X$. Then Φ admits a continuous single-valued selection.*

Theorem 1.1 was proved for constant functions α by Michael [7]. At that time he introduced the notion of α-paraconvexity for the *constant* function α.

Below we denote the open upper half-plane $\{(u, v)\colon\ v > 0\}$ by $\mathbb{R}^2_+$ and the union $\mathbb{R}^2_+ \cup \{(0, 0)\}$ by $\mathbb{R}^2_{+0}$. For any numerical functions f and g on a set A we denote their Cartesian product by (f, g), that is $(f, g)(a) = (f(a), g(a))$, for every $a \in A$. Clearly, the hypotheses $f : A \to \mathbb{R}$, $g : A \to [0, \infty)$ and $g^{-1}(0) \subset f^{-1}(0)$ can be summarized as $(f, g) : A \to \mathbb{R}^2_{+0}$.

Theorem 1.2. *Let $\alpha : (0, \infty) \to [0, 1)$ be any increasing continuous function. Let $E : \mathbb{R}^2 \to \mathbb{R}$ be an open surjection with $E^{-1}(0) = \{(u, v)\colon\ uv = 0\}$ and such that all intersections $E^{-1}(t) \cap \mathbb{R}^2_+$, $t \neq 0$, are α-paraconvex. Let*:

(a) *X be a normal and countably paracompact space and $A \subset X$ any closed subset; or*
(b) *X be a normal space and $A \subset X$ any compact subset, and $h : X \to \mathbb{R}$ a continuous function.*

Then each continuous mapping $(f, g) : A \to \mathbb{R}^2_{+0}$ with $E(f(a), g(a)) = h(a)$, for every $a \in A$, admits an (E, h)-controlled extension $(\hat{f}, \hat{g}) : X \to \mathbb{R}^2_{+0}$.

The situations when all level sets $E^{-1}(t) \cap \mathbb{R}^2_+$, $t \neq 0$, are smooth planar curves are natural areas for applications of Theorem 1.2. Note that the paraconvexity of the connected *graph* Γ of a continuous numerical function of one real variable can be derived from a suitable upper estimate for $\operatorname{dist}(Q, \Gamma)/r$, where Q is the midpoint of the segment $[P, R]$ with $P \in \Gamma$, $R \in \Gamma$ and of length $2r$ (see [8]).

Typical examples are connected graphs of monotone continuous or Lipschitz functions. Other examples of paraconvex subsets of the Euclidean plane will be useful for a more

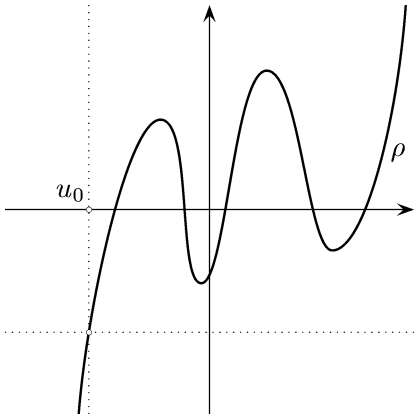

Fig. 1.

concrete version of Theorem 1.2, where we shall work with more analytical properties of the mapping E. In particular, these properties will guarantee the paraconvexity of level sets $E^{-1}(t) \cap \mathbb{R}^2_+$, whenever $t \neq 0$. Note that for normal but not countably paracompact domains X (so-called *Dowker* spaces [10]) and for their closed subsets A we shall state our result only for such more specific mappings $E : \mathbb{R}^2 \to \mathbb{R}$.

Definition 1.3. A function $E : \mathbb{R}^2 \to \mathbb{R}$ is called a *pseudomultiplication* if $E(u, v) = e(u) \cdot v$, where:

(a) $e(\cdot)$ preserves the signs of the arguments;
(b) $e(\cdot)$ is continuously differentiable; and
(c) the derivative $e'(\cdot)$ is positive in some neighborhood of zero and in some neighborhood of the infinity.

For an arbitrary continuously differentiable function $\rho : \mathbb{R} \to \mathbb{R}$ with positive derivative ρ' for all sufficiently large arguments, one can apply two parallel shifts and obtain the function $e(u) = \rho(u + u_0) - \rho(u_0)$, with properties (a)–(c) from Definition 1.3. Polynomials of odd degree provide such examples (see Fig. 1).

We can say in the spirit of [3] that $e(u)$ is an asymptotically increasing function and $e(u^{-1})$ is an asymptotically decreasing function (see Fig. 2).

Lemma 1.4. *Let $E : \mathbb{R}^2 \to \mathbb{R}$ be a pseudomultiplication and let $C_0 > 0$ be any constant. Then there exists an increasing continuous function $\alpha : (0, \infty) \to [0, 1)$ such that all intersections $E^{-1}(t) \cap \mathbb{R}^2_+$, $0 < |t| \leqslant C_0$, are α-paraconvex subsets of the Euclidean plane.*

As a corollary of Theorem 1.2 and Lemma 1.4 we obtain:

Theorem 1.5. *Let $E : \mathbb{R}^2 \to \mathbb{R}$ be a pseudomultiplication. Let $h : X \to \mathbb{R}$ be a continuous function on a normal space X, and A a closed subset of X. Then each continuous mapping*

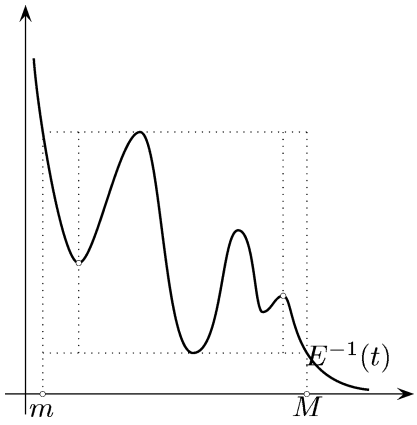

Fig. 2.

$(f, g) : A \to \mathbb{R}^2_{+0}$ such that $E(f(a), g(a)) = h(a)$, $a \in A$, admits an (E, h)-controlled extension $(\hat{f}, \hat{g}) : X \to \mathbb{R}^2_{+0}$.

Observe that the pseudomultiplication $E(u, v) = e(u) \cdot v$ is an open surjection, while the function $u \mapsto e(u)$ is not open at its points of extrema. Pseudomultiplications have an advantage in comparison with open surjections because there is a continuous flow on their level sets. Moreover, all level sets are connected graphs of smooth functions of one real variable. So one can continuously move points along level sets of pseudomultiplications.

One way to find a nonparaconvex variant can be described as follows. Let $H : \mathbb{R} \to \mathrm{Homeo}_+(\mathbb{R})$ be a continuous mapping into the set of all sign-preserving homeomorphisms of the real line, endowed with the topology of uniform convergence. For a function $E : \mathbb{R}^2 \to \mathbb{R}$ we naturally define another function $E_H : \mathbb{R}^2 \to \mathbb{R}$, by setting $E_H(u, v) = E(u, H_u(v))$.

Lemma 1.6. *The existence of (E, h)-controlled extensions implies the existence of (E_H, h)-controlled extensions.*

Shortly, the existence of controlled extensions is a stable property under an action of pointwise (with respect to the first coordinate) homeomorphisms. The level sets $E^{-1}(t)$ in this lemma can clearly be more complicated than paraconvex sets. So on the one hand it generalizes Theorem 1.5. However, in the simplest cases, for example $E(u, v) = u^3 \cdot v$, Lemma 1.6 is not applicable, while Theorem 1.5 works.

2. Proofs

Proof of Lemma 1.4. We begin by determining the function α. Let $v(u) = 1/e(u)$, $u > 0$. Pick $0 < m < M$ so that on the segment $[m, M]$ the graph of the function $v(\cdot)$ lies inside the rectangle $[m, M] \times [v(m), v(M)]$ and v is decreasing on $(0, m)$ and (M, ∞). By setting

$$\alpha(r,t)=\max\left\{\frac{\sqrt{2}}{2},\sin\Big(\arctan\big(t\cdot\max\big\{\big|v'(u)\big|:\ v^{-1}\big(v(m)+2r\big)\leqslant u\leqslant M+2r\big\}\big)\Big)\right\}$$

we obtain a function which is increasing and continuous with respect to both variables r and t. Hence, the function $\alpha(r)=\alpha(r,C_0)$ majorates each function $\alpha(r,t)$. So we only need to check that the function $\alpha(\cdot,t)$ majorates the function of nonconvexity of the curve $E^{-1}(t)\cap\mathbb{R}^2_+$.

To this end we first note that this intersection is the graph of a continuous function on the positive u-ray:

$$E^{-1}(t)\cap\mathbb{R}^2_+=\left\{(u,v):\ u>0,\ v=\frac{t}{e(u)}\right\}.$$

So by [8], we only need to estimate the distances $\operatorname{dist}(Q,E^{-1}(t))$ for the midpoints Q of the segment $[P,R]$ with endpoints in the set $E^{-1}(t)\cap\mathbb{R}^2_+$ and of length $2r$. Let $\pi_1:\mathbb{R}^2\to\mathbb{R}$ be the projection onto the first factor.

(1) If $\pi_1(R)\leqslant m$ or $\pi_1(P)\geqslant M$, then the function $v=t/e(u)$ is decreasing on the segment $[\pi_1(P),\pi_1(R)]$. Hence (see [8]),

$$\frac{\operatorname{dist}(Q,E^{-1}(t))}{r}\leqslant\frac{\sqrt{2}}{2}\leqslant\alpha(r).$$

(2) If $m\leqslant\pi_1(P)\leqslant\pi_1(R)\leqslant M$, then the function $v=t/e(u)$ is Lipschitz on the segment $[\pi_1(P),\pi_1(R)]$ with the constant less than or equal to $t\cdot\max\{|v'(u)|:\ m\leqslant u\leqslant M\}$. Hence (see [8]),

$$\frac{\operatorname{dist}(Q,E^{-1}(t))}{r}\leqslant\sin\Big(\arctan\big(t\cdot\max\big\{\big|v'(u)\big|:\ m\leqslant u\leqslant M\big\}\big)\Big)\leqslant\alpha(r).$$

(3) If $\pi_1(R)\geqslant M\geqslant\pi_1(P)$, then $\pi_1(R)\leqslant M+2r$ and $\pi_1(P)\geqslant v^{-1}(v(m)+2r)$ because the length of $[P,R]$ is $2r$. Hence we obtain for $\operatorname{dist}(Q,E^{-1}(t))/r$ an upper estimate as in the case (2) above with the substitution of the segment $[m,M]$ by the segment $[v^{-1}(v(m)+2r),M+2r]$ (see Fig. 3).

(4) The case $\pi_1(P)\leqslant m\leqslant\pi_1(R)$ and the case of negative parameter t can be treated analogously. □

Observe that for pseudomultiplications E (and for surjections E from Theorem 1.2 as well) the intersection $E^{-1}(0)\cap\mathbb{R}^2_+$ is the open ray $\{(0,v):\ v>0\}$ and the closure of this intersection is paraconvex (since it is convex). We pass now to generalizations of Theorem 0.1.

Proof of Theorem 1.2. The multivalued mapping $E^{-1}\circ h:X\to\mathbb{R}^2$ is LSC due to the openness of E and the continuity of h. The intersection with the open set $\mathbb{R}^2_+$ and the pointwise closure operation preserve the LSC property (see [5]). For $t\neq0$ the intersection $E^{-1}(t)\cap\mathbb{R}^2_+$ is closed because of the α-paraconvexity assumption and $\operatorname{Cl}(E^{-1}(0)\cap\mathbb{R}^2_+)$ is simply a vertical closed ray. So the multivalued mapping $\Psi:X\to\mathbb{R}^2$, defined by $\Psi(x)=\operatorname{Cl}\{E^{-1}(h(x))\cap\mathbb{R}^2_{+0}\}$, is an LSC mapping with α-paraconvex values.

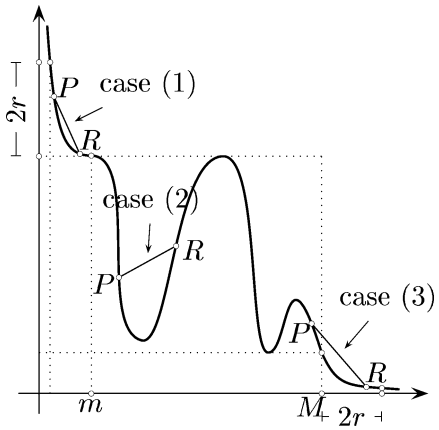

Fig. 3.

The assumed equality $E(f(a), g(a)) = h(a)$, for every $a \in A$, means precisely that the single-valued mapping $s(a) = (f(a), g(a))$, for every $a \in A$, is a partial continuous selection of $\Psi|_A$. Hence the multivalued mapping $\Phi : X \to \mathbb{R}^2$, defined by $\Phi(a) = \{s(a)\}$, for every $a \in A$, and $\Phi(x) = \Psi(x)$, for every $x \in X \setminus A$, is also an LSC mapping and also has α-paraconvex values.

For paracompact domains X we can simply use Theorem 1.1 and extend s to a selection $\hat{s}$ of Φ over the entire X. Clearly, the coordinate projections $\hat{f} = \pi_1 \circ \hat{s}$ and $\hat{g} = \pi_2 \circ \hat{s}$ of such an extension give the desired controlled extensions of f and g.

However, for normal domains X we must be more careful. In fact, we prove an analogue of Theorem 1.1 for normal domains by using local compactness of the plane $\mathbb{R}^2$.

In case (a) we pick arbitrary continuous extensions f_0 and g_0 of f and g and define $s_0 : X \to \mathbb{R}^2$ by the equality $s_0(x) = (f_0(x), g_0(x))$. Observe that $s_0(a) = s(a)$, for all $a \in A$. The distance function $\operatorname{dist}(s_0(x), \Phi(x))$, for every $x \in X$, is an upper semicontinuous numerical function on the normal and countably paracompact space X. Due to the Dowker separation theorem it admits a continuous strong majorant function $r : X \to (0, \infty)$. So in our case we have some continuous r-selection s_0 of Φ, i.e., $\operatorname{dist}(s_0(x), \Phi(x)) < r(x)$, for every $x \in X$. We now inductively proceed with improvement of the precision $s_n(x) \approx \Phi(x)$.

Choose $\alpha(\cdot) < \beta(\cdot) < 1$ with some continuous increasing function $\beta(\cdot)$ and define the multivalued mapping $\Phi_1 : X \to \mathbb{R}^2$ by

$$\Phi_1(x) = \operatorname{Cl}\Big(\operatorname{conv}\big\{\Phi(x) \cap D\big(s_0(x), r(x)\big)\big\}\Big).$$

Clearly, Φ_1 is LSC with nonempty, convex and *compact* values. So by the Michael selection theorem for *normal* domains (see [5] and [9, Part B]), it admits a selection $s_1 : X \to \mathbb{R}^2$. Then the α-paraconvexity guarantees that

$$\operatorname{dist}\big(s_1(x), \Phi(x)\big) \leqslant \alpha\big(r(x)\big) \cdot r(x) < \beta\big(r(x)\big) \cdot r(x) = r_1(x) < r(x),$$
$$\operatorname{dist}\big(s_0(x), s_1(x)\big) \leqslant r(x).$$

The function $r_1 : X \to (0, \infty)$ is continuous. So we can find a selection s_2 of the mapping

$$\Phi_2(x) = \mathrm{Cl}\Big(\mathrm{conv}\big\{\Phi(x) \cap D\big(s_1(x), r_1(x)\big)\big\}\Big)$$

for which

$$\begin{aligned}\mathrm{dist}\big(s_2(x), \Phi(x)\big) &\leqslant \alpha\big(r_1(x)\big) \cdot r_1(x) < \beta\big(r_1(x)\big) \cdot r_1(x) < \beta\big(r(x)\big) \cdot r_1(x)\\ &= \beta^2\big(r(x)\big) \cdot r(x) = r_2(x) < r_1(x),\\ \mathrm{dist}\big(s_1(x), s_2(x)\big) &\leqslant r_1(x).\end{aligned}$$

The obvious continuation of such a procedure yields a sequence of mappings $s_n : X \to \mathbb{R}^2$ with

$$\mathrm{dist}\big(s_n(x), \Phi(x)\big) < \beta^n\big(r(x)\big) \cdot r(x), \qquad \mathrm{dist}\big(s_{n-1}(x), s_n(x)\big) \leqslant \beta^{n-1}\big(r(x)\big) \cdot r(x).$$

For each $x \in X$ the continuous functions $\beta(r(\cdot)) < 1$ and $r(\cdot)$ are bounded on some neighborhood of x. So the sequence $\{s_n\}$ is locally uniformly fundamental and hence it has a limit $\hat{s} : X \to \mathbb{R}^2$ which clearly is a continuous selection of Φ.

In case (b) we first consider the situation when $h : X \to \mathbb{R}$ is bounded. The mapping $t \mapsto \mathrm{Cl}\{E^{-1}(t) \cap \mathbb{R}^2_+\}$, $t \in \mathbb{R}$, admits a continuous selection, due to the cases already studied above. The values of such a selection constitute a bounded set when the parameter t changes from a to b with $[a, b] \supset h(X)$. This means that there exists a point $p \in \mathbb{R}^2$ and a positive r such that the open ball D_r centered at p meets with each value of $\Psi(x)$, for every $x \in X$. The given continuous mapping (f, g) is bounded on A because of the compactness of A. Hence we can assume that the set $(f, g)(A)$ lies inside D_r. So the constant mapping $s_0(\cdot) \equiv p$ is the r-selection of the mapping Φ. Now we repeat the improvement procedure from case (a).

Let h be unbounded on X. Define a strongly increasing sequence $\{X_n\}$ of closed subsets of X by setting $X_n = \{x \in X\colon\ |h(x)| \leqslant n\}$. Then

$$X_1 \subset \operatorname{int} X_2 \subset X_2 \subset \operatorname{int} X_3 \subset X_3 \subset \cdots, \qquad \bigcup X_n = X.$$

Apply the first case to the pair $(X_1, X_1 \cap A)$ and the corresponding restrictions of the functions f, g and h. We obtain an (E, h)-controlled extension $(f_1, g_1) : X_1 \to \mathbb{R}^2_{+0}$ of $f|_{X_1 \cap A}$ and $g|_{X_1 \cap A}$. Hence the case of the bounded function h is applicable to the pair $(X_2, X_1 \cup (X_2 \cap A))$ and so on. Each point $x \in X$ lies in some X_n with its own neighborhood. Thus we obtain a continuous mappings over the whole domain X. □

Proof of Theorem 1.5. The mapping $t \mapsto \mathrm{Cl}\{E^{-1}(t) \cap \mathbb{R}^2_+\}$, $-1 \leqslant t \leqslant 1$, admits a selection due to Lemma 1.4 and Theorem 1.1. Applying the same results for $-2 \leqslant t \leqslant 2$, we extend such a selection onto the segment $[-2, 2]$. A continuation gives a selection $\phi : \mathbb{R} \to \mathbb{R}^2$ of the mapping $t \mapsto \mathrm{Cl}\{E^{-1}(t) \cap \mathbb{R}^2_+\}$, for every $t \in \mathbb{R}$.

For a pseudomultiplication E, its level sets $E^{-1}(t) = \{(u, v)\colon\ v = t/e(u)\}$ look as asymptotically hyperbolic-type curves after intersecting with $\mathbb{R}^2_+$. So on the normal space X we have the mapping $\phi \circ h : X \to \mathbb{R}^2$ with values on the curves $\mathrm{Cl}\{E^{-1}(h(x)) \cap \mathbb{R}^2_+\}$ and on the closed subset $A \subset X$ we have $s : a \mapsto (f(a), g(a))$ with values on the curves $\mathrm{Cl}\{E^{-1}(h(a)) \cap \mathbb{R}^2_+\}$. In general, these mappings are different on A, i.e., $s(a) \neq \phi(h(a))$.

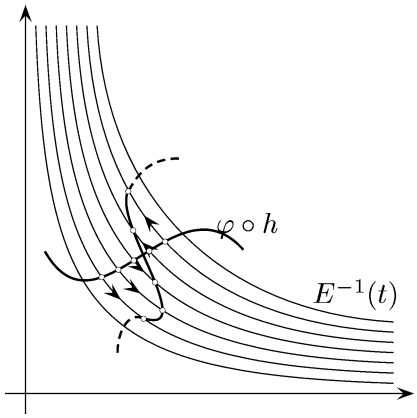

Fig. 4.

But we can assume that all curves $\mathrm{Cl}\{E^{-1}(t) \cap \mathbb{R}^2_+\}$, $t \in \mathbb{R}$, are endowed with the common direction induced by an arbitrary fixed direction on the unique curve $\mathrm{Cl}\{E^{-1}(1) \cap \mathbb{R}^2_+\}$. Next, we can simply move points $\phi(h(a))$ to points $s(a)$ along the corresponding curve $\mathrm{Cl}\{E^{-1}(h(a)) \cap \mathbb{R}^2_+\}$. All others points $\phi(h(x))$ will be transferred continuously on the curves $\mathrm{Cl}\{E^{-1}(h(x)) \cap \mathbb{R}^2_+\}$.

More precisely, for each $a \in A$ we calculate the signed length of the segment of the curve $\mathrm{Cl}\{E^{-1}(h(a)) \cap \mathbb{R}^2_+\}$ between the points $\phi(h(a))$ and $s(a)$. Considering $\phi(h(a))$ as the starting point, we obtain the numerical function $l : A \to \mathbb{R}$. It is clearly continuous. We extend it to some continuous function $\hat{l} : X \to \mathbb{R}$ and then move each point $\phi(h(x))$ along the curve $\mathrm{Cl}\{E^{-1}(h(x)) \cap \mathbb{R}^2_+\}$ exactly for the signed $\hat{l}(x)$ length (see Fig. 4). The result gives a selection of $x \mapsto \mathrm{Cl}\{E^{-1}(h(x)) \cap \mathbb{R}^2_+\}$ which extends s. □

Proof of Lemma 1.6. Let $(f, g) : A \to \mathbb{R}^2_{+0}$ and $E_H(f(a), g(a)) = h(a)$, for every $a \in A$. This means that $E(f(a), H_{f(a)}(g(a))) = h(a)$, for every $a \in A$. Define the continuous mapping $g_H(a) = H_{f(a)}(g(a))$. Clearly, if $g_H(a) = 0$, then $g(a) = 0$. So $f(a) = 0$ and hence (f, g_H) maps A into the set $\mathbb{R}^2_{+0}$. By our assumption it admits an (E, h)-controlled extension $(\hat{f}, \hat{g}_H) : X \to \mathbb{R}^2_{+0}$.

Now, put $\hat{g}(x) = H^{-1}_{\hat{f}(x)}(\hat{g}_H(x)) \geqslant 0$. If $\hat{g}(x) = 0$, then $\hat{g}_H(x) = 0$ and hence $\hat{f}(x) = 0$. Therefore $(\hat{f}, \hat{g}) : X \to \mathbb{R}^2_{+0}$ and

$$E_H\big(\hat{f}(x), \hat{g}(x)\big) = E\Big(\hat{f}(x), H_{\hat{f}(x)}\big(H^{-1}_{\hat{f}(x)}\big(\hat{g}_H(x)\big)\big)\Big) = E\big(\hat{f}(x), \hat{g}_H(x)\big) = h(x),$$

because $(\hat{f}, \hat{g}_H)$ is an (E, h)-controlled extension. □

3. Extensions and selections

Continuous extensions are special cases of continuous selections. In this section we show that this is also true for controlled extensions. So in this section we shall forget about specifics of the upper half plane and shall take a more general point of view.

Definition 3.1. For any mapping $E:\mathbb{R}^n \to \mathbb{R}$ and any open subset $G \subset \mathbb{R}^n$ the *closure of G with respect to E* is defined as

$$\mathrm{Cl}_E G = \bigcup_{t\in\mathbb{R}} \mathrm{Cl}\big(E^{-1}(t) \cap G\big).$$

Definition 3.2. Let X be a topological space, A its subset and G an open subset of $\mathbb{R}^n$. A mapping $E:\mathbb{R}^n \to \mathbb{R}$ is said to be *(X, A)-suitable for extensions to G* if for each continuous function $h: X \to \mathbb{R}$ and $f = (f_1, f_2, \ldots, f_n): A \to \mathrm{Cl}_E G$ with $E \circ f = h|_A$ there exists an (E, h)-controlled extension, i.e., an extension $\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n): X \to \mathrm{Cl}_E G$ of f such that $E \circ f = h$.

Proofs from Section 2 show that we really used only a topological or convexity-like property of the family $\{\mathrm{Cl}(E^{-1}(t) \cap G)\}_{t\in\mathbb{R}}$. So as a purely topological version of the results we state the following:

Theorem 3.3. *Let $E:\mathbb{R}^2 \to \mathbb{R}$ be an open surjection and G an open subset of R^2 such that $\{\mathrm{Cl}(E^{-1}(t) \cap G)\}_{t\in\mathbb{R}}$ is an ELC^0-family, consisting of arcs and singletons. Then E is (X, A)-suitable for extensions to G for an arbitrary paracompact space X and its closed subsets A.*

Proof. The key point of the proof is a recent selection theorem of Cauty [2]. He proved that each LSC mapping on a paracompact space admits a continuous selection whenever its values are arcs or singletons and the family of all values is ELC^0 in some metric space. Therefore, one can repeat the proof of Theorem 1.2 to the point before the proof of case (a) and then use Cauty's theorem. □

Clearly one can apply Cauty's theorem for a mapping from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$.

Question 3.4. Does Theorem 3.3 remain valid for normal, or even for normal countably paracompact domains?

Recall from [11], that a mapping $\phi: Y \to Z$ is said to be *soft with respect to the pair* (X, A) if for each continuous mappings $f: A \to Y$ and $h: X \to Z$ with $\phi \circ f = h|A$ there exists an extension $\hat{f}: X \to Y$ of f such that $\phi \circ \hat{f} = h$.

A mapping which is soft with respect to any pair from a class $\mathcal{L}$ of topological spaces is said to be *$\mathcal{L}$-soft*. Considering the classes of n-dimensional paracompact, finite-dimensional paracompact, all paracompact spaces and so on, we obtain the notions of n-soft, ∞-soft, absolutely soft, etc. mappings. For the case of compact domains there are many different facts concerning soft mappings. For details see [11] and [9, Part C].

Clearly, the $\mathcal{L}$-softness of ϕ means that the multivalued mappings

$$\Phi(a) = \big\{f(a)\big\}, \quad a \in A, \qquad \Phi(x) = \phi^{-1}\big(h(x)\big), \quad x \in X \setminus A,$$

admit a selection for any $(X, A) \in \mathcal{L}$ and arbitrary f and h.

So if one substitutes Y by $\mathbb{R}^n$, Z by $\mathbb{R}$ and ϕ by E, then one gets Definition 3.2.

Theorem 3.5. *Let $\mathrm{Cl}_E G$ be the closure of an open set $G \subset \mathbb{R}^n$ with respect to an open surjection $E : \mathbb{R}^n \to \mathbb{R}$. Then softness with respect to a pair (X, A) of the restricted mapping $E|_{\mathrm{Cl}_E G}$ implies that E is (X, A)-suitable for extensions to G.*

Using Theorem 3.5, each theorem on softness (or each theorem on continuous selections of LSC mappings) gives us a theorem on existence of controlled extensions. For example, if for any function $E : \mathbb{R}^n \to \mathbb{R}$, any open set $G \subset \mathbb{R}^n$ and any function $h : X \to \mathbb{R}$ on any at most $(n+1)$-dimensional paracompact space X, the family $\mathrm{Cl}\{E^{-1}(h(x)) \cap G\}_{x \in X}$ is ELC^n and all its values are C^n then each n-tuple of functions $(f_1, f_2, \ldots, f_n)$ such that $(f_1(A), f_2(A), \ldots, f_n(A)) \subset \mathrm{Cl}_E G$ and $E(f_1(a), f_2(a), \ldots, f_n(a)) = h(a), a \in A$, admits an extensions $(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n)$ such that $(f_1(X), f_2(X), \ldots, f_n(X)) \subset \mathrm{Cl}_E G$ and $E(\hat{f}_1(x), \hat{f}_2(x), \ldots, \hat{f}_n(x)) = h(x), x \in X$.

Our final remark deals with *inequalities* rather than equalities in basic Definitions 0.2 and 3.2.

Lemma 3.6. *Let X be any normal and countably paracompact space and A a closed subset of X. Let $E : \mathbb{R}^n \to \mathbb{R}$ be a continuous function which is (X, A)-suitable for extensions to an open set $G \subset \mathbb{R}^n$. Then for each continuous $h : X \to \mathbb{R}$ and $f = (f_1, f_2, \ldots, f_n) : A \to \mathrm{Cl}_E G$ with*

$$E\big(f_1(a), f_2(a), \ldots, f_n(a)\big) \leqslant h(a), \quad a \in A,$$

there exists a continuous extension $\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n) : X \to \mathrm{Cl}_E G$ of f such that

$$E\big(\hat{f}_1(x), \hat{f}_2(x), \ldots, \hat{f}_n(x)\big) \leqslant h(x), \quad x \in X.$$

Proof. The equality

$$E\big(f_1(a), f_2(a), \ldots, f_n(a)\big) = h_0(a), \quad a \in A,$$

defines the continuous function $h_0 : A \to \mathbb{R}$ such that $h_0(a) \in (-\infty, h(a)]$. The Dowker separation theorem guarantees the existence of an extension $\hat{h}_0 : X \to \mathbb{R}$ of h_0 such that $\hat{h}_0(x) \in (-\infty, h(x)]$. The (X, A)-suitability for extensions to G means precisely that we can extend all functions f_i so that

$$E\big(\hat{f}_1(x), \hat{f}_2(x), \ldots, \hat{f}_n(x)\big) = \hat{h}_0(x) \leqslant h(x), \quad x \in X. \qquad \square$$

Question 3.7. Does Lemma 3.6 also hold for normal domains?

Acknowledgments

The first author was supported in part by the Ministry of Education, Science and Sport of the Republic of Slovenia research program No. 0101-509. The second author was supported in part by the Ministry of Education, Science and Sport of the Republic of Slovenia research grants No. SLO-KIT-04-14 and No. SLO-US-2002/01, and the research program No. 0101-509. The third author was supported in part by the RFBR grant No. 02-01-00014. We thank the referee for several useful comments and remarks.

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