Mathematical Notes, vol. 69, no. 1, 2001, pp. 46–64.
Translated from Matematicheskie Zametki, vol. 69, no. 1, 2001, pp. 52–73.
Original Russian Text Copyright ©2001 by Malešič, Muranov, Repovš

Splitting Obstruction Groups in Codimension 2

J. Malešič, Yu. V. Muranov, and D. Repovš

Received November 4, 1998; in final form, December 1, 1999

Abstract—The splitting obstruction groups depend functorially on the square of fundamental groups. In the paper the problem of splitting along a submanifold of codimension two under some restrictions on the square of fundamental groups is considered. New exact sequences and commutative diagrams containing Wall groups, splitting obstruction groups, and surgery obstruction groups for manifold pairs are obtained. Examples of computation of splitting obstruction groups and natural maps are presented.

Key words: *surgery of manifolds, splitting obstruction groups, Wall groups, submanifolds of codimension 2.*

1. INTRODUCTION

The splitting obstruction groups $LS_{n-q}(F)$ arise naturally in the problem of doing surgery on a submanifold $N \subset M$ of codimension q inside the n-dimensional manifold M. If the codimension of the submanifold N is greater than 2, then the groups $LS_{n-q}(F)$ do not depend on the manifold M and coincide with the abstract surgery obstruction groups $L_{n-q}(\pi_1(N))$, where $\pi_1(N)$ is the fundamental group of the submanifold N, equipped with an orientation homomorphism $w:\pi_1(N)\to\{\pm1\}$ (see [1] and [2]).

Consider a simple homotopy equivalence $f: M\to Y$ of the manifold M and an n-dimensional geometric Poincaré complex Y with a subcomplex X of codimension q. The corresponding problem of splitting the map f along X is to deform f up to homotopy into a map g transversal to X for which the restrictions

$$g|_N : N\to X, \qquad g|_{M\setminus N} : (M\setminus N)\to (Y\setminus X), N = g^{-1}(X)$$

are simple homotopy equivalences. The obstruction to splitting $\sigma(f,Y)$ lies in $LS_{n-q}(F)$, which is a group depending functorially on a pushout square F of fundamental groups with orientations

$$F = \begin{pmatrix} \pi_1(\partial U) & \longrightarrow & \pi_1(Y\setminus X) \\ \downarrow & & \downarrow \\ \pi_1(X) & \longrightarrow & \pi_1(Y) \end{pmatrix}, \tag{1.1}$$

where ∂U is the total space of the normal spherical fibration of X in Y. In addition, the groups $LS_{n-q}(F)$ are 4-periodic, i.e., $n-q$ may be regarded as equal to $0,1,2,3 \bmod 4$ ([1] and [2]). For convenience, we denote the groups with orientations entering in the square F in the following way:

$$F = \begin{pmatrix} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{pmatrix}. \tag{1.2}$$

The groups $LS_*(F)$ fit into the following commutative diagrams of exact sequences (see [1] and [2]):

 0001-4346/2001/6912-0046$25.00 ©2001 Plenum Publishing Corporation

$$\begin{array}{ccccccccccc}
\to & L_{n+q}(C) & & \longrightarrow & & L_{n+q}(D) & & \longrightarrow & & LS_{n-1}(F) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & LP_n(F) & & & & L_{n+q}(C \to D) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & LS_n(F) & & \longrightarrow & & L_n(B) & & \longrightarrow & & L_{n+q-1}(C) & \to
\end{array}, \tag{1.3}$$

$$\begin{array}{ccccccccccc}
\to L_{n+1}(B) & & \longrightarrow & & L_{n+q+1}(C \to D) & & \longrightarrow & & L_{n+q+1}(F) \to \\
\nearrow \quad \searrow & & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & L_{n+q+1}(A \to B) & & & & LS_n(F) & & & \\
\searrow \quad \nearrow & & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to L_{n+q+2}(F) & & \longrightarrow & & LS_n(\Psi) & & \longrightarrow & & L_n(B) \to
\end{array}, \tag{1.4}$$

where $LP_*(F)$ are the surgery obstruction groups for manifold pairs, and Ψ denotes the pushout square of the groups

$$\Psi = \begin{pmatrix} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \end{pmatrix}. \tag{1.5}$$

Note that the map $LS_*(\Psi) \to LS_*(F)$ in the diagram (1.4) is induced by the natural map of squares $\Psi \to F$.

In the case of codimension 1, the methods of investigation of the splitting problem are essentially different in two cases. Rather complete results for the case of a two-sided submanifold N with connected or disconnected complement are given in [1, 2], and [3].

In the case of a one-sided submanifold of codimension $q = 1$ when the horizontal maps in the square F are isomorphisms, the groups $LS_*(F)$ coincide with the Browder–Livesay groups $LN_*(A \to B)$, which are the Wall groups of the ring with antistructure (see, for example, [2] and [3]). Using the Quinn–Ranicki spectra from the papers [4, 5] and [6] for the case of one-sided submanifold and epimorphic horizontal maps in the square F, the new exact sequences and commutative diagrams are obtained. These diagrams determine a close connection between the groups $LS_*(F)$, $LP_*(F)$, and the Wall groups.

In this paper we consider the splitting obstruction groups for the case of codimension $q = 2$. A quite complete survey of the results and the applications to the geometric problems in this case is given in [2] and [7]. In general, we use the methods developed in [4, 5], and [6] for the case of one-sided submanifolds to investigate the splitting problem in the case of codimension 2. In this case,, under some natural restrictions on the square F, we obtain new exact sequences and diagrams containing the groups $LS_*(F)$ and $LP_*(F)$. Also we compute some splitting obstruction groups and natural maps in diagrams that are interesting for geometric applications. In particular, we consider an example of a nontrivial map, the map $\Theta : L_n(D) \to LS_{n-3}(F)$, which is an analog of the Browder–Livesay invariant (see [8] and [9]). Therefore, it makes sense to determine an analog of the iterated Browder–Livesay invariants (see [8]), using a sequence of embedded submanifolds of codimension 2. A surgery spectral sequence based on a sequence of submanifolds in codimension 2 is defined as well. The construction of this spectral sequence formally coincides with the construction from [10]. In this case, the question of how to compute differentials and to find nontrivial differentials of higher order is open.

Note as well that it is possible to consider the groups L_*, LS_*, LN_*, and LP_* with different decorations "s", "h", "p" (see [1] and [2]). In this paper we shall consider only the case of groups decorated by "s", which corresponds to the problem of obtaining a simple homotopy equivalence by doing surgery. All the results of sections 3 and 5 are valid as well for groups decorated by "h" and "p". Further we assume that all groups are equipped with the decoration "s", unless otherwise stipulated.

2. THE SPLITTING OBSTRUCTION GROUPS AND THE WALL GROUPS

Recall the geometric definition of the splitting obstruction groups [1] for the case of codimension 2 and the connection of the splitting obstruction groups with the Wall groups of rings equipped with antistructures.

Suppose that for the pushout square of groups F (1.2) we are given a universal fibration $p: X \to Y$ with fiber S^1 which induces the map of fundamental groups $A \to B$. Denote by Z the topological space $K(C,1)$. We assume that the mapping cylinder M_p of the map p intersects $K(C,1)$ in the subspace X. Then the fundamental group of the topological space $M_Y \cup Z$ is equal to D. Denote by $K(F)$ the triad $(M_{\cup}Z\,;\, M_Y, Z\,;\, X)$. The group $LS_n(F)(n \geq 5)$ is the cobordism group of mappings of triads $(W\,;\, N, S)$ into the triad $K(F)$, assuming that all maps are compatible with the orientation homomorphisms.

Here W is a manifold of dimension $n{+}2$ and N is a finite Poincaré complex of dimension n given with a Poincaré embedding $N \to W$. By the definition from [1], the Poincaré embedding consists of a fibration $p: E \to N$ with fiber S^1, a finite Poincaré pair (S, E), and a simple homotopy equivalence $h: S \cup M_p \to W$, where $S \cap M_p = E$. In the case of a manifold with boundary and Poincaré pair, we assume, as usual, that the Poincaré embedding is already smoothed on the boundary.

According to [1], there exists an exact sequence functorially depending on the square F,

$$\to L_{n+1}(B) \to L_{n+3}(C \to D) \to LS_n(F) \to L_n(B) \to \tag{2.1}$$

in which the map $L_{n+1}(B) \to L_{n+3}(C \to D)$ is the composition of the transfer map $p^!$ induced by the fibration p and the natural map of the relative Wall groups induced by the horizontal maps of the square F:

$$L_{n+1}(B) \xrightarrow{p^!} L_{n+3}(A \to B) \longrightarrow L_{n+3}(C \to D). \tag{2.2}$$

Note that the exact sequence (2.1) is contained in the commutative diagrams (1.3) and (1.4).

Consider the square Ψ (1.5), which corresponds to the pair of manifolds (U, X) of codimension 2 obtained from the pair (Y, X), where U is a tubular neighborhood of X in Y. For the fibration

$$\xi : S^1 \to \partial U \to X,$$

we have the following part of the exact sequence of the fibration

$$\Pi : \mathbb{Z} \xrightarrow{l} A \xrightarrow{p} B \to 1. \tag{2.3}$$

Let $w_1 : B \to \{\pm 1\}$ be the first Stiefel–Whitney class of the fibration ξ, and let the pair (Π, w_1) denote the exact sequence (2.3) with the homomorphism w_1. Then, according to [2],

$$gtg^{-1} = t^{w_1 p(g)}, \qquad t = l(1), \quad g \in A.$$

Denote by $w_Y : B \to \{\pm 1\}$ the orientation homomorphism given on the space Y. Then we take the restriction $w_Y|_{\pi_1(U)}$ as the orientation homomorphism on the space U, which is homotopy equivalent to X. The orientation homomorphism on the subspace X is given by the condition (see [1], 2)

$$w^{\xi} = w_Y w : B \to \{\pm 1\}.$$

Thus the orientation on the left group $B = \pi_1(X)$ in the square Ψ may be different from the orientation on the right group $B = \pi_1(U)$. To avoid misunderstandings, in what follows we shall denote by B^{ξ} the left group B with the orientation w^{ξ} mentioned above. In particular, the exact sequence (2.1) for the square Ψ has the form

$$\to L_{n+1}(B^{\xi}) \to L_{n+3}(A \to B) \to LS_n(\Psi) \to L_n(B^{\xi}) \to . \tag{2.4}$$

Note that in codimension 2, as before, the groups $LS_n(\Psi)$ are denoted by $LN_n(A \to B)$, just as the Browder–Livesay groups.

According to [1] and [2], there exists a universal S^1-fibration

$$S^1 \to E \to V \tag{2.5}$$

for which the part of the exact sequence of the fibration

$$\mathbb{Z} \to \pi_1(E) \to \pi_1(V) \to 1 \tag{2.6}$$

coincides with the exact sequence Π, and the first Stiefel–Whitney class coincides with w_1.

If two pairs (Π, w) and (Π', w') are given, then a morphism $(f, g) : (\Pi, w) \to (\Pi', w')$ in the category of such pairs is defined in a natural way (see [2]) as a commutative diagram

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 1 \\ \| & & \downarrow f & & \downarrow g & & \\ \mathbb{Z} & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & 1 \end{array} , \tag{2.7}$$

in which the map g commutes with the orientation homomorphisms, $w'g = w$. If the pairs (Π, w) and (Π', w') correspond to some S^1-fibrations, then any morphism of pairs

$$(f, g) : (\Pi, w) \to (\Pi', w')$$

is induced by an morphism of universal S^1-fibrations (see [1] and [2]).

In the general case (see [11, 12], and [13]), the Wall groups are defined for an arbitrary ring with antistructure (R, α, u), where R is the ring with 1, $u \in R$ is an invertible element of the ring, and $\alpha : R \to R$ is an antiautomorphism for which

$$\alpha(u) = u^{-1}, \alpha^2(x) = uxu^{-1} \quad \forall x \in R.$$

According to [2], for the case of a submanifold of codimension 2 the groups $LN_*(A \to B)$ are the Wall groups of the group ring $\mathbb{Z}A$ equipped with certain antistructure. Recall the necessary results from the [2].

Let (Π, w) be the exact sequence (2.3) with the homomorphism $w_1 = w$. Define homomorphism $w_B : B \to \{\pm 1\}$ corresponding geometrically to the orientation w_Y of the ambient manifold. In this case,, the antistructure $\mathbb{Z}A, \beta, t)$ on the group ring $\mathbb{Z}A$ is defined in the following way:

$$\beta\colon \sum_{g \in A} n_g g \to \sum_{wp(g)=+1} n_g w_B p(g) g^{-1} - \sum_{wp(g)=-1} n_g w_B p(g) t g^{-1} \quad \forall g \in A, \quad t = l(1). \tag{2.8}$$

In addition, we have the morphism of antistructures $(\mathbb{Z}A, \beta, t) \to (\mathbb{Z}B, \alpha^\xi, 1)$ induced by the homomorphism p, where α^ξ denotes the standard involution

$$\sum n_g g \to \sum n_g w^\xi(g) g^{-1}, \qquad n_g \in \mathbb{Z}, \quad g \in B,$$

on the group ring $\mathbb{Z}B$ with the orientation homomorphism $w^\xi : B \to \{\pm 1\}$.

According to [2], we have the isomorphism

$$LN_n(A \to B, w_B) \cong L_n(\mathbb{Z}A, \beta, t). \tag{2.9}$$

Note that the morphism $A \to B$ in the notation for the group LN_* for the splitting problem in codimension 2 is a part of the exact sequence (2.3), given with the orientation homomorphism w. Thus the groups LN_* in formula (2.9) depend functorially on the triple $((\Pi, w), w_B)$.

In the case of the square Ψ when the fibration $\partial U \to X$ is trivial, we have an isomorphism $A = B \times \mathbb{Z}$, and according to [1] and [7]

$$LS_n(\Psi) = LN_n(A \to B) = H^{n+1}(Wh(B)),$$

where $H^{n+1}(Wh(B))$ are the Tate cohomology groups of the Whitehead group of the group B.

3. NATURAL MAPS

In this section, we consider the groups $LS_*(F)$ in codimension 2. Under some natural restrictions on the square F, we obtain new diagrams and exact sequences connecting these groups with the Wall groups and the groups LN_*. For the case of one-sided submanifolds in codimension 1, analogous results were obtained in [4, 5] and [6].

Consider the square F (1.2) corresponding to some splitting problem in codimension 2. Denote the groups and the maps in the following way:

$$F = \begin{pmatrix} A & \xrightarrow{f} & C \\ \downarrow i & & \downarrow j \\ B^{\xi} & \xrightarrow{g} & D \end{pmatrix}. \tag{3.1}$$

All groups in the square F are equipped with orientations. From now on we make the following agreement about the orientations of these groups. We fix a certain orientation homomorphism $w : D \to \{\pm 1\}$ on the group D. The composition of w with the natural maps from the square F gives orientations on all groups appearing in the square. We denote these orientations by w and do not include them into the notation. Besides, following section 2, we denote by B^{ξ} the group B with the orientation w^{ξ}. Denote by (Π, w_1) the exact sequence

$$\Pi : \mathbb{Z} \to A \xrightarrow{i} B \to 1 \tag{3.2}$$

given by the left column of the square F with the orientation $w_1 = w^{\xi} w$.

For the square F, let the following conditions be satisfied:

i) the map j in the square F is an epimorphism;

ii) there exists a pair (Π', w_1'):

$$\Pi' : \mathbb{Z} \to C \xrightarrow{j} D \to 1 \tag{3.3}$$

such that the pair of maps (f, g) induces a morphism $(\Pi, w_1) \to (\Pi', w_1')$.

Definition. The pushout square of the groups F for the splitting problem in codimension 2 for which conditions i), ii) are satisfied is said to be a *geometric diagram* in codimension 2.

The notion of a geometric diagram was introduced in [14] for the square F arising in the splitting problem along a one-sided submanifold when the horizontal maps are epimorphisms. In particular, the horizontal maps give a morphism of quadratic extensions (see [4, 5] and [6]). Thus this definition preserves the analogy with codimension 1.

Remark. Under the assumption that the map j is an epimorphism, condition ii) is satisfied if there exists a homomorphism w_1' on the group D for which $w_1' g = w_1$. In particular, if j is an epimorphism, this condition ii) is satisfied automatically for $w_1 \equiv 1$.

For the geometric diagram F, there exist natural maps

$$\Psi \xrightarrow{\mathcal{S}} F \xrightarrow{\mathcal{P}} \Phi$$

of pushout squares of the groups with orientations

$$\begin{pmatrix} A & \longrightarrow & A \\ \downarrow i & & \downarrow i \\ B^{\xi} & \longrightarrow & B \end{pmatrix} \to \begin{pmatrix} A & \xrightarrow{f} & C \\ \downarrow i & & \downarrow j \\ B^{\xi} & \xrightarrow{g} & D \end{pmatrix} \to \begin{pmatrix} C & \longrightarrow & C \\ \downarrow j & & \downarrow j \\ D^{\xi} & \longrightarrow & D \end{pmatrix}, \tag{3.4}$$

where the orientation on the group D^{ξ} is defined as $w^{\xi} = w_1' w : D^{\xi} \to \{\pm 1\}$.

Similarly to (2.9), there exists an antistructure $(\mathbb{Z}C, \beta', t')$ for which the isomorphism

$$LN_n(C \to D, w) \cong L_n(\mathbb{Z}C, \beta', t')$$

take place. The composition of the maps $\mathcal{PS}$ gives rise to a morphism of antistructures

$$(\mathbb{Z}B, \beta, t) \to (\mathbb{Z}C, \beta', t').$$

Hence the relative groups

$$LN_n(\Psi \to \Phi) \cong L_n(\mathbb{Z}B \to \mathbb{Z}C, \beta, t)$$

are defined. These groups fit into the long exact sequence

$$\to LN_n(A \to B, w) \longrightarrow LN_n(C \to D, w) \longrightarrow LN_n(\Psi \to \Phi) \to . \tag{3.5}$$

In what follows, we shall not indicate the orientation w in the groups LN whenever it is clear from the context.

Theorem 1. *Let F be a geometric diagram of groups corresponding to a pair of manifolds for a splitting problem in codimension 2. Then we have the following commutative diagrams of exact sequences:*

$$\begin{array}{ccccccccc}
\to L_{n+1}(B^\xi) & & \longrightarrow & & L_{n+3}(C \to D) & & \longrightarrow & & LN_n(C \to D) \to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
& & L_{n+1}(D^\xi) & & & & LS_n(F) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to LN_{n+1}(C \to D) & & \longrightarrow & & L_{n+1}(B^\xi \to D^\xi) & & \longrightarrow & & L_n(B^\xi) \to
\end{array}, \tag{3.6}$$

$$\begin{array}{ccccccccc}
\to LN_{n+1}(C \to D) & & \longrightarrow & & L_{n+1}(B^\xi \to D^\xi) & & \longrightarrow & & L_{n+3}(F) \to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
& & LN^{rel}_{n+1} & & & & LS_n(F) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to L_n(F) & & \longrightarrow & & LN_n(A \to B) & & \longrightarrow & & LN_n(C \to D) \to
\end{array}, \tag{3.7}$$

where we denote by LN^{rel}_n the relative groups $LN_n(\Psi \to \Phi)$.

Proof. There exist simplicial Ω-spectra with homotopy groups isomorphic to the corresponding L-groups or LN-groups (see, for example, [13]). According to the results of [1], the map $p_0 : L_{n+1}(B^\xi) \to L_{n+3}(A \to B)$ from the sequence (2.2) is realized on the level of spectra. Since the natural map of relative groups is realized on the spectrum level as well (see [1] and [13]), we obtain the fibration of spectra

$$\mathbb{L}(B^\xi) \to \Omega^2 \mathbb{L}(C \to D). \tag{3.8}$$

For the homotopy groups of the homotopy cofiber $\mathbb{LS}(F)$ of the map (3.8), we have an isomorphism $\pi_n(\mathbb{LS}(F)) = LS_n(F)$. In addition, the exact sequence (2.1) is the homotopy long exact sequence of the fibration of spectra (3.8). The map $\mathcal{P}$ induces a map of the corresponding fibrations. Thus we obtain the homotopy commutative diagram of spectra

$$\begin{array}{ccc}
\mathbb{LS}(F) & \to & \mathbb{L}(B^\xi) \\
\downarrow \mathbb{P} & & \downarrow \\
\mathbb{LN}(C \to D) & \to & \mathbb{L}(D^\xi)
\end{array}, \tag{3.9}$$

in which the vertical maps are induced by $\mathcal{P}$. The cofibers of the horizontal maps in the diagram (3.9) are homotopy equivalent to $\Omega^2 \mathbb{L}(C \to D)$, and the map $\mathcal{P}$ induces a homotopy equivalence between them. Thus the square (3.9) is the pushout square of spectra, and, hence, the cofibers of the vertical maps are homotopy equivalent to the spectrum $\mathbb{L}(B^\xi \to D^\xi)$ as well. Now the exact long homotopy sequences of the maps from the square (3.9) give the commutative diagram (3.6).

Consider the diagram of spectra

$$\begin{array}{ccc} \mathbb{LN}(A \to B) \xrightarrow{\mathbb{S}} & \mathbb{LS}(F) & \longrightarrow \Omega^3\mathbb{L}(F) \\ & \downarrow \mathbb{P} & \\ & \mathbb{LN}(C \to D) & \end{array},$$

in which the maps $\mathbb{P}$ and $\mathbb{S}$ are induced by the maps $\mathcal{P}$ and $\mathcal{S}$ of the pushout squares respectively. Passing to the spectra in the commutative diagram (1.4), we see that the homotopy cofiber of the map $\mathbb{S}$ is homotopy equivalent to the spectrum $\Omega^3\mathbb{L}(F)$.

The homotopy cofiber of the composition $\mathbb{PS}$ is the spectrum $\mathbb{LN}(\Psi \to \Phi)$, and the exact sequence (3.5) is the exact homotopy sequence of the fibration $\mathbb{PS}$. Thus we obtain a pushout square of spectra

$$\begin{array}{ccc} \mathbb{LS}(F) & \to & \Omega^3\mathbb{L}(F) \\ \downarrow \mathbb{P} & & \downarrow \\ \mathbb{LN}(C \to D) & \to & \mathbb{LN}(\Psi \to \Phi) \end{array}.$$

Passing to the exact long homotopy sequences of fibrations of the maps in the obtained square, we obtain a commutative diagram of exact sequences.

The diagrams obtained in Theorem 1 are similar to the diagrams (22.1), (22.2) from [4].

Now we shall describe the connections between the groups LP_* for the squares Ψ, F, Φ. We assume, as before, that F is a geometric diagram of groups. Consider the diagrams (1.3) for the squares Ψ and Φ. We see that the groups $LP_*(\Psi), LP_*(\Phi)$ fit into exact sequences

$$\to LP_n(\Psi) \to L_n(B^\xi) \to L_{n+1}(A) \to LP_{n-1}(\Psi) \to, \tag{3.10}$$

$$\to LP_n(\Phi) \to L_n(D^\xi) \to L_{n+1}(C) \to LP_{n-1}(\Phi) \to . \tag{3.11}$$

Thus the groups $LP_*(\Psi)$ are the relative groups for the composition

$$L_{n+1}(B^\xi) \xrightarrow{p^!} L_{n+3}(A \to B) \xrightarrow{\partial} L_{n+2}(A)$$

of the transfer map $p^!$ and the map $\partial : L_{n+3}(A \to B) \to L_{n+2}(A)$ from the relative exact sequence of L-groups for the map $A \to B$ (see [2]). Since the maps $p^!$ and ∂ are realized on the spectrum level (see [1] and [2]), we can define the spectrum $\mathbb{LP}(\Psi)$ as the homotopy fiber of the map of spectra

$$\mathbb{L}(B) \to \Omega\mathbb{L}(A), \tag{3.12}$$

which induces the map $\partial p^!$. Then we have an isomorphism $\pi_n(\mathbb{LP}(\Psi) \cong LP_n(\Psi)$, and the exact sequence (3.10) is the exact long homotopy sequence of the fibration (3.12). The spectrum $\mathbb{LP}(\Phi)$ is defined similarly. In particular, the relative groups $LP_*(\Psi \to \Phi)$ fitting into the long exact sequence

$$\cdots \to LP_n(\Psi) \to LP_n(\Phi) \to LP_n(\Psi \to \Phi) \to \cdots \tag{3.13}$$

are defined.

Theorem 2. *Let F be the geometric diagram of groups corresponding to the pair of manifolds for a splitting problem in codimension 2. Then we have the following commutative diagrams of exact sequences:*

$$\begin{array}{cccccccccccc}
 & \to L_{n+1}(B^{\xi}) & & \longrightarrow & & L_{n+2}(C) & & \longrightarrow & & L_{n+2}(A\to C)\to & & \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & L_{n+2}(A) & & & & LP_n(F) & & & & , \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & \to L_{n+3}(A\to C) & & \longrightarrow & & LP_n(\Psi) & & \longrightarrow & & L_n(B^{\xi})\to & &
\end{array} \qquad (3.14)$$

$$\begin{array}{cccccccccccc}
 & \to L_{n+1}(B^{\xi}\to D^{\xi}) & & \longrightarrow & & L_n(B^{\xi}) & & \longrightarrow & & L_{n+1}(C)\to & & \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & LP_n(F) & & & & L_n(D^{\xi}) & & & & , \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & \to L_{n+2}(C) & & \longrightarrow & & LP_n(\Phi) & & \longrightarrow & & L_n(B^{\xi}\to D^{\xi})\to & &
\end{array} \qquad (3.15)$$

$$\begin{array}{cccccccccccc}
 & \to L_{n+1}(B^{\xi}\to D^{\xi}) & & \longrightarrow & & L_{n+2}(A\to C) & & \longrightarrow & & LP_{n-1}(\Psi)\to & & \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & LP_n(F) & & & & LP_n^{rel} & & & & , \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & \to LP_n(\Psi) & & \longrightarrow & & LP_n(\Phi) & & \longrightarrow & & L_n(B^{\xi}\to D^{\xi})\to & &
\end{array} \qquad (3.16)$$

$$\begin{array}{cccccccccccc}
 & \to L_{n+1}(B^{\xi}\to D^{\xi}) & & \longrightarrow & & LP_n(F) & & \longrightarrow & & L_{n+2}(D)\to & & \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & LS_n(F) & & & & LP_n(\Phi) & & & & , \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & \to L_{n+3}(D) & & \longrightarrow & & LN_n(C\to D) & & \longrightarrow & & L_n(B^{\xi}\to D^{\xi})\to & &
\end{array} \qquad (3.17)$$

where LP_n^{rel} denote relative groups $LP_n(\Psi\to\Phi)$.

Proof. Using the diagram (1.3), we can define the spectrum $\mathbb{LP}(F)$. Thus we have the pushout square of spectra for which the exact long homotopy sequences of the maps give the diagram (1.3). Consider the homotopy commutative square of spectra

$$\begin{array}{ccc}
\Omega^2\mathbb{L}(A) & \to & \mathbb{LP}(\Psi) \\
\downarrow & & \downarrow \\
\Omega^2\mathbb{L}(C) & \to & \mathbb{LP}(F)
\end{array},$$

in which the vertical maps are induced by $\mathcal{S}$, and the horizontal maps are the fibrations of spectra from the diagram (1.3). This square is pushout, since the cofibers of the horizontal maps are naturally homotopy equivalent, and they are homotopy equivalent to the spectrum $\mathbb{L}(B^{\xi})$. Hence the cofibers of the vertical maps are homotopy equivalent to the spectrum $\Omega^2\mathbb{L}(A\to C)$. The exact long homotopy sequences of the maps of this square give the diagram (3.14). The other diagrams are obtained in a similar way. The theorem is proved. □

The diagrams obtained in Theorem 2 are analogs of the diagrams (3.6) from [6]. There also exists a diagram of another type connecting the groups LS_* and LP_* in the case considered.

Proposition 1. *Under the assumptions of Theorem 2 we have the following commutative diagram in which the rows and columns are exact sequences*

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & LN_n(A\to B) & \longrightarrow & LP_n(\Psi) & \longrightarrow & L_{n+2}(B) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & LS_n(F) & \longrightarrow & LP_n(F) & \longrightarrow & L_{n+2}(D) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_{n+3}(F) & \longrightarrow & L_{n+2}(A\to C) & \longrightarrow & L_{n+2}(B\to D) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\quad . \qquad (3.18)
$$

Proof. The map $\mathcal{S}:\Psi\to F$ in (3.4) induces a homotopy commutative diagram of spectra

$$
\begin{array}{ccc}
\mathbb{LN}(A\to B) & \longrightarrow & \mathbb{LP}(\Psi) \\
\downarrow & & \downarrow \\
\mathbb{LS}(F) & \longrightarrow & \mathbb{LP}(F)
\end{array}
\quad .
$$

Hence, considering the exact long homotopy sequences of maps, we obtain the commutative diagram (3.18). □

The diagrams obtained in Theorems 1, 2, together with the diagrams (1.3), (1.4) from [1], give very effective methods for computations of the splitting obstruction groups and natural maps. Thus in [15] the squares of geometric antistructures are considered. These squares naturally generalize the squares of groups in the case of a one-sided submanifold. In the case of a finite 2-group in [15], the projective groups LS_* and LP_* and natural maps in the diagrams are completely computed.

4. COMPUTATION OF SOME LS_*-GROUPS

One of the possible appoaches to the closed manifold surgery problem is based on the diagram (1.3) in the case of a one-sided submanifold. In this case,, the closed manifold surgery problem is closely connected with the action of Wall groups on the set of homotopy smoothings (triangulations) of the given manifold (see, for example, [8]). Then the diagram (1.3) is considered for the square Ψ (1.5) in which horizontal maps are isomorphisms, and vertical maps are inclusions of index 2. In this case,, the groups $LS_*(\Psi)$ coincide with the Browder–Livesay groups $LN_*(A\to B)$. In many cases this approach yields complete results (see [8, 9, 16, 17], and [18]).

In this section,, we describe similar methods that use submanifolds in codimension 2 and compute splitting obstruction groups for some pairs of manifolds.

First, recall the necessary results from papers [1] and [2]. The homotopy triangulation or s-triangulation of a simple n-dimensional geometric Poincaré complex Y is a simple homotopy equivalence $f:M\to Y$, where M is a compact n-dimensional triangulated topological manifold. Up to the end of this section, we assume that $n\geq 5$. Two triangulations $f_i:M_i\to Y$, $i=1,2$, are equivalent if there exists a homeomorphism $h:M_1\to M_2$ such that the maps f_1 and f_2h

are homotopic. The set of equivalence classes of s-triangulations of the complex Y is denoted by $S^{TOP}(Y)$.

Assume that Y is already a triangulated topological manifold of dimension n. Then we have the Sullivan exact sequence

$$\xrightarrow{\sigma_*} L_{n+1}(\pi_1(Y)) \longrightarrow S^{TOP}(Y) \longrightarrow [Y, G/TOP] \xrightarrow{\sigma_*} L_n(\pi_1(Y)). \tag{4.1}$$

We do not explicitly display the homomorphism w in the notation for the L-groups if this does not lead to misunderstandings. From the construction of the exact sequence (4.1), it follows that the elements of the group $L_n(\pi_1(Y))$ lying in the image of the map σ_* are realized by normal maps of closed manifolds (see [1] and [2]). In addition (see [2]), if $X \subset Y$ is a closed submanifold of codimension $q = 1, 2$, then we have the commutative diagram

$$\begin{array}{ccccccccc} \xrightarrow{\sigma_*} & L_{n+1}(\pi_1(Y)) & \longrightarrow & S^{TOP}(Y) & \longrightarrow & [Y, G/TOP] & \xrightarrow{\sigma_*} & L_n(\pi_1(Y)) \\ & \| & & \downarrow & & \downarrow & & \| \\ \longrightarrow & L_{n+1}(\pi_1(Y)) & \xrightarrow{\Theta} & LS_{n-q}(F) & \longrightarrow & LP_{n-q}(F) & \longrightarrow & L_n(\pi_1(Y)) \end{array}, \tag{4.2}$$

in which the bottom row is the exact sequence from the diagram (1.3). From the diagram (4.2) it follows immediately that elements $x \in L_{n+1}(\pi_1(Y))$ with $\Theta(x) \neq 0$ act nontrivially on the set of homotopy triangulations $S^{TOP}(Y)$. Furthermore, all elements lying in the image of σ_*, lie in the image of the map

$$LP_{n-q}(F) \to L_n(\pi_1(Y))$$

from the diagrams (1.3) and (3.17).

In the case of a Browder–Livesay pair (a one-sided submanifold of codimension 1, $B = D$) the map

$$\Theta : L_n(D) \to LN_{n-2}(A \to B)$$

determines the Browder–Livesay invariant $\Theta(x)$, which is the first obstruction to realization by closed manifolds (see [8] and [16]). In this case,, if $\Theta(x) \neq 0$, then the element X is not realized by normal maps of closed manifolds and acts nontrivially on any homotopy triangulation of a manifold M with $\pi_1(M) = D$.

Example 1. Consider the natural embedding of the real projective spaces $S^1 = RP^1 \subset RP^3$.

Lemma 1. *The normal fibration of the submanifold $S^1 \subset RP^3$ is trivial.*

Proof. The manifolds S^1 and RP^1 are parallelizable. Denote by ν_{S^1} a normal fibration of S^1 in RP^3. Let τ_{S^1} be a tangential fibration of S^1, and $\tau_{RP^3} \cong \epsilon^3$ be a tangential fibration of RP^3. We have the isomorphisms

$$\nu_{S^1} \oplus \tau_{S^1} \cong \tau_{RP^3} \cong \epsilon^3.$$

Since $\tau_{S^1} \cong \epsilon^1$, we have $\nu_{S^1} \oplus \epsilon^1 \cong \epsilon^3$.

Now we can use the following result from [19]. Let ξ^m is m-dimensional vector fibration over the CW-complex K of dimension $k < m$. Then ξ^m is a trivial vector fibration if and only if the fibration $\xi^m \oplus \epsilon^1$ is trivial. The lemma is proved. □

Lemma 2. *Let $S^1 \to RP^3$ be the standard embedding of projective spaces inducing an epimorphism of fundamental groups. Then we have the isomorphism $\pi_1(RP^3 \setminus S^1) \cong \mathbb{Z}$.*

Proof. Consider as a model of $RP^3 \setminus S^1$ a closed three-dimensional ball D^3 with deleted segment connecting the poles and identified antipodal points on the boundary. The obtained space is homotopy equivalent to a closed two-dimensional ball D^2 with deleted center and identified antipodal points of the boundary. Hence $\pi_1(RP^3 \setminus S^1) \cong \mathbb{Z}$. The lemma is proved. □

Multiply the pair (RP^3, S^1) by a simply connected manifold M of high dimension. We obtain the pair of manifolds $(Y, X) = (RP^3 \times M, S^1 \times M)$ of codimension 2. It follows from Lemma 1 and Lemma 2 that the pushout square F in the splitting problem for the submanifold $S^1 \times M$ is

$$F = \begin{pmatrix} A & \xrightarrow{f} & C \\ \downarrow i & & \downarrow j \\ B & \xrightarrow{g} & D \end{pmatrix} = \begin{pmatrix} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \\ \downarrow i & & \downarrow j \\ \mathbb{Z} & \xrightarrow{g} & \mathbb{Z}/2 \end{pmatrix}. \tag{4.3}$$

The map i is the projection onto the first summand,

Orientations on the all groups in the square F are trivial by Lemma 1. The maps j and g are reductions modulo 2. The map f is given by the formula $f(a,b) = a - 2b$, which follows from Lemma 2 and the universal property. It is easy to verify that the square F is a geometric diagram in codimension 2.

Now we compute some groups LS_* for the square (4.3). As corollaries, we present some results of some geometric interest.

All Wall groups of the groups from the square (4.3) are known (see [1]). We have the isomorphisms:

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0,1 \mod 4, \\ \mathbb{Z}/2 & \text{for } n = 2,3 \mod 4, \end{cases}$$

$$L_n(\mathbb{Z}/2) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{for } n = 0 \mod 4, \\ 0 & \text{for } n = 1 \mod 4, \\ \mathbb{Z}/2 & \text{for } n = 2,3 \mod 4, \end{cases}$$

$$L_n(\mathbb{Z}\oplus\mathbb{Z}) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2 & \text{for } n = 0,2 \mod 4, \\ \mathbb{Z}\oplus\mathbb{Z} & \text{for } n = 1 \mod 4, \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & \text{for } n = 3 \mod 4. \end{cases}$$

Theorem 3. *Let F be the geometric diagram of groups* (4.3) *in codimension* 2. *Then we have the following isomorphisms*

$$LS_n(F) \cong \begin{cases} \mathbb{Z}/2 & \textit{for } n = 2 \mod 4, \\ \mathbb{Z}\oplus\mathbb{Z} & \textit{for } n = 1 \mod 4. \end{cases}$$

The groups $LS_3(F)$ and $LS_0(F)$ fit into the exact sequence

$$0 \longrightarrow LS_0(F) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow LS_3(F) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Proof. Consider the long exact sequence

$$\to LS_n(\Psi) \to LS_n(F) \to L_{n+3}(F) \to$$

from the diagram (1.4). The left vertical map in the square F is the projection. Hence, according to [1], we have the isomorphism

$$LS_n(\Psi) = LN_n(\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}) \cong H^{n+1}(Wh(\mathbb{Z})),$$

where $H^{n+1}(Wh(\mathbb{Z}))$ are the Tate cohomology groups of the Whitehead group (see [1]). Since the Whitehead group $Wh(\mathbb{Z})$ is trivial, we obtain isomorphisms

$$LS_n(F) \cong L_{n+3}(F) \quad \forall n = 0,1,2,3 \mod 4.$$

The groups $L_n(F)$ fit into the commutative diagram of exact sequences (see [1] and [2])

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow & \\ \to & L_n(A) & \xrightarrow{f_*} & L_n(C) & \to & L_n(f) & \to \\ & \downarrow i_* & & \downarrow j_* & & \downarrow & \\ \to & L_n(B) & \xrightarrow{g_*} & L_n(D) & \to & L_n(g) & \to, \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & L_n(i) & \to & L_n(j) & \to & L_n(F) & \to \\ & \downarrow, & & \downarrow & & \downarrow & \end{array} \tag{4.4}$$

where the maps f_*, g_*, i_*, and j_* are induced by the corresponding maps from the square F. The maps i, f have right inverse maps. Hence the maps i_*, f_* give projections of the Wall groups on the direct summand for arbitrary dimension. Thus we have isomorphisms

$$L_n(i) \cong L_n(f) \cong \begin{cases} \mathbb{Z}/2 & \text{for } n = 0, 1 \mod 4, \\ \mathbb{Z} & \text{for } n = 2, 3 \mod 4. \end{cases}$$

The maps g_*, j_* are isomorphisms in dimensions 2 and 3 according to [1]. In dimension 1 they are trivial, since the group $L_1(\mathbb{Z}/2)$ is trivial.

In dimension 0, the maps g_* and j_* are inclusions in a direct summand, since they have right inverse maps (see [5]). It is clear that $L_n(j) \cong L_n(g)$. Now consider the relative exact sequence of the L-groups for the map j:

$$\begin{array}{ccccccccc} L_0(\mathbb{Z}) & \xrightarrow{mono} & L_0(\mathbb{Z}/2) & \longrightarrow & L_0(\mathbb{Z} \to \mathbb{Z}/2) & \longrightarrow & L_3(\mathbb{Z}) & \xrightarrow{\cong} & L_3(\mathbb{Z}/2) \to \\ \big\downarrow{=} & & \big\downarrow{=} & & & & \big\downarrow{=} & & \big\downarrow{=} \\ \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & & & \mathbb{Z}/2 & & \mathbb{Z}/2 \end{array}$$

$$\begin{array}{ccccccccc} \to L_3(\mathbb{Z} \to \mathbb{Z}/2) & \longrightarrow & L_2(\mathbb{Z}) & \xrightarrow{\cong} & L_2(\mathbb{Z}/2) & \longrightarrow & L_2(\mathbb{Z} \to \mathbb{Z}/2) & \xrightarrow[epi]{} & L_1(\mathbb{Z}) \\ & & \big\downarrow{=} & & \big\downarrow{=} & & & & \big\downarrow{=} \\ & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & & & \mathbb{Z} \end{array} .$$

From this we obtain the isomorphisms

$$L_n(g) \cong L_n(j) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, 2 \mod 4, \\ 0 & \text{for } n = 1, 3 \mod 4. \end{cases}$$

From the diagram (4.4), we obtain the exact sequence

$$\begin{array}{ccccccccccc} L_1(g) & \longrightarrow & L_1(F) & \longrightarrow & L_0(f) & \longrightarrow & L_0(g) & \xrightarrow{mono} & L_0(F) & \xrightarrow[epi]{} & L_3(f) \\ \big\downarrow{=} & & & & \big\downarrow{=} & & \big\downarrow{=} & & & & \big\downarrow{=} \\ 0 & & & & \mathbb{Z}/2 & & \mathbb{Z} & & & & \mathbb{Z} \end{array} ,$$

in which the map $L_0(F) \to L_3(f)$ is epimorphic, since the group $L_3(g)$ is trivial. Hence

$$L_1(F) \cong \mathbb{Z}/2 \qquad \text{and} \quad L_0(F) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

By virtue of the isomorphism

$$LS_n(F) \cong L_{n+3}(F) \quad \forall n = 0, 1, 2, 3 \bmod 4,$$

we obtain the result of the theorem for $n = 1, 2$. The exact sequence of the theorem now follows from the commutative diagram (1.3). $\square$

Corollary 1. *For the square* (4.3), *we have the following isomorphisms*:

$$LP_1(F) \cong LP_2(F) \cong \mathbb{Z} \oplus \mathbb{Z}/2, \quad LP_3(F) \cong LS_3(F), \quad LP_0(F) \cong LS_0(F) \oplus \mathbb{Z}/2.$$

Proof. It suffices to consider the diagram (1.3) for the square F and use the results of Theorem 3. □

Corollary 2. *For the square F under the assumptions of Theorem* 3, *the map*

$$\Theta : \mathbb{Z} \oplus \mathbb{Z} \cong L_0(\mathbb{Z}/2) \to LS_1(F) \cong \mathbb{Z} \oplus \mathbb{Z}$$

has the image $\mathbb{Z}$.

Proof. Consider the following part of the diagram (1.3):

$$\begin{array}{ccccccccc} L_0(C) & & \longrightarrow & & L_0(D) & & \xrightarrow{\Theta} & & LS_1(F) \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & LP_2(F) & & & & L_0(C \to D) & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ LS_2(F) & & \longrightarrow & & L_2(B) & & \longrightarrow & & L_3(C) \end{array} .$$

From Theorem 3 and Corollary 1, we see that this part has the following form:

$$\begin{array}{ccccccccc} \mathbb{Z} & & \xrightarrow{mono} & & \mathbb{Z} \oplus \mathbb{Z} & & \xrightarrow{\Theta} & & \mathbb{Z} \oplus \mathbb{Z} \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & \mathbb{Z} \oplus \mathbb{Z}/2 & & & & \mathbb{Z} & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ \mathbb{Z}/2 & & \xrightarrow{\cong} & & \mathbb{Z}/2 & & \xrightarrow{0} & & \mathbb{Z}/2 \end{array} .$$

The top and bottom rows are chain complexes with isomorphic homology groups. This implies the assertion of the lemma. □

Corollary 3. *Elements of the group $L_0(\mathbb{Z}/2) \cong \mathbb{Z} \oplus \mathbb{Z}$ that do not lie in the subgroup*

$$\mathbb{Z} = Im\{L_0(1) \to L_0(\mathbb{Z}/2)\}$$

act nontrivially on the set $S^{TOP}\left(RP^3 \times M^{4k}\right)$, where M^{4k} is a simply connected manifold of dimension $4k$, $k \geq 1$.

Proof. The image of $L_0(1)$ coincides with the image of $L_0(\mathbb{Z})$ in the group $L_0(\mathbb{Z}/2)$. Now the result follows from Corollary 2 and the commutative diagram (4.2). □

Remark. The result of Corollary 3 is known. It is interesting that we obtained this result using the map Θ for the manifold of codimension 2. It is possible to obtain this result by considering the one-sided submanifold $RP^2 \times M^{4k} \subset RP^3 \times M^{4k}$ and using the Browder–Livesay invariant (see [17]).

Consider one more example of a geometric diagram, for which we compute the groups LS_* and LP_* in all dimensions.

Let

$$F = \begin{pmatrix} A & \xrightarrow{f} & C \\ \downarrow i & & \downarrow j \\ B & \xrightarrow{g} & D \end{pmatrix} = \begin{pmatrix} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/2 \oplus \mathbb{Z} \\ \downarrow i & & \downarrow j \\ \mathbb{Z} & \xrightarrow{g} & \mathbb{Z}/2 \end{pmatrix} \tag{4.5}$$

be a commutative square of groups, in which the maps i and j are projections on the first summand, the map g is reduction modulo 2, and $f(a, b) = (a \mod 2, b)$. Thus F is the geometric diagram of groups in codimension 2.

Theorem 4. *Let F be the commutative square of groups* (4.5). *Then we have the following isomorphisms*

$$LS_n(F) \cong \begin{cases} 0 & \text{for } n = 0, 2 \mod 4, \\ \mathbb{Z} & \text{for } n = 1, 3 \mod 4. \end{cases}$$

Proof. There exist isomorphisms [1]:

$$L_n(\mathbb{Z}/2 \oplus \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 & \text{for } n = 0 \mod 4, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \mod 4, \\ \mathbb{Z}/2 & \text{for } n = 2 \mod 4, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } n = 3 \mod 4. \end{cases}$$

Consider the commutative diagram (see, for example, [2])

$$\begin{array}{ccc} L_n(\mathbb{Z}) & \stackrel{g_*}{\to} & L_n(\mathbb{Z}/2) \\ \downarrow \alpha & & \downarrow \beta \\ L_n(\mathbb{Z} \oplus \mathbb{Z}) & \stackrel{f_*}{\to} & L_n(\mathbb{Z}/2 \oplus \mathbb{Z}) \\ \downarrow & & \downarrow \\ L_{n-1}(\mathbb{Z}) & \stackrel{g_*}{\to} & L_{n-1}(\mathbb{Z}/2) \end{array},$$

in which the vertical columns give the natural decompositions of the groups of the middle row into direct sums

$$L_n(\mathbb{Z} \oplus \mathbb{Z}) = L_n(\mathbb{Z}) \oplus L_{n-1}(\mathbb{Z}), \qquad L_n(\mathbb{Z}/2 \oplus \mathbb{Z}) = L_n(\mathbb{Z}/2) \oplus L_{n-1}(\mathbb{Z}/2).$$

In this case, $i_*\alpha = Id$ and $j_*\beta = Id$ are the identity maps. This yields the isomorphism

$$L_n(f) = L_n(g) \oplus L_{n-1}(g).$$

The map g_* is described in the proof of Theorem 3. In the same way as in Theorem 3, we obtain isomorphisms

$$LS_n(F) \cong L_{n+3}(F) \quad \forall n = 0, 1, 2, 3 \mod 4.$$

Now from the diagram (4.4) we obtain the exact sequence

$$\begin{array}{ccccccccccc} L_1(g) & \longrightarrow & L_1(F) & \longrightarrow & L_0(f) & \xrightarrow{epi} & L_0(g) & \xrightarrow{mono} & L_0(F) & \xrightarrow[epi]{} & L_3(f) \\ \downarrow = & & & & \downarrow = & & \downarrow = & & & & \downarrow = \\ 0 & & & & \mathbb{Z} & & \mathbb{Z} & & & & \mathbb{Z} \end{array},$$

in which the map $L_0(F) \to L_3(f)$ is epimorphic since the group $L_3(g)$ is trivial. Hence $L_1(F) \cong 0$ and $L_0(F) \cong \mathbb{Z}$. Similarly, the other part of the exact sequence gives $L_3(F) \cong 0$ and $L_2(F) \cong \mathbb{Z}$. Now it is sufficient to use the isomorphism

$$LS_n(F) \cong L_{n+3}(F) \quad \forall n = 0, 1, 2, 3 \mod 4.$$

The theorem is proved. $\square$

Corollary 4. *For the square* (4.5), *we have the isomorphisms*

$$LP_n(F) = \begin{cases} \mathbb{Z}/2 & \text{for } n = 0 \mod 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{for } n = 1 \mod 4, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 2 \mod 4, \\ \mathbb{Z} & \text{for } n = 3 \mod 4. \end{cases}$$

Proof. It is sufficient to consider the diagram (1.3) for the square F and use Theorem 4. $\square$

Note also that for the square (4.5) the maps $LP_{2k}(F) \to L_{2k+2}(\mathbb{Z}/2)$, which correspond to forgetting the submanifold, are isomorphisms. This follows easily from the diagram (1.3).

We give yet one more example in which the map Θ is nontrivial even for the groups LN in codimension 2. This result is very unexpected, since the kernel of the map Θ is usually very large for a submanifold of codimension 2.

Let

$$\Phi = \begin{pmatrix} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow j & & \downarrow j \\ \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \end{pmatrix},$$

then by definition $LN_n(\mathbb{Z} \to \mathbb{Z}/2) \cong LS_n(\Phi)$.

Theorem 5. *We have the isomorphisms*

$$LN_n(\mathbb{Z} \to \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z} & \text{for } n = 1 \mod 4, \\ \mathbb{Z}/2 & \text{for } n = 2 \mod 4 \end{cases}$$

and the exact sequence

$$0 \to LN_0(\mathbb{Z} \to \mathbb{Z}/2) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to LN_3(\mathbb{Z} \to \mathbb{Z}/2) \to \mathbb{Z}/2 \to 0.$$

Proof. The rows of the diagram (1.3) for the square Φ are chain complexes with isomorphic homology groups for the corresponding members. The map in the top row

$$\mathbb{Z} \cong L_0(\mathbb{Z}) \to L_0(\mathbb{Z}/2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

is an inclusion in a direct summand. In the corresponding place of the bottom row, we have the map $LN_2 \to L_2(\mathbb{Z}/2) \cong \mathbb{Z}/2$. This is a monomorphism since $L_1(\mathbb{Z}/2) = 0$, and it is an epimorphism since the map $L_3(\mathbb{Z}) \to L_3(\mathbb{Z}/2)$ is an isomorphism, according to [1]. Hence, $LN_2 \cong \mathbb{Z}/2$. This implies that the map $L_0(\mathbb{Z}/2) \cong \mathbb{Z} \oplus \mathbb{Z} \to LN_1$ is an epimorphism with kernel $\mathbb{Z}$, which is a direct summand. The exact sequence for the groups LN for others dimensions now follows immediately from the diagram. The theorem is proved. $\square$

Corollary 5. *Under the assumptions of Theorem* 5 *for the square* Φ, *the map*

$$\Theta : \mathbb{Z} \oplus \mathbb{Z} \cong L_0(\mathbb{Z}/2) \to LN_1(\mathbb{Z} \to \mathbb{Z}/2) \cong \mathbb{Z}$$

is an epimorphism.

Proof. This follows from Theorem 5. $\square$

In [1], the groups $LN_*(\mathbb{Z} \to \mathbb{Z}/n)$ are applied to investigate the false lens spaces. For odd n it was proved that they are trivial in even dimensions, and in odd dimensions they are isomorphic to the groups $L_1(\mathbb{Z}/n)$, which are trivial according to [8]. Thus we see that for even n the splitting obstruction groups $LN_*(\mathbb{Z} \to \mathbb{Z}/n)$ are nontrivial already in the case $n = 2$.

5. SUBMANIFOLDS OF CODIMENSION 2 AND A SURGERY SPECTRAL SEQUENCE

A surgery spectral sequence was first constructed in [10]. For this construction, the case of one-sided submanifolds and Browder–Livesay groups in the diagram (1.3) was considered. From the algebraic point of view, the diagram (1.3) in this case is constructed for a pair $\pi \subset G$, where π is a subgroup of index 2 in the group G equipped with an orientation homomorphism. The realization of the diagram (1.3) on the spectrum level is the main tool used to construct the spectral sequence. Using this property in [18], spectral sequences of Tate cohomology groups of K-groups for quadratic extensions of antistructures were constructed and the first differentials in these spectral sequences were described.

In this section, we study the surgery spectral sequence constructed on the basis of the diagram (1.3) for the case of a submanifold of codimension 2. Since we shall use constructions from [9], we can perform the construction in the general case.

Let Ψ be the universal square (1.5) of groups in the splitting problem along a submanifold of codimension $q = 1, 2$. In this case, the groups $LS_*(\Psi)$ are denoted by $LN_*(A \to B)$. We assume that the right group B in the square Ψ is equipped with an orientation homomorphism. We denote by B^ξ the left group B together with the orientation homomorphism. The whole diagram (1.3) is realized on the spectrum level (see [1] and [10]). Consider the central square of the diagram (1.3) realized on the spectrum level:

$$\begin{array}{ccccc} & & \mathbb{L}(B) & & \\ & \nearrow & & \searrow & \\ \Sigma^q \mathbb{LP}(\Psi) & & & & \mathbb{L}(A \to B)\,, \\ & \searrow & & \nearrow & \\ & & \Sigma^q \mathbb{L}(B^\xi) & & \end{array} \tag{5.1}$$

where Σ denotes a functor from the category of spectra into itself. This functor is given for any spectrum $\mathbb{A} = \{\mathbb{A}_n\}$ by the condition $(\Sigma\mathbb{A})_n = \mathbb{A}_{n+1}$. The square (5.1) is a homotopy push-out square of spectra for which the fibers and the cofibers of parallel maps are homotopy equivalent. Denote by Ψ^- the push-out square of groups with the orientation reversed

$$\Psi^- = \begin{pmatrix} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B^\xi \end{pmatrix}. \tag{5.2}$$

Now we can construct the diagram of spectra:

$$\begin{array}{ccccc} & & \mathbb{L}(B) & & \\ & \nearrow & & \searrow & \\ \Sigma^q \mathbb{LP}(\Psi) & & & & \mathbb{L}(A \to B) \\ & \searrow & & \nearrow & \\ & & \Sigma^q \mathbb{L}(B^\xi) & & \\ & \nearrow & & \searrow & \\ \Sigma^{2q} \mathbb{LP}(\Psi^-) & & & & \Sigma^q \mathbb{L}(A \to B^\xi)\,. \\ & \searrow & & \nearrow & \\ & & \Sigma^{2q} \mathbb{L}(B) & & \\ & \nearrow & & \searrow & \\ \Sigma^{3q} \mathbb{LP}(\Psi) & & & & \Sigma^{2q} \mathbb{L}(A \to B) \\ & \searrow & & \nearrow & \\ & & \Sigma^{3q} \mathbb{L}(B^\xi) & & \\ & & \cdots & & \end{array} \tag{5.3}$$

The squares (5.1) are written in the vertical column of the diagram (5.3) for Ψ and Ψ^- with a shift of dimension. The diagram (5.3) is extended naturally up and down. Using push-out squares of spectra, it is possible to extend this diagram to the left and to the right (see [10]). Let us introduce the following notation (see [10] and [19]):

$$\begin{array}{rr} X_{0,0} = \mathbb{L}(B), & X_{1,1} = \Sigma^q \mathbb{L}(B^\xi), \\ X_{2,2} = \Sigma^{2q} \mathbb{L}(B), & X_{3,3} = \Sigma^{3q} \mathbb{L}(B^\xi), \\ \dots & \dots \\ X_{1,0} = \Sigma^q \mathbb{L}\mathbb{P}(\Psi), & X_{2,1} = \Sigma^{2q} \mathbb{L}\mathbb{P}(\Psi^-), \\ X_{3,2} = \Sigma^{3q} \mathbb{L}\mathbb{P}(\Psi), & X_{4,3} = \Sigma^{4q} \mathbb{L}\mathbb{P}(\Psi^-), \\ \dots & \dots \\ X_{0,1} = \mathbb{L}(A \to B), & X_{1,2} = \Sigma^q \mathbb{L}(A \to B^\xi), \\ X_{2,3} = \Sigma^{2q} \mathbb{L}(A \to B), & X_{3,4} = \Sigma^{3q} \mathbb{L}(A \to B^\xi), \\ \dots & \dots \end{array} .$$

Now the spectrum $X_{k,k-2}$ is constructed by means of known maps

$$X_{k-1,k-2} \to X_{k-1,k-1} \leftarrow X_{k,k-1}$$

in such a way that the square

$$\begin{array}{ccc} X_{k,k-2} & \to & X_{k-1,k-2} \\ \downarrow & & \downarrow \\ X_{k,k-1} & \to & X_{k-1,k-1} \end{array}$$

is a homotopy pull-back square. A similar construction with homotopy push-out squares is used to construct the spectrum $X_{k,k+2}$. The further construction of spectra $X_{m,l}$ is analogous. Thus we obtain an infinite homotopy commutative diagram of spectra

$$\begin{array}{cccccccc} & & & & & & \searrow & \\ & & & & & & & X_{-1,0} \\ & & & & \searrow & & \nearrow & \\ & & & & & X_{0,0} & & \\ & & \searrow & & \nearrow & & \searrow & \\ & & & X_{1,0} & & & & X_{0,1} \\ \searrow & & \nearrow & & \searrow & & \nearrow & \\ & X_{2,0} & & & & X_{1,1} & & \\ \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & & X_{2,1} & & & & X_{1,2} \\ & & \nearrow & & \searrow & & \nearrow & \\ & & & & & X_{2,2} & & \\ & & & & \nearrow & & \searrow & \\ & & & & & & & X_{2,3} \\ & & & & & & \nearrow & \end{array} . \qquad (5.4)$$

To construct the surgery spectral sequence, we consider the filtration (see [10])

$$\dots \to X_{3,0} \to X_{2,0} \to X_{1,0} \to X_{0,0} \to X_{-1,0} \to \dots \qquad (5.5)$$

Then, by definition (see [10]),

$$E_1^{p,s} = \pi_{s-p}(X_{p,0}, X_{p+1,0}) = \pi_{s-p}(X_{p,i}, X_{p+1,i}) \quad \forall i.$$

The differential

$$d_1^{p,s} : E_1^{p,s} \to E_1^{p+1,s}$$

is the natural composition

$$\pi_{s-p}(X_{p,p}, X_{p+1,p}) \overset{\cong}{\to} \pi_{s-p}(X_{p,p+1}, X_{p+1,p+1}) \to \pi_{s-p-1}\left(X_{p+1,p+1}, X_{p+2,p}\right).$$

Thus

$$\begin{aligned} E_1^{p,s} &= \pi_{s-p}(X_{p,p}, X_{p+1,p}) = \pi_{s-p}(X_{p-1,p}) \\ &= \pi_{s-p}\big(\Sigma^{(p+1)q+1}\mathbb{LS}(\Psi^{(-)^p})\big) = \pi_{s-(q+1)(p+1)}\mathbb{LS}(\Psi^{(-)^p}). \end{aligned}$$

Recall that for $q=1$ we have the isomorphism $E_1^{p,s} = LN_{s+2}(A \to B)$ (see [10]). For $q=2$, the group $E_1^{p,s}$ already depends on p and we have the isomorphism

$$E_1^{p,s} = LS_{s+p+1}(\Psi^{(-)^p}) = LN_{s+p+1}(A \to B^{(\xi)^p}).$$

In the case $q=1$, the differential d_1 is sufficiently simple and is completely described in algebraic terms in [10] and [16].

Proposition 2. *For $q = 2$, the first differential is the composition of maps from the diagram* (1.3):

$$d_1^{p,s} : LS_{s+p+1}(\Psi^{(-)^p}) \to L_{s+p+1}(B^{(\xi)^{p+1}}) \to LS_{s+p-2}(\Psi^{(-)^{p+1}}).$$

The first map appears in the diagram (1.3) *for the square $\Psi^{(-)^p}$, and the second map appears in the diagram* (1.3) *for the square $\Psi^{(-)^{p+1}}$.*

Proof. This follows from the construction of the spectral sequence as in [10] and [20]. □

The general properties of the surgery spectral sequence for $q=1$ described in [10] are preserved for $q=2$, since these properties only use the diagram (5.4) and the filtration (5.5), which is infinite in both directions. In particular, maps analogous to the iterated Browder–Livesay invariants are defined.

Even for $q=1$, this spectral sequence was not studied very extensively. Thus in [10] an example of a nontrivial second differential is given, but it is not known whether there exist pairs of groups with a nontrivial differential of higher order. In the case of finite abelian 2-groups, all differentials except the first one are trivial (see [10] and [17]).

In the case $q=2$, the isomorphism (2.9) allows to write splitting obstruction groups as Wall groups of certain rings with antistructures. The question of describing the first differential in algebraic terms on the level of rings with antistructures as in the case $q=1$ remains open. It is natural to expect that the second differential (and all differentials of higher order) will be always trivial in the case $q=2$. But at the present time the authors do not even have an example of a square Ψ in codimension 2 for which the first differential is nontrivial.

ACKNOWLEDGMENTS

The authors acknowledge the referee for carefully reading the manuscript and for helpful corrections and remarks.

The work of the first and the third authors was performed with the support of the Ministry of Science and Technology of the Republic of Slovenia under grant no. J1–0885–0101–98. The work of the second author was performed with the support of the Russian Foundation for Basic Research under grant no. 99–01–00009.

REFERENCES

1. C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, London, 1970.
2. A. A. Ranicki, "Exact Sequences in the Algebraic Theory of Surgery," in: *Matem. Notes*, vol. 26, Princeton Univ. Press, Princeton, N.J., 1981.
3. A. A. Ranicki, *Algebraic L-Theory and Topological Manifolds*, Cambridge Univ. Press, Cambridge, 1992.
4. Yu. V. Muranov, "Obstruction groups to splitting and quadratic extensions of antistructures," *Izv. Ross. Akad. Nauk Ser. Mat.* [*Russian Acad. Sci. Izv. Math.*], **59** (1995), no. 6, 107–132.
5. P. M. Akhmet′ev and Yu. V. Muranov, "Obstructions to the splitting of manifolds with infinite fundamental group," *Mat. Zametki* [*Math. Notes*], **60** (1996), no. 2, 163–175.
6. Yu. V. Muranov and D. Repovš, "Groups of obstructions to surgery and splitting for a manifold pair," *Mat. Sb.* [*Russian Acad. Sci. Sb. Math.*], **188** (1997), no. 3, 127–142.
7. A. A. Ranicki, *High-Dimensional Knot Theory*, Springer, Berlin–Heidelberg, 1998.
8. S. E. Cappell and J. L. Shaneson, "Pseudo-free actions," *Lecture Notes in Math.*, **763** (1979), 395–447.
9. A. F. Kharshiladze, "Surgery on manifolds with finite fundamental groups," *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **42** (1987), 55–85.
10. I. Hambleton and A. F. Kharshiladze, "A spectral sequence in surgery theory," *Mat. Sb.* [*Russian Acad. Sci. Sb. Math.*], **183** (1992), 3–14.
11. C. T. C. Wall, "Foundations of algebraic L-theory," *Lecture Notes in Math.*, **343** (1973), 266–300.
12. C. T. C. Wall, "On the classification of hermitian forms. VI. Group Rings," *Ann. of Math.*, **103** (1976), 1–80.
13. I. Hambleton, A. Ranicki, and L. Taylor, "Round L-theory," *J. Pure and Appl. Algebra*, **47** (1987), 131–154.
14. P. M. Akhmet′ev, "Splitting homotopy equivalences along a one-sided submanifold of codimension 1," *Izv. Akad. Nauk SSSR Ser. Mat.* [*Math. USSR-Izv.*], **51** (1987), 211–241.
15. Yu. V. Muranov and I. Hambleton, "Projective splitting obstruction groups for one-sided submanifolds," *Mat. Sb.* [*Russian Acad. Sci. Sb. Math.*], **190** (1999), 65–86.
16. I. Hambleton, "Projective surgery obstructions on closed manifolds," *Lecture Notes in Math.*, **967** (1982), 101–131.
17. Yu. V. Muranov and A. F. Kharshiladze, "Browder–Livesay groups of abelian 2-groups," *Mat. Sb.* [*Math. USSR-Sb.*], **181** (1990), 1061–1098.
18. Yu. V. Muranov and D. Repovš, "Surgery of closed manifolds with dihedral fundamental group," *Mat. Zametki* [*Math. Notes*], **64** (1998), no. 2, 238–250.
19. J. W. Milnor, "A procedure for killing the homotopy groups of differentiable manifolds," *Symposia in Pure Math.*, vol. 3, Amer. Math. Soc., Providence, R.I., 1961, pp. 39–55.
20. A. Cavicchioli, Y. V. Muranov, and D. Repovš, "Spectral sequences in K-theory for a twisted quadratic extension," *Yokohama Math. J.*, **46** (1998), 1–13.

(J. Malešič, D. Repovš) University of Ljubljana, Slovenia
E-mail: (J. Malešič) joze.malesic@fmf.uni-lj.si, (D. Repovš) dusan.repovs@fmf.uni-lj.si
(Yu. V. Muranov) Vitebsk State Technical University
E-mail: mur@vstu.unibel.by