On Eversion of Spheres

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Abstract—The celebrated Smale–Hirsch classification of immersions allows one to obtain several nice applications of algebraic topology to differential topology. Unfortunately, these applications are still not presented in books or survey papers either in Russian or in English. The purpose of this paper is to expose the most simple and fundamental of these applications: the Smale–Kaiser theorem on the dimension of spheres that can be turned inside out, the Haefliger–Hirsch classification of immersions by means of equivariant maps, and its corollary concerning embeddings of highly connected manifolds (in particular, of spheres).

1. INTRODUCTION

The celebrated Smale–Hirsch classification of immersions allows one to obtain several nice applications of algebraic topology to differential topology. Unfortunately, these applications are still not presented in books or survey papers either in Russian or in English. The purpose of this paper is to expose the simplest and most fundamental of these applications: the Smale–Kaiser theorem on the dimension of spheres that can be turned inside out, the Haefliger–Hirsch classification of immersions by means of equivariant maps, and the latter's corollary on embeddings of highly connected manifolds (in particular, of spheres). Another interesting result is the main lemma in Section 3.

We are working in the smooth category. A mapping $f \colon N \to \mathbb{R}^m$ of the manifold N is said to be an *immersion* if $df(x) \neq 0$ for each point $x \in N$. Two immersions are said to be *regularly homotopic* if they are homotopic by the homotopy that is an immersion itself. The Smale theorem on immersions of spheres into Euclidean space reduces the classification of immersions to a homotopy problem (see also [3]).

Theorem 1.1 [21]. *There is a 1–1 correspondence between the set of immersions $S^n \to \mathbb{R}^m$ up to regular homotopy and the group $\pi_n(V_{mn})$.*

Note that $V_{n+1,n} = SO_{n+1}$. The 1–1 correspondence from Theorem 1.1 is explicitly constructed in the proof of the main lemma in Section 3. The celebrated corollary to Theorem 1.1 asserts that the two-dimensional sphere can be turned inside out in the three-dimensional Euclidean space. It was generalized by U. Kaiser in his diploma thesis written under the supervision of Prof. U. Koschorke.

Theorem 1.2 [11; 13, Problem 4.61]. *The sphere S^n can be turned inside out (more exactly, the standard embedding of S^n into the Euclidean space $\mathbb{R}^{n+1}$ is regularly homotopic to the composition of the standard embedding and of the reflection with respect to an $(n-1)$-dimensional hyperplane) if and only if $n \in \{0, 2, 6\}$.*

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This result is related to the parallelizability of spheres S^1, S^3, and S^7 [2].

Hirsch generalized Theorem 1.1 to the case of arbitrary manifolds.

Theorem 1.3 [9]. *Let N be a smooth n-dimensional manifold and $m \geq n + 1$. Then, the mapping $f \mapsto df$ induces a 1–1 correspondence between the set of immersions $N \to \mathbb{R}^m$ up to regular homotopy and the set of linear monomorphisms $\Phi \colon TN \to \mathbb{R}^m$ up to the homotopy in the class of linear monomorphisms.*

In the metastable dimension range, Haefliger and Hirsch obtained the following description of the set of immersions. Let $O\Delta$ be a neighborhood of the diagonal set Δ in the product $N \times N$. Denote $SN = O\Delta - \Delta$. If N is a polyhedron, then the equivariant homotopy type of the space SN does not depend on $O\Delta$. For an immersion $h \colon N \to \mathbb{R}^m$, the equivariant mapping $\widetilde{h} \colon SN \to S^{m-1}$ is well defined by the formula $\widetilde{h}(x, y) = \frac{hx - hy}{|hx - hy|}$. If immersions h_0 and h_1 are regularly homotopic, then it is clear that $\widetilde{h}_0 \simeq_{\mathrm{eq}} \widetilde{h}_1$. A pair $(N, \partial N)$ is said to be *homologically k-connected* if $H_i(N, \partial N) = 0$ for each $i = 0, \ldots, k$.

Theorem 1.4 [6]. *Let N be a smooth n-dimensional manifold.*

(a) *Suppose that either $m \geq \frac{3n+1}{2}$ or the pair $(N, \partial N)$ is homologically $(3n - 2m)$-connected. If there exists an equivariant mapping $\Phi \colon SN \to S^{m-1}$, then there exists an immersion $h \colon N \to \mathbb{R}^m$ such that $\widetilde{h} \simeq_{\mathrm{eq}} \Phi$.*

(b) *Suppose that either $m \geq \frac{3n}{2} + 1$ or the pair $(N, \partial N)$ is homologically $(3n - 2m + 1)$-connected. If a pair of immersions $h_0, h_1 \colon N \to \mathbb{R}^m$ satisfies $\widetilde{h}_0 \simeq_{\mathrm{eq}} \widetilde{h}_1$, then h_0 and h_1 are regularly homotopic.*

It follows from Theorem 1.4(b) that, for $m \geq \frac{3n}{2} + 1$, each embedding $S^n \to S^m$ is regularly homotopic to the standard embedding (hence, its normal bundle is trivial). This result was obtained originally by Kervaire with different methods [12]. The restriction on dimensions $m \geq \frac{3n}{2} + 1$ in this result (and therefore in Theorem 1.4(b)) cannot be removed: this result is false for $n = 4l - 1$ and each $m = 4l + 2, 4l + 3, \ldots, 6l - 1$ [5, 6.8]. More generally, if N is a closed homologically d-connected n-manifold ($d \leq \frac{n}{2}$), then, for $m \geq 2n - d + 1$, any two *embeddings* $N \to \mathbb{R}^m$ are regularly homotopic (and therefore have equal normal bundles). These normal bundles may be nontrivial even for a 16-dimensional homotopic sphere N and $m = 29$ [16]. The restriction on dimensions $m \geq 2n - d + 1$ in the Kervaire result cannot be removed essentially since, for each odd $n \neq 3, 5, 9$, there exists an embedding $S^2 \times S^{n-2} \to \mathbb{R}^{2n-2}$ that has a nontrivial normal bundle [15].

Theorem 1.4 can also be applied to the proof of the famous Haefliger theorem on embeddings [4]. Analogues of Theorems 1.1, 1.3, and 1.4 for the case of piecewise linear immersions of polyhedra and piecewise linear manifolds are proved in [7, 8, 19]. For the piecewise linear version of the Haefliger theorem, see [23, 8, 19, 20, 22, 18].

In Section 2, Theorem 1.4 is proved and proofs of Theorems 1.1 and 1.3 are sketched. In Section 3, Theorem 1.2 is proved. The proof of Theorem 1.4 is based on Theorem 1.3 and on the Freudenthal suspension theorem.

Conjecture 1.5. *For certain $n \not\equiv -1 \mod 4$ and for closed manifolds, the conditions on dimensions in Theorems $1.4(a)$ and $1.4(b)$ can be weakened using the strengthened Freudenthal suspension theorem ("the hard part" of Whitehead, the James theorem on double suspension, and the ENR-sequence [10]).*

2. PROOFS OF THE SMALE–HIRSCH AND THE HAEFLIGER–HIRSCH THEOREMS

Sketch of the proof of Theorem 1.1 [17]. Let $m > n + s$. Denote by X_{ns}^m the space of framed immersions $D^n \to \mathbb{R}^m$ such that, in a neighborhood of a given point of the boundary ∂D^n, both the immersion and the framing coincide with the standard ones. Obviously, the space X_{ns}^m is contractible. Introduce the space Y_{ns}^m by analogy with X_{ns}^m, by changing D^n with S^n. Consider the

mapping $X_{ns}^m \to Y_{n-1,s+1}^m$ that takes an immersion of a disk D to the restriction of that immersion onto the boundary ∂D, completing the framing of immersion by a vector field oriented from the boundary of D into its interior. The fibers of this mapping are homeomorphic to Y_{ns}^m. This mapping is a Serre fibration (this fact is nontrivial, see [3]). The exact homotopy sequence of this Serre fibration implies that $\pi_j(Y_{ns}^m) \cong \pi_{j+1}(Y_{n-1,s+1}^m)$. Hence, $\pi_0(Y_{n0}^m) \cong \pi_n(Y_{0n}^m) \cong \pi_n(V_{mn})$. Theorem 1.1 is proved. $\square$

Theorem 1.3 can be proved by induction on the number of handles of the handlebody decomposition of the manifold N. To this end, a relative version of the Smale theorem should be applied.

Proof of Theorem 1.4 [6]. Introduce the *equivariant Stiefel manifold* V_{mn}^{eq} as the space of equivariant mappings $S^{n-1} \to S^{m-1}$ with respect to the antipodal involution. A bundle ϕ with fibers V_{mn}^{eq} over N can be associated with the tangential bundle TN. There is a 1–1 correspondence between the sections of the bundle ϕ up to fiber-preserving homotopy and the equivariant mappings $\Phi\colon SN \to S^{m-1}$ up to equivariant homotopy. By Theorem 1.3, there is a 1–1 correspondence between immersions $N \to \mathbb{R}^m$ up to regular homotopy and sections of the V_{mn}-subbundle of the bundle ϕ. The obstructions to a deformation of sections of the V_{mn}^{eq}-bundle ϕ to sections of the V_{mn}-subbundle are elements of

$$H^i(N, \pi_i(V_{mn}^{\mathrm{eq}}, V_{mn})_T) \cong H_{n-i}(N, \partial N, \pi_i(V_{mn}^{\mathrm{eq}}, V_{mn}))$$

(the coefficients are twisted corresponding to the orienting double cover of N; it is possible that $\partial N = \varnothing$). By the universal coefficient theorem, it suffices to prove that $\pi_i(V_{mn}^{\mathrm{eq}}, V_{mn}) = 0$ for $0 \le i \le 2(m-n)-1$. The last fact follows from Lemma 2.1 and from the exact homotopy sequence of the pair $V_{mn}^{\mathrm{eq}} \supset V_{mn}$. $\square$

Lemma 2.1 [6, (1.1)]. *The homomorphism $\rho_i\colon \pi_i(V_{mn}) \to \pi_i(V_{mn}^{\mathrm{eq}})$ induced by the inclusion is an isomorphism for $0 \le i \le 2(m-n)-2$ and is an epimorphism for $i = 2(m-n)-1$.*

Proof [6, (1.1)]. Apply induction on n. For $n = 1$, the assertion of the lemma is true since $V_{m1} \cong V_{m1}^{\mathrm{eq}} \cong S^{m-1}$. Now, let $n \ge 2$. It is known that there exists a Serre bundle $S^{m-n} \to V_{mn} \xrightarrow{r} V_{m,n-1}$, where r stands for the restriction map. It is easy to verify that there exists a Serre bundle $\Omega_{n-1}S^{m-1} \to V_{mn}^{\mathrm{eq}} \xrightarrow{r_2} V_{m,n-1}^{\mathrm{eq}}$, where r_2 stands for the restriction map (it can be proved by induction on n that the fiber of the mapping r_2 is homeomorphic to $\Omega_{n-1}S^{m-1}$). It is clear that $r\rho_n = \rho_{n-1}r_2$ and that the homomorphism

$$\pi_i(S^{m-n}) \xrightarrow{\Sigma^{n-1}} \pi_{i+n-1}(S^{m-1}) \cong \pi_i(\Omega_{n-1}S^{m-1})$$

is induced by ρ_n. Hence, the mapping ρ_n induces homomorphisms between the following exact homotopy sequences:

$$
\begin{array}{ccccccccc}
\pi_{i+1}(V_{m,n-1}) & \longrightarrow & \pi_i(S^{m-n}) & \longrightarrow & \pi_i(V_{mn}) & \longrightarrow & \pi_i(V_{m,n-1}) & \longrightarrow & \pi_{i-1}(S^{m-n}) \\
\downarrow{\scriptstyle \rho_{n-1}} & & \downarrow{\scriptstyle \Sigma^{n-1}} & & \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_{n-1}} & & \downarrow{\scriptstyle \Sigma^{n-1}} \\
\pi_{i+1}(V_{m,n-1}^{\mathrm{eq}}) & \longrightarrow & \pi_{i+n-1}(S^{m-1}) & \longrightarrow & \pi_i(V_{mn}^{\mathrm{eq}}) & \longrightarrow & \pi_i(V_{m,n-1}^{\mathrm{eq}}) & \longrightarrow & \pi_{i+n-2}(S^{m-1})
\end{array}
$$

Suppose first that $i \ge 1$. ρ_{n-1} is an isomorphism by the inductive assumption. Σ^{n-1} is an isomorphism by the Freudenthal suspension theorem. For $i \le 2(m-n)-2$, ρ_n is an isomorphism by the five-lemma; analogously, ρ_n is an isomorphism for $i = 2(m-n)-1$. For $i = 0$, the proof is the same except that the right column consists of zeroes since r and r_2 are mappings onto. $\square$

Note that, in [6], the roles of m and n are interchanged. There are some misprints in [6]: when expressing condition (1.1_{n-1}) and when proving assertion (1.2), the equality $\pi_i(V_{mn}^{\mathrm{eq}}, V_{mn}) = 0$ should hold for $0 \le i \le 2(m-n)-1$ rather than only for $0 < i < 2(m-n)-1$ as in [6].

3. THE PROOF OF THE SMALE–KAISER THEOREM

Proof that the spheres S^0, S^2, and S^6 can be turned inside out. By the Smale Theorem 1.1, it suffices to prove that $\pi_0(SO_1) = \pi_2(SO_3) = \pi_6(SO_7) = 0$. First, $\pi_0(SO_1) = \pi_0(\mathrm{pt}) = 0$. Furthermore, $\pi_2(SO_3) = \pi_2(\mathbb{R}\mathrm{P}^3) = \pi_2(S^3) = 0$. Note that it is obvious how to turn S^0 inside out in $\mathbb{R}^1$, but the explicit turning of S^2 inside out in $\mathbb{R}^3$ is far from being trivial. To prove that $\pi_6(SO_7) = 0$, consider the standard bundle $SO_7 \to S^6$ with the fiber SO_6. Consider a segment of the homotopy exact sequence corresponding to this bundle:

$$\pi_6(SO_6) \to \pi_6(SO_7) \to \pi_6(S^6) \to \pi_5(SO_6) \to \pi_5(SO_7).$$

The unitary Bott periodicity and $\mathrm{Spin}_6 = SU_4$ [2] imply

$$\pi_6(SO_6) = \pi_6(\mathrm{Spin}_6) = \pi_6(SU_4) = \pi_6(U_4) = \pi_6(U) = 0.$$

Furthermore, $\pi_5(SO_6) = \pi_5(U) = \mathbb{Z}$. By the orthogonal Bott periodicity, $\pi_5(SO_7) = 0$. Since $\pi_6(S^6) = \mathbb{Z}$, we obtain the exact sequence $0 \to \pi_6(SO_7) \to \mathbb{Z} \to \mathbb{Z} \to 0$; hence, $\pi_6(SO_7) = 0$. Note that an explicit turning of S^6 inside out in $\mathbb{R}^7$ is unknown.   □

For the sake of completeness, let us state the results on $\pi_n(SO_{n+1})$ (in other words, on the groups of immersions $S^n \to \mathbb{R}^{n+1}$) for some small values of n:

$$\pi_3(SO_4) = \pi_3(S^3 \times \mathbb{R}\mathrm{P}^3) = \mathbb{Z} \oplus \mathbb{Z}, \qquad \pi_4(SO_5) = \pi_4(\mathrm{Spin}_5) = \pi_4(Sp_2) = \pi_4(S^3) = \mathbb{Z}_2,$$

$$\pi_5(SO_6) = \pi_5(\mathrm{Spin}_6) = \pi_5(SU_4) = \pi_5(U_4) = \pi_5(U) = \mathbb{Z}.$$

All these calculations are well known. Since $\pi_4(SO_5) = \mathbb{Z}_2$, each immersion $S^4 \to \mathbb{R}^5$ is regularly homotopic to either the standard embedding or the standard embedding composed with a reflection. The results on $\pi_n(SO_{n+1})$ for arbitrary values of n are unknown (it is known that all of them are unstable).

Now, prove that the sphere S^n cannot be turned inside out for $n \neq 0, 2, 6$: three different proofs will be given. In the first proof, the Smale Theorem 1.1 is not applied; in the second and the third ones, only the trivial part of that theorem is applied, namely, that if immersions of spheres are regularly homotopic, then the corresponding spheroids are homotopic.

An idea of the first proof that the sphere S^n cannot be turned inside out for $n \neq 0, 2, 6$. Consider a regular homotopy $F \colon S^n \times I \to \mathbb{R}^{n+1} \times I$ between the standard embedding and the standard embedding composed with the reflection with respect to a hyperplane. Take a unit normal vector field v corresponding to the immersion F such that $v(x, 0) \in \mathbb{R}^n \times 0$ is oriented outward from the sphere $F(S^n \times 0) \subset \mathbb{R}^n \times 0$ and $v(x, 1) = -v(x, 0) \in \mathbb{R}^n \times 1$ for arbitrary $x \in S^n$. Now, we have

$$S^{n+1} = D_+^{n+1} \bigcup_{\partial D_+^{n+1} = S^n \times 1} S^n \times I \bigcup_{\partial D_-^{n+1} = S^n \times 0} D_-^{n+1}.$$

Extend F over $D_+^{n+1} \cup D_-^{n+1}$ to get the immersion $F \colon S^{n+1} \to \mathbb{R}^{n+1}$ (symmetrically over D_+^{n+1} and D_-^{n+1}) so that the field v is extended over $D_-^{n+1} \cup D_+^{n+1}$ (symmetrically over D_+^{n+1} and D_-^{n+1}) and the vector $v(y)$ is not oriented upward for any $y \in S^{n+1}$. Consider the Gauss mapping $G \colon S^{n+1} \to S^{n+1}$ corresponding to the immersion $F \colon S^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}$. The range of G is a subset of $\mathbb{R}^{n+1} = S^{n+1} \setminus$ (the north pole). Take a set of $n + 1$ linearly independent tangential vector fields on $\mathbb{R}^{n+1}$. For each point $x \in S^{n+1}$, we obtain a set of $n + 1$ linearly independent vectors at the point $G(x) \in \mathbb{R}^{n+1}$. These vectors are tangential since G is a Gauss mapping. Hence, the sphere S^{n+1} is parallelizable; therefore, by the Adams theorem on the Hopf invariant, $n = 0, 2, 6$ □

The degree of a normal (Gauss) mapping distinguishes the standard embedded sphere from the reflected one; therefore, this degree is an obstruction to turning the spheres inside out. The

development of this idea serves as a second proof that the sphere S^n cannot be turned inside out for $n \neq 0, 2, 6$.

Main lemma. *Consider the standard bundle $SO_{n+2} \to S^{n+1}$ (that assigns the last column to a matrix) with fiber SO_{n+1} and consider a segment of the exact homotopy sequence corresponding to this bundle (for simplicity, the base points are neglected):*

$$\pi_{n+1}(SO_{n+2}) \xrightarrow{\alpha} \pi_{n+1}(S^{n+1}) \xrightarrow{\beta} \pi_n(SO_{n+1}). \tag{1}$$

Then, the element of the group $\pi_n(SO_{n+1})$ corresponding by the Smale Theorem 1.1 to the standard embedding $i\colon S^n \to \mathbb{R}^{n+1}$ is the zero element. The element of the group $\pi_n(SO_{n+1})$ corresponding to the standard embedding composed with the reflection with respect to a hyperplane is the element $\beta(1) \in \pi_n(SO_{n+1})$, where 1 denotes the generator of the group $\pi_{n+1}(S^{n+1}) \cong \mathbb{Z}$.

The second proof that the sphere S^n cannot be turned inside out for $n \neq 0, 2, 6$. For α entering the exact sequence (1), it is true that $1 \in \operatorname{im}\alpha$ if and only if the sphere S^{n+1} is parallelizable. Since the sequence is exact, it follows from the Adams theorem on the Hopf invariant that $\beta(1) = 0$ if and only if n differs from 0, 2, and 6. By the Main lemma, the sphere S^n cannot be turned inside out for $n \neq 0, 2, 6$.

Let us prove the above assertion concerning $\operatorname{im}\alpha$ and the parallelizability [1]. If the sphere S^{n+1} is parallelizable, one can choose an orthonormal frame in the tangent space at each point $x \in S^{n+1}$ such that it depends on x continuously. Extend this frame to an orthonormal frame in the tangent space of $\mathbb{R}^{n+2}$ by adding the normal vector to the sphere as the last vector. This construction gives a mapping $S^{n+1} \to SO_{n+2}$ such that, composed by the natural bundle mapping $SO_{n+2} \to S^{n+1}$, it results in the identical mapping $S^{n+1} \to S^{n+1}$. Now, suppose that $1 \in \operatorname{im}\alpha$. Take a mapping $h\colon S^{n+1} \to SO_{n+2}$ such that the composition $S^{n+1} \xrightarrow{h} SO_{n+2} \to S^{n+1}$ has degree 1. By the covering homotopy theorem, the mapping h is homotopic to a mapping $\widehat{h}$ such that the composition $S^{n+1} \xrightarrow{\widehat{h}} SO_{n+2} \to S^{n+1}$ is the identity mapping. Hence, the first $n+1$ columns in the matrix $\widehat{h}(x)$ give an orthonormal frame in $T_x S^{n+1}$. $\square$

Proof of the Main lemma. First, for each immersion $f\colon S^n \to \mathbb{R}^{n+1}$, construct a spheroid $\sigma_f\colon S^n \to SO_{n+1}$ such that the spheroids corresponding to regularly homotopic immersions are homotopic. Consider the standard sphere $S^n \subset \mathbb{R}^{n+1}$ given by the equation $\sum x_i^2 = 1$. The direct sum $TS^n \oplus \nu$ of its tangent bundle TS^n and its normal bundle ν is a trivial bundle. Fix a trivialization of the tangent bundle $T\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ taking vectors parallel to the coordinate vectors as an oriented base in the tangent space. The obtained trivialization of $T\mathbb{R}^{n+1}$ induces a trivialization of $TS^n \oplus \nu \to S^n$. Denote by $e(x)$ the unit normal vector at the point $x \in S^n$ and by $\mathbb{R}^{n+1}(x)$ the fiber of the bundle $TS^n \oplus \nu \to S^n$ at the point $x \in S^n$.

Take an immersion $f\colon S^n \to \mathbb{R}^{n+1}$. Extend the differential $df(x)\colon T_x S^n \to T_{f(x)}\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$ to a linear operator $\Sigma_f(x)\colon \mathbb{R}^{n+1}(x) \to \mathbb{R}^{n+1}$ by the following construction. Let $\Sigma_f(x)$ coincide with the differential $df(x)$ on the subspace $T_x S^n \subset \mathbb{R}^{n+1}(x)$. Let $\Sigma_f(x)$ map the vector $e(x)$ to a vector that is orthogonal to $df(x)$ and is such that the determinant corresponding to the operator $\Sigma_f(x)$ equals 1. The operator $\Sigma_f(x)$ is well defined since the range of the differential $df(x)$ is a hyperplane (this follows from the fact that f is an immersion). The determinant of the operator $\Sigma_f(x)$ is determined uniquely since the trivialization is fixed. Since $\Sigma_f(x) \in SL_{n+1}$, we have a mapping $\Sigma_f\colon S^n \to SL_{n+1}$ that takes a point $x \in S^n$ into an operator $\Sigma_f(x)$. Now, there is a natural retraction $SL_{n+1} \to SO_{n+1}$ induced by the Gram–Schmidt orthogonalization (each operator in SL_{n+1} is factorized uniquely into a product of an upper triangular operator having positive diagonal elements with an operator in SO_{n+1}). Denote by σ_f the composition $S^n \to SL_{n+1} \to SO_{n+1}$.

Now, the Smale Theorem 1.1 can be formulated more precisely. Immersions $f, g\colon S^n \to \mathbb{R}^{n+1}$ are regularly homotopic if and only if the corresponding spheroids σ_f and σ_g (Σ_f and Σ_g) are homotopic.

Let $i\colon S^n \to \mathbb{R}^{n+1}$ be the standard embedding. Then, the corresponding σ_i is the constant mapping that takes an arbitrary point on the sphere to the unit matrix in SO_{n+1}. (Note that, in the case $n = 1$, the spheroid σ_f corresponding to an immersion f with the index equal to k is homotopic to the $(k-1)$-multiple of the generator of the group $\pi_1(SO_2) = \mathbb{Z}$.)

Now, let H be a hyperplane in $\mathbb{R}^{n+1}$ that passes through the origin. Denote by S the reflection with respect to H. Investigate the spheroid $\sigma = \sigma_{S\circ i}$ corresponding to the embedding $S \circ i$. Note that the fibers of the bundle $TS^n \oplus \nu \to S^n$, as well as the fibers of the tangent bundle to $\mathbb{R}^{n+1}$, can be identified with $\mathbb{R}^{n+1}$.

Assertion. *The operator $\sigma(x)$ maps a vector v into the vector Sv when v is tangential at the point $x \in S^n$ and maps a vector u into the vector $-Su$ when u is normal:*

$$(v \in T_x S^n) \xrightarrow{\sigma(x)} Sv, \qquad (u \perp T_x S^n) \xrightarrow{\sigma(x)} -Su. \qquad (*)$$

Proof. Since $S \circ i$ is an isometry, $\sigma(x)$ coincides with $\Sigma_{S\circ i}(x)$; hence, $\sigma(x)$ acts as $d(S \circ i)(x)$ on the tangent space $T_x S^n$. Furthermore, $d(S \circ i)(x)$ coincides with S on $T_x S^n$. Since the determinant of the operator $\Sigma_{S\circ i}(x)$ equals 1, a vector u, which is perpendicular to $T_x S^n$, is mapped to $-Su$ (each reflection changes the orientation). $\square$

Proof of the Main lemma continued. Consider the space $\mathbb{R}^{n+2}$ equipped with orthogonal coordinates $(x_1, \ldots, x_{n+2})$. Let the subspace $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ be defined by the equation $x_{n+2} = 0$. Introduce the hyperplane Γ defined by the equation $x_1 = 0$ and denote by H the intersection $\Gamma \cap \mathbb{R}^{n+1}$. Denote by $\widehat{S}$ the reflection of the space $\mathbb{R}^{n+2}$ with respect to the hyperplane Γ, and denote by S the reflection of the space $\mathbb{R}^{n+1}$ with respect to the hyperplane H. Let S^{n+1} and S^n be unit spheres in the spaces $\mathbb{R}^{n+2}$ and $\mathbb{R}^{n+1}$, respectively (see figure).

Identify the space SO_{n+1} with the preimage of the vector $(0, \ldots, 0, 1)$ under the bundle projection $SO_{n+2} \to S^{n+1}$. Therefore, we can assume that the spheroid $\sigma = \sigma_{S\circ i}$ lies in the space SO_{n+2}.

Consider the spheroid $\sigma_{\widehat{S}\circ i}\colon S^{n+1} \to SO_{n+2}$ corresponding to the immersion $\widehat{S}\circ i\colon S^{n+1} \to \mathbb{R}^{n+2}$. The restriction of $\sigma_{\widehat{S}\circ i}$ onto S^n coincides with σ. Indeed, the restriction of $\widehat{S}$ onto the subspace $\mathbb{R}^{n+1}$ coincides with S, and a vector normal to S^n in the space $\mathbb{R}^{n+1}$ is a normal vector to S^{n+1}. Therefore, the operators $\sigma(x) \in SO_{n+2}$ and $\sigma_{\widehat{S}\circ i}(x) \in SO_{n+2}$ coincide for all $x \in S^n$; they are defined by the same formula $(*)$ (note that $\sigma_{\widehat{S}\circ i}(x)$ maps the vector $(0, \ldots, 0, 1)$ to itself).

The sphere S^{n+1} and the half-space $x_{n+2} \geq 0$ intersect in a closed disk D^{n+1}. Consider the restriction $F\colon D^{n+1} \to SO_{n+2}$ of the mapping $\sigma_{\widehat{S}\circ i}\colon S^{n+1} \to SO_{n+2}$ on the disk D^{n+1}. It coincides
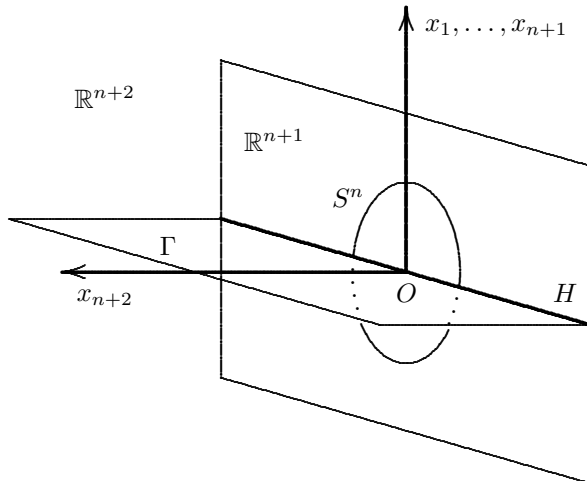
Figure

with σ on the boundary $\partial D^{n+1} = S^n$. The mapping F, composed with the bundle projection $SO_{n+2} \to S^{n+1}$, maps the whole sphere $\partial D^{n+1} = S^n$ to the point $(0,\ldots,0,1) \in S^{n+1}$. To prove the lemma, it suffices to prove that the mapping $F\colon (D^{n+1}, S^n) \to (S^{n+1}, (0,\ldots,0,1))$ provides a generator of the group $\pi_{n+1}(S^{n+1}, (0,\ldots,0,1))$. Indeed, in this case, the group element corresponding to the spheroid σ is equal to the image of the generator of the group $\pi_{n+1}(S^{n+1}, (0,\ldots,0,1))$ by the homomorphism β [2]. To prove that F provides a generator of the group $\pi_{n+1}(S^{n+1}, (0,\ldots,0,1))$, it suffices to prove that the degree of the mapping F equals ± 1.

To this end, it suffices to prove that F maps bijectively $D^{n+1} \setminus S^n$ onto $S^{n+1} \setminus \{(0,\ldots,0,1)\}$. Calculate the image $F(x)$. Taking $v = (0,\ldots,0,1)$ in $(*)$, we obtain

$$F(x) = \sigma_{\widehat{S}\circ i}(x)v = -\widehat{S}\big(\langle v, e(x)\rangle e(x)\big) + \widehat{S}\big(v - (\langle v, e(x)\rangle e(x))\big) = \widehat{S}\big(v - 2(\langle v, e(x)\rangle e(x))\big),$$

where $e(x) = x$ denotes the unit normal vector to S^{n+1} at the point x. In our coordinates, we obtain

$$F(x_1,\ldots,x_{n+2}) = (-2x_{n+2}x_1, -2x_{n+2}x_2, \ldots, -2x_{n+2}x_{n+1}, 1 - 2x_{n+2}^2).$$

The equation

$$(-2x_{n+2}x_1, -2x_{n+2}x_2, \ldots, -2x_{n+2}x_{n+1}, 1 - 2x_{n+2}^2) = (y_1,\ldots,y_{n+2})$$

has obviously a unique solution lying in the half-space $x_{n+2} > 0$ for given $(y_1,\ldots,y_{n+2}) \in S^{n+1} \setminus \{(0,\ldots,0,1)\}$. This solution lies in the disk D^{n+1}. Indeed, putting $x_i^2 = \frac{y_i^2}{4x_{n+2}^2}$ for $i = 1,\ldots,n+1$ and $x_{n+2}^2 = \frac{1-y_{n+2}}{2}$, we get

$$x_1^2 + \ldots + x_{n+2}^2 = \frac{y_1^2}{2(1 - y_{n+2})} + \ldots + \frac{y_{n+1}^2}{2(1 - y_{n+2})} + \frac{1 - y_{n+2}}{2} = \frac{1 - y_{n+2}^2}{2(1 - y_{n+2})} + \frac{1 - y_{n+2}}{2} = 1.$$

We proved that F maps $D^{n+1} \setminus S^n$ bijectively onto $S^{n+1} \setminus \{(0,\ldots,0,1)\}$. Hence, by the Sard theorem, there exists a point having exactly one regular preimage (moreover, the restriction of F on the set $D^{n+1} \setminus S^n$ is a diffeomorphism; however, it takes more effort to verify this fact).   □

It can be proved that the degree of the mapping $S^n \xrightarrow{\sigma} SO_{n+1} \to S^n$ equals 0 for n even and equals ± 2 for n odd (see the remark before the Main lemma).

Idea of the third proof that the sphere S^n cannot be turned inside out for $n \neq 0, 2, 6$. Let $\Sigma\colon \pi_{2n+1}(S^{n+1}) \to \pi_{2n+2}(S^{n+2})$ be the suspension homomorphism. Denote by $*$ the north pole of the sphere S^{n+1}. Let F_{n+1} be the space $\{f\colon S^{n+1} = S^{n+1} \mid f(*) = *\}$ and let $\eta\colon \pi_n(F_{n+1}) \to \pi_{2n+1}(S^{n+1})$ be the Hurewicz isomorphism [10]. It can be proved analogously [10, §3] that a Hopf homomorphism $H_1\colon \ker\Sigma \to \pi_n(SO_{n+1}) \cong \pi_n(SO_{n+1}, SO_1)$ can be defined such that $\eta \circ \iota_{n+1} \circ H_1 = \mathrm{id}$; hence, H_1 is a monomorphism. Applying the geometric interpretation of the ENR-sequences [14], one can prove that $H_1[\iota_{n+1}, \iota_{n+1}] = \sigma$. Therefore, $\sigma \neq 0$ if and only if $[\iota_{n+1}, \iota_{n+1}] \neq 0$. By a corollary to the Adams theorem, this is valid if and only if $n = 0, 2, 6$.   □

ACKNOWLEDGMENTS

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Translated by the authors