ON CHARACTERIZATION OF LIPSCHITZ MANIFOLDS

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Abstract. We construct an example of a wild Cantor set in $\mathbb{R}^3$ which is Lipschitz ambientally homogeneous in $\mathbb{R}^3$, thereby showing that Lipschitz homogeneity does not characterize Lipschitz submanifolds od $\mathbb{R}^3$ (contrary to the smooth homogeneity).

1. Introduction

In 1989, working on a problem of Arnol'd [1] concerning one-parameter group actions on $\mathbb{R}^2$, Dimovski, Repovš and Ščepin [4] introduced the concept of C^∞-homogeneity for locally compact subsets of $\mathbb{R}^2$. This notion was later generalized by Repovš, Skopenkov and Ščepin [8][9] to C^r-homogeneity in an arbitrary smooth manifold:

Definition 1 *A subset $K \subset M^n$ is said to be C^r-homogeneous in a smooth n-manifold M^n, $r > 0$, if for every pair of points $a, b \in K$ there exist neighbourhoods $O_a, O_b \subset M^n$ of a and b, respectively, and a C^r-diffeomorphism*

$$h : (O_a, O_a \cap K, a) \longrightarrow (O_b, O_b \cap K, b)$$

It was proved in [9] that this property characterizes the C^r-submanifolds of C^r-manifolds, for every $r > 0$:

Theorem 1 *(Repovš, Skopenkov and Ščepin, 1996) Let K be a locally compact (possibly nonclosed) subset of a smooth n-manifold M^n. Then K is C^r-homogeneous in M^n, $r > 0$, if and only if K is a C^r-submanifold of M^n.*

As an interesting application one obtains a simple geometric proof of the classical result of Bochner and Montgomery [3] that the Hilbert- Smith

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conjecture is true for actions by diffeomorphisms (for some new results concerning this conjecture see [6] and [10]). Namely, suppose to the contrary, that the group A_p of p-adic integers acted freely on a smooth manifold M by diffeomorphisms. Then every orbit would be diffeomorfic to the group A_p. At the same time, every orbit would also be C^∞-homogeneous in M, hence by Theorem 1 itself a smooth manifold. Contradiction.

Obviously Theorem 1 is not valid for topological homogeneity as the example of the standard ternary Cantor set in $\mathbb{R}^2$ demonstrates. It was expected however, that Theorem 1 could nevertheless be generalized to the case of Lipschitz submanifolds of $\mathbb{R}^n$ (where Lipschitz homogeneity is defined analogously to the C^r-homogeneity in Definition 1).

However, as we prove in the present paper, this is not true already in the plane $\mathbb{R}^2$:

Theorem 2 *The standard ternary Cantor set, lying on the x-axis in $\mathbb{R}^2$, is Lipschitz homogeneous in $\mathbb{R}^2$.*

In 1995 Ščepin asked whether in $\mathbb{R}^3$ Lipschitz homogeneity of Cantor sets would imply their tameness. The main result of this paper, stated below, answers his question in the negative:

Theorem 3 *There exists a wild Cantor set in $\mathbb{R}^3$ which is Lipschitz homogeneous in $\mathbb{R}^3$.*

We believe that our methods can be generalized to higher dimensions, using the techniques of Blankinship [2]:

Conjecture 1 *There exists a wild Cantor set in $\mathbb{R}^n$, for every $n \geq 4$, which is Lipschitz homogeneous in $\mathbb{R}^n$.*

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2. Preliminaries

Recall that a map $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be *Lipschitz* if there exists a constant λ such that

$$|S(x) - S(y)| \leq \lambda |x - y| \quad \text{for every } x, y \in \mathbb{R}^n$$

and the smallest such λ is called the *Lipschitz constant of S*. In the special case when

$$|S(x) - S(y)| = \lambda |x - y| \quad \text{for every } x, y \in \mathbb{R}^n$$

the map S is called a *similarity* and the number λ is called the *coefficient of similitude*. Finally, when $\lambda = 1$ the map S is called an *isometry*.

Let G be a finite index set and let $\mathcal{S} = \{S_g : \mathbb{R}^n \longrightarrow \mathbb{R}^n | g \in G\}$ be a set of similarities having the same coefficient of similitude. Additionally, suppose that there exists a compact set $X \subset \mathbb{R}^n$ such that

(i) $S_g(X) \subset \mathrm{Int}(X)$ for each $g \in G$; and
(ii) the sets $S_g(X)$ are pairwise disjoint, $g \in G$.

For each multiindex $\gamma = (g_1, g_2, \dots g_k) \in G^k = G \times G \times \dots \times G$ denote:

$$S_\gamma = S_{g_1} \circ S_{g_2} \circ \dots \circ S_{g_k}$$

and

$$X_\gamma = S_\gamma(X).$$

In particular,

$$X_g = S_g(X) \text{ for } g \in G.$$

The number of components of a multiindex γ is called the *dimension of* γ:

$$\dim(\gamma) = k \quad \text{if } \gamma \in G^k.$$

Denote

$$X_k = \bigcup_{\dim(\gamma)=k} X_\gamma.$$

It is well-known (cf. [5]) that the intersection of the sequence of sets $X \supset X_1 \supset X_2 \supset \dots$ is a self-similar Cantor set and it does not depend on the choice of X. Therefore it depends only on the set $\mathcal{S}$ and so it can be denoted by $|\mathcal{S}|$.

For an infinite multiindex $\gamma = (g_1, g_2, g_3, \dots) \in G^\infty$ denote

$$\gamma^k = (g_1, g_2, \dots g_k)$$

and

$$X_\gamma = \bigcap_{k=1}^{\infty} X_{\gamma^k}.$$

Obviously, each X_γ is a singleton, consisting of a point from the Cantor set $|\mathcal{S}|$ and for each point from $|\mathcal{S}|$ there exists exactly one such multiindex γ. The components of γ are called *coordinates* of the corresponding point from the Cantor set $|\mathcal{S}|$.

3. Sufficient conditions for Lipschitz homogeneity

Let G be a finite cyclic group written additively and let 0 and 1 be the neutral element and the generator of G, respectively.

Lemma 1 *Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a Lipschitz homeomorphism such that*

(i)

$$f|_{\mathbb{R}^n - Int(X)} = id_{\mathbb{R}^n - Int(X)}$$

(ii)

$$f(X_g) = X_{g+1} \quad \textit{for each } g \in G$$

and the following diagram commutes

$$\begin{array}{ccc} & X & \\ S_g \swarrow & & \searrow S_{g+1} \\ X_g & \xrightarrow{f} & X_{g+1} \end{array}$$

Then the Cantor set $|S|$ is Lipschitz homogeneous in $\mathbb{R}^n$.

Proof: Define a juxtaposition of multiindices: if $\delta = (d_1, d_2, \dots d_k)$ is a finite multiindex and $\gamma = (g_1, g_2, \dots)$ is finite or infinite then let

$$\delta\gamma = (d_1, d_2, \dots d_k, g_1, g_2, \dots).$$

In the special case when $\dim(\gamma) = 1$, hence $\gamma = g_1$ and

$$\delta g_1 = (d_1, d_2, \dots d_k, g_1).$$

In order to prove Lemma 1 we introduce several homeomorphisms and describe their properties in the subsequent lemmas. For an arbitrary finite multiindex $\gamma = (g_1, g_2, \dots g_k) \in G^k$ define the homeomorphism $f_\gamma = S_\gamma \circ f \circ S_\gamma^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Lemma 2 *The homeomorphism f_γ is Lipschitz with the Lipschitz constant equal to the Lipschitz constant of f and the following holds:*

(i)

$$f_\gamma|_{\mathbb{R}^n - Int(X_\gamma)} = id_{\mathbb{R}^n - Int(X_\gamma)}$$

(ii) For arbitrary $g_{k+1} \in G$

$$f_\gamma(X_{\gamma g_{k+1}}) = X_{\gamma(1+g_{k+1})}$$

and the following diagram commutes:

$$\begin{array}{ccc} & X & \\ S_{\gamma g_{k+1}} \swarrow & & \searrow S_{\gamma(1+g_{k+1})} \\ X_{\gamma g_{k+1}} & \xrightarrow{f_\gamma} & X_{\gamma(g+1_{k+1})} \end{array}$$

Therefore, $f_\gamma|_{X_{\gamma g_{k+1}}}$ is an isometry.

(iii) For arbitrary indices $g_{k+1}, g_{k+2}, g_{k+3}, \dots$

$$f_\gamma(X_{(g_1,g_2,\dots g_k,g_{k+1},g_{k+2},g_{k+3},\dots)}) = X_{(g_1,g_2,\dots g_k,1+g_{k+1},g_{k+2},g_{k+3},\dots)}.$$

Proof: Proposition (i) follows directly from the condition (i) of Lemma 1. Proposition (ii) follows from condition (ii). Finally, (ii) implies (iii). □

For an arbitrary pair of points $a, b \in |\mathcal{S}|$ we now construct a homeomorphism

$$h : (\mathbb{R}^n,\ |\mathcal{S}|,\ a) \longrightarrow (\mathbb{R}^n,\ |\mathcal{S}|,\ b)$$

and we prove that h and h^{-1} are Lipschitz maps.

Let $\alpha = (a_1, a_2, \dots) \in G^\infty$, $\beta = (b_1, b_2, \dots) \in G^\infty$ be coordinates of the points a, b, respectively. Introduce infinite sequences of homeomorphisms

$$\{f_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n | k \in \mathbb{N}\}, \{g_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n | k \in \mathbb{N}\}, \{h_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n | k \in \mathbb{N}\}$$

given by

$$f_1 = f^{b_1 - a_1}, \quad f_2 = f_{b_1}^{b_2 - a_2}, \quad f_3 = f_{(b_1,b_2)}^{b_3 - a_3}, \quad f_{k+1} = f_{\beta^k}^{b_{k+1} - a_{k+1}}$$

where $f^{b-a} = f \circ f \circ \dots \circ f \quad (b - a \text{ times})$,

$$g = f^{-1},\ g_1 = g^{b_1 - a_1}, \quad g_2 = g_{a_1}^{b_2 - a_2}, \quad g_3 = g_{(a_1,a_2)}^{b_3 - a_3}, \quad g_{k+1} = g_{\alpha^k}^{b_{k+1} - a_{k+1}}$$

$$h_k = f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1$$

Lemma 3 *The homemomorphisms h_k possess the following properties:*

(i)

$$h_k^{-1} = g_1 \circ g_2 \circ \dots \circ g_{k-1} \circ g_k$$

(ii)

$$h_k(X_{\alpha^k}) = X_{\beta^k} \ \textit{and, moreover}\ h_k(X_{\alpha^k\gamma}) = X_{\beta^k\gamma}$$

for arbitrary multiindex γ, finite or infinite.

(iii)

The restriction $h_k|_{X_{\alpha^k a_{k+1}}} : X_{\alpha^k a_{k+1}} \longrightarrow X_{\beta^k a_{k+1}}$ *is an isometry.*

(iv) The following restrictions coincide:

$$h_k|_{\mathbb{R}^n - \mathrm{Int}\, X_{\alpha^k}} = h_{k+1}|_{\mathbb{R}^n - \mathrm{Int}\, X_{\alpha^k}} = h_{k+2}|_{\mathbb{R}^n - \mathrm{Int}\, X_{\alpha^k}} = \dots$$

Proof: Property (i) can be proved directly by examining the construction of h_k. Property (ii) follows from Lemma 2, (ii) and (iii). Property (iii) holds since $f_\gamma|_{X_{\gamma g_{k+1}}}$ is an isometry. Property (iv) holds because of Lemma 2 (i). □

Lemma 4 *The homeomorphisms h_k and h_k^{-1} are Lipschitz maps with equal Lipschitz constants for all values of k.*

Proof: Having fixed the sequence $\alpha = (a_1, a_2, \ldots)$ of coordinates of the point $a \in |\mathcal{S}|$ introduce the notion of *degree* of a point $x \in \mathbb{R}^n$:

$$\deg x = j \text{ if } x \in X_{\alpha^j} - \mathrm{Int}(X_{\alpha^{j+1}}).$$

Additionally, let

$$\deg x = 0 \text{ if } x \in X - \mathrm{Int}(X_{a_1}) \text{ and } \deg x = -1 \text{ if } x \in \mathbb{R}^n - \mathrm{Int}(X).$$

For arbitrary points $x, y \in \mathbb{R}^n$ we now estimate the expression $h_k(x) - h_k(y)$.

Step 1 Let the Lipschitz constant of the homeomorphism f be denoted by λ. Hence the Lipschitz constants of the homeomorphisms $f_1, f_2, \ldots, g_1, g_2, \ldots$ do not exceed the number $\lambda^{|G|}$, where $|G|$ denotes the number of elements of G. Let $|\deg x - \deg y| \leq 1$, i.e.

$$\deg x \in \{j, j+1\}, \quad \deg y = j+1$$

for some $j \in \mathbb{N}$. By Lemma 3, (iii) and (iv), and because of the construction of h_k,

$$|h_k(x) - h_k(y)| = |f_{j+1} \circ f_j(x) - f_{j+1} \circ f_j(y)| \leq \lambda^{2|G|}|x - y|.$$

Step 2 Let now $|\deg x - \deg y| \geq 2$. First let the degrees be nonnegative, i.e.

$$\deg x = j \geq 0 \text{ and } \deg y \geq j+2$$

for some $j \in \mathbb{N}$. Then

$$x \in X_{\alpha^j} - \mathrm{Int}(X_{\alpha^{j+1}}), \quad y \in X_{\alpha^{j+2}}.$$

For arbitrary disjoint compact sets $C_1, C_2 \subset \mathbb{R}^n$ denote:

$$\mathrm{d_{min}}(C_1, C_2) = \min\{|x - y|; x \in C_1, \ y \in C_2\}$$

and

$$\mathrm{d_{max}}(C_1, C_2) = \max\{|x - y|; x \in C_1, \ y \in C_2\}.$$

The sets $X - \mathrm{Int}(X_1)$ and X_2 are compact and disjoint, hence the numbers

$$\mathrm{d}_X = \mathrm{d}_{\min}(X - \mathrm{Int}(X_1), X_2)$$

and

$$\mathrm{D}_X = \mathrm{d}_{\max}(X - \mathrm{Int}(X_1), X_2)$$

exist. Since the similarity S_{α^k} maps the triple $(X, X_{a_1}, X_{(a_1,a_2)})$ onto the triple $(X_{\alpha^k}, X_{\alpha^k a_1}, X_{\alpha^k(a_1,a_2)})$, for each $k \in \mathbb{N}$, the following holds:

$$\frac{\mathrm{d}_{\max}(X_{\alpha^k} - \mathrm{Int}\, X_{\alpha^k a_1}, X_{\alpha^k(a_1,a_2)})}{\mathrm{d}_{\min}(X_{\alpha^k} - \mathrm{Int}\, X_{\alpha^k a_1}, X_{\alpha^k(a_1,a_2)})} \leq \frac{\mathrm{D}_X}{\mathrm{d}_X}.$$

Hence

$$|h_k(x) - h_k(y)| \leq \frac{\mathrm{D}_X}{\mathrm{d}_X}|x - y|.$$

Finally, let $\deg x = -1$ and $\deg y \geq 1$, i.e. $x \in \mathbb{R}^n - \mathrm{Int}\, X$ and $y \in X_1$. Then $h_k(x) = x$ and

$$\frac{|h_k(x) - h_k(y)|}{|x - y|} \leq \frac{|x - y| + |y - h_k(y)|}{|x - y|} \leq 1 + \frac{\mathrm{diam}\, X_1}{m}$$

where

$$m = \inf\{|x - y|; x \in \mathbb{R}^n - \mathrm{Int}\, X,\ \ y \in X_1\}$$

(it is easy to show that $m > 0$). To conclude, denote

$$L = \max\{\lambda^{2|G|}, \frac{\mathrm{D}_X}{\mathrm{d}_X}, 1 + \frac{\mathrm{diam}\, X_1}{m}\}$$

Then

$$|h_k(x) - h_k(y)| \leq L|x - y|$$

for an arbitrary $k \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$.

The estimate

$$|h_k^{-1}(x) - h_k^{-1}(y)| \leq L|x - y|$$

can be proved analogously, using Lemma 3 (i). □

It follows immediately by Lemma 3 (iv) that the sequences of homeomorphisms $h_1, h_2, \ldots$ and $h_1^{-1}, h_2^{-1}, \ldots$ converge pointwisely at all points different from the point a and b, respectively. The convergence of the sequences at the point a and at the point b follows from Lemma 3, (ii). Denote the limits of the sequences by $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\tilde{h} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, respectively. It also follows from Lemma 3 that $h(a) = b$, that $h(|\mathcal{S}|) = |\mathcal{S}|$, and that $h \circ \tilde{h} = \tilde{h} \circ h = \mathrm{id}_{\mathbb{R}^n}$. It follows from Lemma 4 that h and $\tilde{h}$ are Lipschitz. Thus Lemma 1 is proved. □

4. Proofs of Theorems 2 and 3

Proof of Theorem 2: Set $G = \mathbb{Z}_2 = \{0, 1\}$ and consider the similarities

$$S_{(0)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ defined by } (x, y) \mapsto \frac{1}{3}(x, y)$$

and

$$S_{(1)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ defined by } (x, y) \mapsto (0, \frac{2}{3}) + \frac{1}{3}(x, y).$$

For X = circular disk $\{(x, y) | (x - \frac{1}{2})^2 + y^2 \leq (\frac{3}{4})^2\}$ and for the disks $X_{(0)} = S_{(0)}(X)$ and $X_{(1)} = S_{(1)}(X)$ the conditions

$$X_{(0)} \subset \mathrm{Int}(X), \quad X_{(1)} \subset \mathrm{Int}(X), \quad X_{(0)} \cap X_{(1)} = \emptyset$$

are satisfied (see figure 1). Hence $|\mathcal{S}| = |\{S_{(0)}, S_{(1)}\}|$ is the standard ternary Cantor set [5].

A diffeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ can be constructed such that

(i)
$$f|_{\mathbb{R}^2 - \mathrm{Int}(X)} = \mathrm{id}_{\mathbb{R}^2 - \mathrm{Int}(X)}$$

(ii)
$$f(X_g) = X_{g+1} \quad \text{for each } g \in \{0, 1\}$$

and the following diagram commutes

$$\begin{array}{ccc} & X & \\ S_g \swarrow & & \searrow S_{g+1} \\ X_g & \xrightarrow{f} & X_{g+1} \end{array}$$

To construct the diffeomorphism f, introduce the polar coordinate system with the point $(\frac{1}{2}, 0)$ as the center and with the x-axis as the polar axis. In polar coordinates r and ϕ the set X is given by the equation $r \leq \frac{3}{4}$. Take a smooth function $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\Phi(r) = \begin{cases} 0, & r \geq \frac{3}{4} \\ \pi, & r < \frac{5}{8} \end{cases}$$

and introduce a diffeomorphism $\tilde{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by:

$$\tilde{f}(r, \phi) = (r, \phi + \Phi(r)).$$

Then obviously,

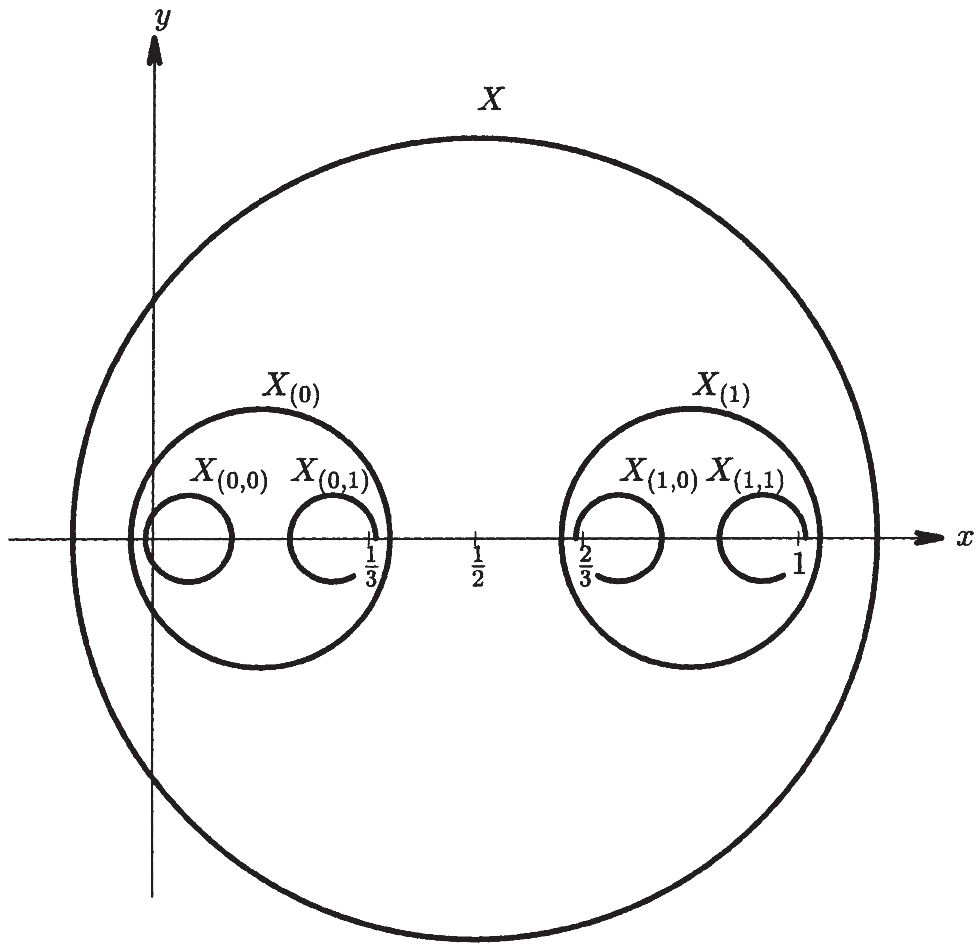

Figure 1. **First three steps in the construction of the ternary Cantor set**

(i)

$$\tilde{f}|_{\mathbb{R}^2-\mathrm{Int}(X)} = \mathrm{id}_{\mathbb{R}^2-\mathrm{Int}(X)}$$

(ii)

$$\tilde{f}(X_g) = X_{g+1} \quad \text{for each } g \in \{0,1\}.$$

However, the following diagram

$$\begin{array}{ccc} & X & \\ S_g \swarrow & & \searrow S_{g+1} \\ X_g & \xrightarrow{\tilde{f}} & X_{g+1} \end{array}$$

does not commute.

Introduce the homeomorphisms $\tilde{f}_{(0)}, \tilde{f}_{(1)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined as follows:

$$\tilde{f}_g = S_g \circ \tilde{f} \circ S_g^{-1}, \quad g \in G$$

Then the diffeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by:

$$f|_{\mathbb{R}^2 - X_{(0)} - X_{(1)}} = \tilde{f}|_{\mathbb{R}^2 - X_{(0)} - X_{(1)}}$$

and

$$f|_{X_g} = \tilde{f}_g \circ \tilde{f}|_{X_g}, \quad g \in G$$

satisfies all the conditions requested in Lemma 1. Hence, the standard ternary Cantor set is indeed Lipschitz homogeneous in $\mathbb{R}^2$.□

Proof of Theorem 3: Let n be an even number and let

$$G = \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$$

Let $\{c_g | g \in G\}$ be the vertices of a regular n-gon $C \subset \mathbb{R}^3$ having its centre at the origin O. Let $R_g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the rotation by angle $\frac{\pi}{2}$ about the axis of symmetry of n-gon C which is perpendicular to the vector c_g. Choose a positive number λ and for each $g \in G$ introduce a map $S_g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined for each $x \in \mathbb{R}^3$ as follows:

$$S_g : x \mapsto \begin{cases} c_g + \lambda x & , g \text{ even} \\ c_g + \lambda R_g(x) & , g \text{ odd} \end{cases}$$

Then S_g is a similarity with λ as the coefficient of similitude.

Let D be a circular disc D with center $c_{(0)}$, parallel to the vector $c_{(0)}$ and orthogonal to C. Let $X \subset \mathbb{R}^3$ be the solid torus obtained by rotation of the disk D about the axis through the origin and orthogonal to C. The diameter of D is called the *thickness* of the solid torus X. It can be verified by elementary calculations that for all sufficiently big numbers n, there exist values of λ - not too big and not too small - and corresponding thicknesses of X such that

(i) $X_g = S_g(X) \subset \text{Int}(X)$ for each $g \in G$ and
(ii) tori X_g are pairwise disjoint and mutually linked as shown in figure 2.

Hence $|\mathcal{S}| = |\{S_d | g \in G\}|$ is Cantor set in $\mathbb{R}^3$. Since it is an example of the Antoine necklace it is wild [9].

A diffeomorphism $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ can be constructed such that

(i)

$$f|_{\mathbb{R}^3 - \text{Int}(X)} = \text{id}_{\mathbb{R}^3 - \text{Int}(X)} \quad \text{and}$$

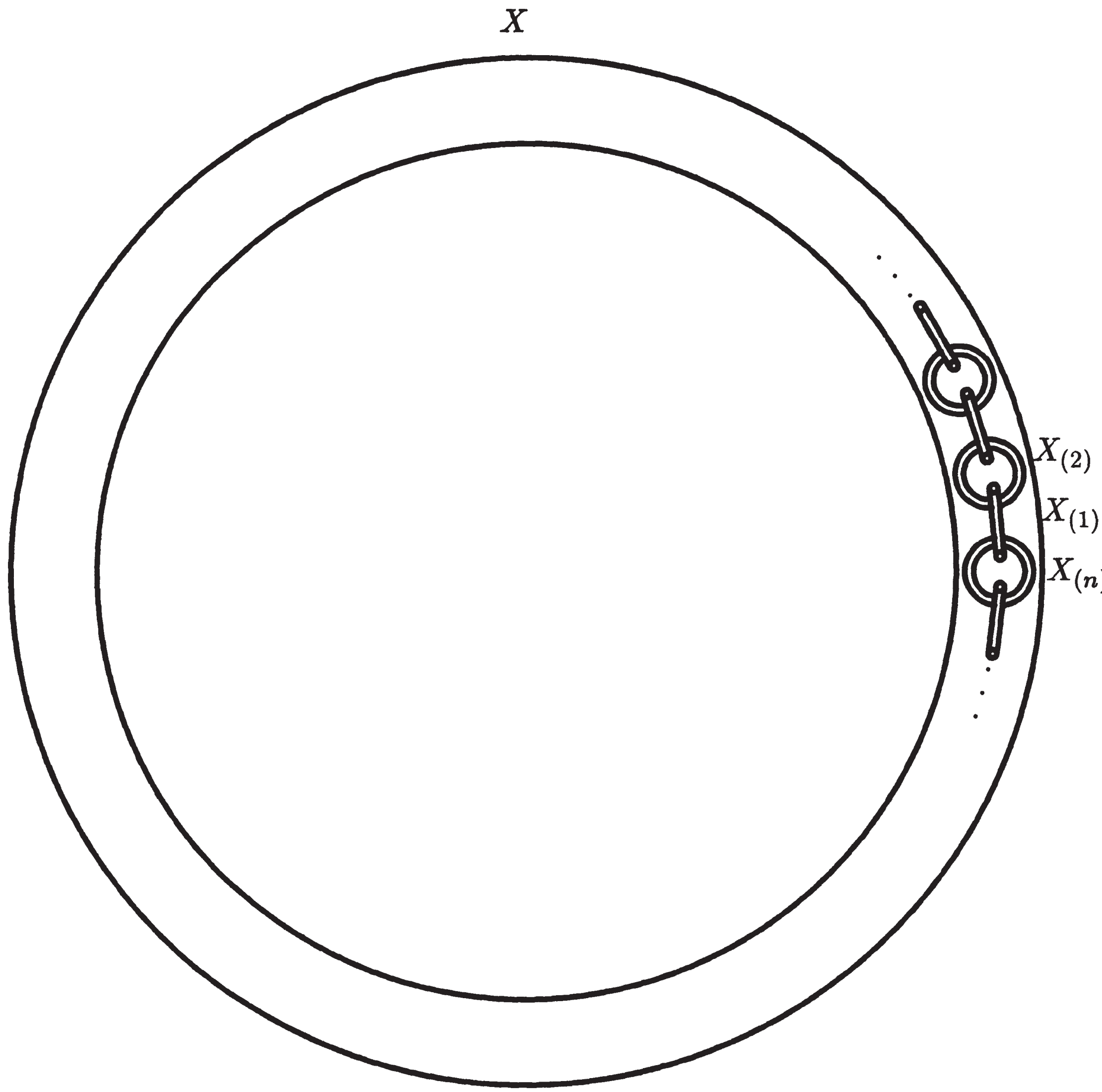

Figure 2. **First two steps in the construction of the Antoine necklace**

(ii)

$$f(X_g) = X_{g+1} \quad \text{for each } g \in G$$

and the following diagram commutes

$$\begin{array}{ccc} & X & \\ S_g \swarrow & & \searrow S_{g+1} \\ X_g & \xrightarrow{f} & X_{g+1} \end{array}$$

To construct the diffeomorphism f, introduce the torical coordinate system in the solid torus X, defined as follows. Let the point $x \in X$ come from a point $x_0 \in D$ by rotation by an angle ψ and let r and ϕ be polar coordinates of the point x_0. Then r, ϕ and ψ are torical coordinates of the point x. Suppose w.l.o.g. that $r = 1$ for the points lying on the boundary of the solid torus X. Choose a real number $0 < r_0 < 1$ such that $r < r_0$ for all points in the solid tori $X_g, \quad g \in G$. Take smooth functions $\Phi, \Psi : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\Phi(r) = \begin{cases} 0, & r \geq 1 \\ \frac{\pi}{2}, & r \leq r_0 \end{cases}$$

and

$$\Psi(r) = \begin{cases} 0, & r \geq 1 \\ \frac{\pi}{n}, & r \leq r_0. \end{cases}$$

Introduce the diffeomorphism $\tilde{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by:

$$\tilde{f}|_{\mathbb{R}^3 - X} = \mathrm{id}|_{\mathbb{R}^3 - X}$$

and

$$\tilde{f}(r, \phi, \psi) = (r, \phi + \Phi(r), \psi + \Psi(r))$$

where r, ϕ and ψ are toric coordinates of a point in X.

Similarly as in the proof of Theorem 2, but with much more labor, $\tilde{f}$ can be improved to obtain a diffeomorphism $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which satisfies the conditions:

(i)
$$f|_{\mathbb{R}^3 - \mathrm{Int}(X)} = \mathrm{id}_{\mathbb{R}^3 - \mathrm{Int}(X)} \quad \text{and}$$

(ii)
$$f(X_g) = X_{g+1} \quad \text{for each } g \in G$$

and the following diagram commutes

$$\begin{array}{ccc} & X & \\ S_g \swarrow & & \searrow S_{g+1} \\ X_g & \xrightarrow{f} & X_{g+1} \end{array}$$

Then by Lemma 1, the Antoine necklace $|S|$ is indeed Lipschitz homogeneous in $\mathbb{R}^3$. □

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