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Fractional magnetic Schrödinger-Kirchhoff problems with convolution and critical nonlinearities

Sihua Liang[1,2] | **Dušan D. Repovš**[3,4] | **Binlin Zhang**[5]

[1]College of Mathematics, Changchun Normal University, Changchun, 130032, China

[2]College of Mathematics and Computer Science, Fujian Normal University, Qishan Campus, Fuzhou, 350117, China

[3]Department of Mathematics, University of Ljubljana, Ljubljana, 1000, Slovenia

[4]Institute of Mathematics, Physics, and Mechanics, Ljubljana, 1000, Slovenia

[5]College of Mathematics System Science, Shandong University of Science Technology, Qingdao, 266590, China

Correspondence
Binlin Zhang, College of Mathematics System Science, Shandong University of Science Technology, Qingdao 266590, China.
Email: zhangbinlin2012@163.com

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In this paper, we are concerned with the existence and multiplicity of solutions for the fractional Choquard-type Schrödinger-Kirchhoff equations with electromagnetic fields and critical nonlinearity:

$$\begin{cases} \varepsilon^{2s}M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = (|x|^{-\alpha} * F(|u|^2))f(|u|^2)u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

where $(-\Delta)_A^s$ is the fractional magnetic operator with $0 < s < 1$, $2_s^* = 2N/(N-2s)$, $\alpha < \min\{N, 4s\}$, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function, $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential, $F(|u|) = \int_0^{|u|} f(t)dt$, and $\varepsilon > 0$ is a positive parameter. The electric potential $V \in C(\mathbb{R}^N, \mathbb{R}_0^+)$ satisfies $V(x) = 0$ in some region of $\mathbb{R}^N$, which means that this is the critical frequency case. We first prove the $(PS)_c$ condition, by using the fractional version of the concentration compactness principle. Then, applying also the mountain pass theorem and the genus theory, we obtain the existence and multiplicity of semiclassical states for the above problem. The main feature of our problems is that the Kirchhoff term M can vanish at zero.

KEYWORDS
Choquard-type equation, critical nonlinearity, fractional magnetic operator, variational method

MSC CLASSIFICATION
35J10; 35B99; 35J60; 47G20

1 | INTRODUCTION AND MAIN RESULTS

In this paper, we consider the fractional Choquard-Kirchhoff–type problem with electromagnetic fields and critical nonlinearity:

$$\begin{cases} \varepsilon^{2s}M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = (\mathcal{K}_\alpha * F(|u|^2))\, f(|u|^2)u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a positive parameter, $N > 2s$, $0 < s < 1$, $2_s^* = 2N/(N-2s)$ is the critical Sobolev exponent, $V \in C(\mathbb{R}^N, \mathbb{R}_0^+)$ is an electric potential, $\mathcal{K}_\alpha(x) = |x|^{-\alpha}$, $\alpha < \min\{N, 4s\}$, $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ is a magnetic potential, and

$$[u]_{s,A}^2 := \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} dxdy.$$

If A is a smooth function, the fractional operator $(-\Delta)_A^s$, which up to normalization constants can be defined on smooth functions u as

$$(-\Delta)_A^s u(x) := 2\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

has recently been introduced in d'Avenia and Squassina.[1] Hereafter, $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and of radius $\varepsilon > 0$. For details on fractional magnetic operators, we refer to d'Avenia and Squassina,[1] and for the physical background, we refer to previous studies.[2-4]

This paper was motivated by some works concerning the magnetic Schrödinger equation

$$-(\nabla u - iA)^2 u + V(x)u = f(x, |u|)u, \tag{1.2}$$

which have appeared in recent years (see other works[5-9]) and have extensively studied 1.2, when the above magnetic

$$-(\nabla u - iA)^2 u = -\Delta u + 2iA(x) \cdot \nabla u + |A(x)|^2 u + iu\mathrm{div}A(x).$$

As stated in Squassina and Volzone,[10] up to correcting the operator by the factor $(1-s)$, it follows that $(-\Delta)_A^s u$ converges to $-(\nabla u - iA)^2 u$ as $s \to 1$.

Thus, up to normalization, the nonlocal case can be seen as an approximation of the local one. The motivation for its introduction was described in the literature[1,10] and relies essentially on the Lévy-Khintchine formula for the generator of a general Lévy process. If the magnetic field $A \equiv 0$, the operator $(-\Delta)_{A_\varepsilon}^s$ can be reduced to the fractional Laplacian operator $(-\Delta)^s$, which may be viewed as the infinitesimal generator of a Lévy stable diffusion processes.[11] This operator arises in the description of various phenomena in applied sciences, such as phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, and plasma physics, see Di Nezza et al[12] and references therein.

The study of fractional and nonlocal operators of elliptic type has recently attracted a lot of attention. For the cases in which bounded domains and the entire space are involved, we refer the readers, eg, to previous works[13-19] and the references therein. When the interaction between the particles is considered, ie, when the nonlinear term $f(u)$ is of type $(\mathcal{K}_\alpha * |u|^p)|u|^{p-2}u$, this type of problem is usually called the Choquard-type equation and has been investigated by many authors, see, eg, other studies.[20,21]

Another strong motivation for studying problem (1.1) is the significant feature of Kirchhoff-type problems. More precisely, in 1883 Kirchhoff proposed the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{\lambda} + \frac{Y}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

as a generalization of the well-known D'Alembert's wave equation for free vibrations of elastic strings. Here, L is the length of the string, λ is the area of the cross section, Y is the Young modulus of the material, ρ is the mass density, and p_0 is the initial tension. Essentially, Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. For recent results in this direction, we refer the reader, eg, to previous studies.[22,23]

Recently, Fiscella and Valdinoci[24] first deduced a stationary fractional Kirchhoff model which considered the non-local aspect of the tension arising from nonlocal measurements of the fractional length of the string (see Fiscella and Valdinoci,[24] appendix for more details). More precisely, the following Kirchhoff-type problem involving critical exponent was studied in Fiscella and Valdinoci[24]:

$$\begin{cases} M([u]_s^2)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

where Ω is an open bounded domain in $\mathbb{R}^N$. By using the mountain pass theorem and the concentration compactness principle, together with a truncation technique, the existence of nonnegative solutions for problem (1.4) was obtained.

Here, we point out that $M(0) > 0$ in (1.4), this is called the nondegenerate case. Otherwise, the problem is called degenerate if $M(0) = 0$. In recent years, there has been a lot of interest in studying fractional Kirchhoff-type problems, here we just list some references, eg, see the literature[16,25,26] for recent results on the nondegenerate case, previous works[21,27-30] for recent results on the degenerate case, and other studies[31-33] for discussions of both cases.

Next, let us mention some enlightening works related to problem (1.1). Mingqi et al[31] first studied the following Schrödinger-Kirchhoff–type equation involving the fractional p-Laplacian and the magnetic operator

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = f(x,|u|)u \quad \text{in} \quad \mathbb{R}^N, \tag{1.5}$$

where the right-hand term in (1.5) satisfies the subcritical growth. By using variational methods, they obtained several existence results for problem (1.5). Using similar methods, for $M(t) = a + bt$ with $a \in \mathbb{R}_0^+$ and $b \in \mathbb{R}^+$, Wang and Xiang[21] proved the existence of two solutions and infinitely many solutions for fractional Schrödinger-Choquard-Kirchhoff–type equations with external magnetic operator and critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

Binlin et al[34] first considered the following singularly perturbed fractional Schrödinger equations:

$$\varepsilon^{2s}(-\Delta)_{A_\varepsilon}^s u + V(x)u = f(x,|u|)u + K(x)|u|^{2_\alpha^*-2}u \ \text{in} \ \mathbb{R}^N, \tag{1.6}$$

where $V(x)$ satisfies some assumptions. By using variational methods, they proved the existence of solutions u_ε which tends to the trivial solution as $\varepsilon \to 0$. Moreover, they proved the existence of infinite many solutions and sign-changing solutions for problem (1.6) without magnetic field under some additional assumptions.

Subsequently, Liang et al[25] investigated the existence and multiplicity of solutions for problem (1.1) without Choquard-type term in the non-degenerate Kirchhoff case. Very recently, by employing variational methods, Ambrosio[35] obtained the existence and concentration of nontrivial solutions for a singularly perturbed fractional Choquard problem with a subcritical nonlinearity and an external magnetic field.

Inspired by the above works, in particular the literature,[25,31,34,36] we consider in this article the existence and multiplicity of solutions for the fractional Choquard-type problems with electromagnetic fields and critical nonlinearity in the possibly degenerate Kirchhoff context. It is worthwhile to remark that in the arguments developed in other studies,[34,36] one of the key points is to prove the $(PS)_c$ condition. Here, we use the fractional version of Lions' second concentration compactness principle and concentration compactness principle at infinity to prove that the $(PS)_c$ condition holds, which is different from methods used in previous works.[34,36]

In fact, the appearance of the magnetic field also brings additional difficulties into the study of our problem, eg, the effects of the magnetic fields on the linear spectral sets and on the solution structure, and the possible interactions between the magnetic fields and the linear potentials. Therefore, we need to develop new techniques to conquer difficulties induced by these new features as well as the possibly degenerate nature of the Kirchhoff coefficient.

Suppose that functions $V(x)$, $M(t)$ and $f(t)$ satisfy the following conditions:

(V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\min_{x\in\mathbb{R}^N} V(x) = 0$ and there is $\tau_0 > 0$ such that the set $V^{\tau_0} = \{x \in \mathbb{R}^N : V(x) < \tau_0\}$ has finite Lebesgue measure.

(M) (M_1) there exists $\sigma \in (1, 2_s^*/2)$ satisfying $\sigma\tilde{M}(t) \geq M(t)t$ for all $t \geq 0$, where $\tilde{M}(t) = \int_0^t M(s)ds$;
(M_2) there exists $m_1 > 0$ such that $M(t) \geq m_1 t^{\sigma-1}$ for all $t \in \mathbb{R}^+$ and $M(0) = 0$.

(F) (f_1) $f \in C(\mathbb{R}^+, \mathbb{R})$;
(f_2) there exist $c_0 > 0$ and $\max\{\sigma, 2\} < p < 2_s^*$ such that $|f(t)| \leq c_0|t|^{\frac{p-1}{2}}$;
(f_3) there exist $2\sigma < \mu < 2_s^*$ such that $0 < \mu F(t) \leq f(t)t$ for all $t \in \mathbb{R}^+$, where $F(t) = \int_0^t f(s)ds$.

The following is our first main result, the existence theorem for problem (1.1).

Theorem 1. *Let the conditions (V), (M) and (F) be satisfied. Then for any $\kappa > 0$, there is $\mathcal{E}_\kappa > 0$ such that if $0 < \varepsilon < \mathcal{E}_\kappa$, then problem (1.1) has at least one solution u_ε satisfying*

$$\frac{2\mu-\sigma}{4\sigma}\int\int_{\mathbb{R}^{2N}} \frac{F(|u_\varepsilon(x)|^2)F(|u_\varepsilon(y)|^2)}{|x-y|^\alpha}dxdy + \left(\frac{1}{2\sigma} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} |u_\varepsilon|^{2_s^*}dx \leq \kappa\varepsilon^{\frac{2s2_s^*}{2_s^*-4\sigma}}, \tag{1.7}$$

$$\left(\frac{1}{2\sigma}-\frac{1}{\mu}\right)\alpha_0\varepsilon^{2s}[u_\varepsilon]_{s,A}^{2\sigma}+\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}V(x)|u_\varepsilon|^2dx\leq\kappa\varepsilon^{\frac{2s2_s^*}{2_s^*-4\sigma}}. \tag{1.8}$$

Moreover, $u_\varepsilon\to 0$ *in* E *as* $\varepsilon\to 0$.

The following is our second main result, the multiplicity theorem for problem (1.1).

Theorem 2. *Let the conditions (V), (M) and (F) be satisfied. Then for any* $m\in\mathbb{N}$ *and* $\kappa>0$, *there is* $\mathcal{E}_{m\kappa}>0$ *such that if* $0<\varepsilon<\mathcal{E}_{m\kappa}$, *then problem (1.1) has at least m pairs of solutions* $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i=1,2,\ldots,m$ *which satisfy estimates (1.7) and (1.8). Moreover,* $u_{\varepsilon,i}\to 0$ *in* E *as* $\varepsilon\to 0$, $i=1,2,\ldots,m$.

2 | FUNCTIONAL SETTING

In this paper, we will use Banach space E defined by

$$E=\left\{u\in H_A^s(\mathbb{R}^N,\mathbb{C}):\int_{\mathbb{R}^N}V(x)|u|^2dx<\infty\right\}$$

with the norm

$$\|u\|_E:=\left([u]_{s,A}^2+\int_{\mathbb{R}^N}V(x)|u|^2dx\right)^{\frac{1}{2}},$$

where V is nonnegative, $H_{A_\varepsilon}^s(\mathbb{R}^N,\mathbb{C})$ is the fractional Sobolev space defined by

$$H_A^s(\mathbb{R}^N,\mathbb{C})=\left\{u\in L^2(\mathbb{R}^N,\mathbb{C}):[u]_{s,A}<\infty\right\},$$

where $s\in(0,1)$ and $[u]_{s,A}$ denotes the so-called Gagliardo semi-norm, that is

$$[u]_{s,A}=\left(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}$$

and $H_A^s(\mathbb{R}^N,\mathbb{C})$ is endowed with the norm

$$\|u\|_{H_A^s(\mathbb{R}^N,\mathbb{C})}=\left([u]_{s,A}^2+\|u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

By assumption (V), we know that the embedding $EH_A^s(\mathbb{R}^N,\mathbb{C})$ is continuous. Note that the norm $\|\cdot\|_E$ is equivalent to the norm $\|\cdot\|_\varepsilon$ defined by

$$\|u\|_\varepsilon:=\left([u]_{s,A}^2+\varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u|^2dx\right)^{\frac{1}{2}},$$

for each $\varepsilon>0$. It is obvious that for each $\theta\in[2,2_s^*]$, there is $c_\theta>0$, independent of $0<\varepsilon<1$, such that

$$|u|_\theta\leq c_\theta\|u\|_E\leq c_\theta\|u\|_\varepsilon. \tag{2.1}$$

Hereafter, we shortly denote by $\|\cdot\|_\nu$ the norm of Lebesgue space $L^\nu(\Omega)$ with $\nu\geq 1$.
We first recall the following embedding theorem:

Proposition 1. *(see d'Avenia and Squassina[1], lemma 3.5). Let* $A\in C(\mathbb{R}^N,\mathbb{R}^N)$. *Then the embedding*

$$H_A^s(\mathbb{R}^N,\mathbb{C})\hookrightarrow L^\theta(\mathbb{R}^N,\mathbb{C}),$$

is continuous for any $\theta\in[2,2_s^*]$. *Moreover, the embedding*

$$H_A^s(\mathbb{R}^N,\mathbb{C})\hookrightarrow\hookrightarrow L_{loc}^\theta(\mathbb{R}^N,\mathbb{C})$$

is compact for any $\theta \in [1, 2_s^*)$.

We will use the following diamagnetic inequality:

Lemma 1. *(see lemma 3.5[1], lemma 3.3). For every* $u \in H_A^s(\mathbb{R}^N, \mathbb{C})$,

$$|u| \in H^s(\mathbb{R}^N).$$

More precisely,

$$[|u|]_s \le [u]_{s,A}.$$

By Di Nezza et al,[12] proposition 3.6, we have

$$[u]_s = \|(-\Delta)^{\frac{s}{2}}\|_{L^2(\mathbb{R}^N)}$$

for any $u \in H^s(\mathbb{R}^N)$, ie,

$$\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx.$$

Thus,

$$\int\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(x) \cdot (-\Delta)^{\frac{s}{2}} v(x) dx.$$

To obtain the solution of problem (1.1), we will use the following equivalent form

$$\begin{cases} M([u]_{s,A}^2)(-\Delta)_A^s u + \varepsilon^{-2s} V(x) u = \varepsilon^{-2s} \int_{\mathbb{R}^N} \frac{F(|u|^2)}{|x-y|^\alpha} dy f(|u|^2) u + \varepsilon^{-2s} |u|^{2_s^*-2} u, \ x \in \mathbb{R}^N, \\ u(x) \to 0, \quad as \quad |x| \to \infty, \end{cases} \tag{2.2}$$

for $\varepsilon \to 0$. The energy functional $J_\varepsilon : E \to \mathbb{R}$ associated with problem (2.2)

$$J_\varepsilon(u) := \frac{1}{2}\tilde{M}\left([u]_{s,A}^2\right) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{\varepsilon^{-2s}}{4}\int\int_{\mathbb{R}^{2N}} \frac{F(|u(x)|^2)F(|u(y)|^2)}{|x-y|^\alpha} dxdy - \frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^N} |u|^{2_s^*} dx$$

is well defined. Under the assumptions, it is easy to check that as shown in the literature,[37,38] $J_\varepsilon \in C^1(E, \mathbb{R})$ and its critical points are weak solutions of problem (2.2).

By condition (f_2), we have

$$F(|u|^2) \le C(|u|^2 + |u|^p), \quad \text{for all} \quad u \in H_A^s(\mathbb{R}^N, \mathbb{C}).$$

Note that, by the Hardy-Littlewood-Sobolev inequality, the integral

$$\int\int_{\mathbb{R}^{2N}} \frac{F(|u(x)|^2)F(|u(y)|^2)}{|x-y|^\alpha} dxdy$$

is well defined if $F(|u|^2) \in L^r(\mathbb{R}^N)$ for some $r > 1$ satisfying

$$\frac{2}{r} + \frac{\alpha}{N} = 2,$$

that is $r = 2N/(2N-\alpha)$. Actually, by $\alpha < \min\{N, 4s\}$, it follows that $2 < 2r < 2_s^*$. Moreover, from $2 < pr < 2_s^*$, we can deduce

$$\int_{\mathbb{R}^N} |F(|u|^2)|^r dx \le 2^{r-1} C^r \left(\int_{\mathbb{R}^N} |u|^{2r} dx + \int_{\mathbb{R}^N} |u|^{pr} dx\right) \tag{2.3}$$

$$\le 2^{r-1} C^r \left(C_{2r}^{2r} \|u\|^{2r} + C_{pr}^{pr} \|u\|^{pr}\right) \quad \text{for all} \ \ u \in H_A^s(\mathbb{R}^N, \mathbb{C}). \tag{2.4}$$

By a standard argument, one can show that $J_\varepsilon(u)$ is of class C^1 and

$$\langle J_\varepsilon'(u), v\rangle = M\left([u]_{s,A}^2\right) \mathrm{Re} \int\int_{\mathbb{R}^{2N}} \frac{(u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u(y))\overline{(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} v(y))}}{|x-y|^{N+2s}} dxdy$$

$$+\varepsilon^{-2s} \mathrm{Re} \int_{\mathbb{R}^N} V(x) u \bar{v} dx - \varepsilon^{-2s} \mathrm{Re} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * F(|u|^2)) f(|u|^2) u \bar{v} dx - \varepsilon^{-2s} \mathrm{Re} \int_{\mathbb{R}^N} |u|^{2_s^*-2} u \bar{v} dx,$$

for all $u, v \in E$. Hence, a critical point of J_ε is a weak solution of problem (1.1).

Now, we recall the general version of the mountain pass theorem in Rabinowitz[37] which will be used later.

Theorem 3. *Let $\mathcal{J}$ be a functional on a Banach space Y and $\mathcal{J} \in C^1(Y, \mathbb{R})$. Let us assume that there exist $\zeta, \rho > 0$ such that*

(i) *$\mathcal{J}(u) \geq \zeta$, for every $u \in Y$ with $\|u\| = \rho$;*
(ii) *$\mathcal{J}(0) = 0$ and $\mathcal{J}(e) < \zeta$ for some $e \in Y$ with $\|e\| > \rho$.*

Let us define $\Gamma = \{\gamma \in C([0,1]; Y) : \gamma(0) = 0, \gamma(1) = e\}$ and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{J}(\gamma(t)).$$

Then there exists a sequence $\{u_n\}_n \subset Y$ such that $\mathcal{J}(u_n) \to c$ and $\mathcal{J}'(u_n) \to 0$ in Y' (dual of Y).

By the assumptions (V), (M) and (F), one can see that $J_\varepsilon(u)$ has the mountain pass geometry.

Lemma 2. *Assume that conditions (V), (M) and (F) hold. Then the functional J_ε satisfies the conclusions (i)-(ii) of Theorem 3.*

Proof. For each $\varepsilon > 0$, by the fractional Sobolev embedding, (M_2) and (f_2), we have

$$\begin{aligned} J_\varepsilon(u) &:= \frac{1}{2}\tilde{M}\left([u]_{s,A}^2\right) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{\varepsilon^{-2s}}{4}\int\int_{\mathbb{R}^{2N}} \frac{F(|u(x)|^2)F(|u(y)|^2)}{|x-y|^\alpha} dxdy - \frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^N} |u|^{2_s^*} dx \\ &\geq \min\left\{\frac{m_1}{2\sigma}, \frac{1}{2}\right\} \|u\|_\varepsilon^{2\sigma} - \varepsilon^{-2s} C \|u\|_\varepsilon^{2p} - \frac{\varepsilon^{-2s}}{2_s^*} S^{\frac{-2_s^*}{2}} \|u\|_\varepsilon^{2_s^*}, \end{aligned}$$

for all $u \in E$. It follows from $\max\{2, \sigma\} < p$ that there exist small enough $\varrho_\varepsilon > 0$ and $\alpha_\varepsilon > 0$ such that $J_\varepsilon(u) \geq \alpha_\varepsilon > 0$ for all $u \in E$ with $\|u\|_\varepsilon = \varrho_\varepsilon$, and all $\varepsilon > 0$. Hence, (i) in Theorem 3 holds.

Now, we verify condition (*ii*) in Theorem 3. Let $\varphi_0 \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with $\|\varphi_0\|_\varepsilon = 1$. By (M_2), we have

$$\tilde{M}(t) \leq \tilde{M}(1) t^\sigma \quad \text{for all } t \geq 1. \tag{2.5}$$

Then by (f_3), the following holds

$$\begin{aligned} J_\varepsilon(t\varphi_0) &\leq \tilde{M}(1)t^{2\sigma} + \frac{1}{2}t^2 - \frac{\varepsilon^{-2s}}{4}\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|t\varphi_0|^2))F(|t\varphi_0|^2)dx - \frac{\varepsilon^{-2s}}{2_s^*} t^{2_s^*} |\varphi_0|_{2_s^*}^{2_s^*} \\ &\leq \tilde{M}(1)t^{2\sigma} + \frac{1}{2}t^2 - \frac{\varepsilon^{-2s}}{2_s^*} t^{2_s^*} |\varphi_0|_{2_s^*}^{2_s^*}, \end{aligned}$$

and hence $J_\varepsilon(t\varphi_0) \to -\infty$ as $t \to \infty$, since $2\sigma < 2_s^*$. Therefore, there exists large enough t_0 such that $J_\varepsilon(t_0\varphi_0) < 0$. Then we take $e = t_0\varphi_0$ and $J_\varepsilon(e) < 0$. Hence (*ii*) in Theorem 3 holds. The proof is thus complete. □

3 | VERIFICATION OF $(PS)_C$ CONDITION

In this section, we recall the fractional version of concentration compactness principle in the fractional Sobolev space, see other studies[30,39,40] for more details.

Lemma 3. *(see Palatucci and Pisante[39], theorem 1.5) Let $\Omega \subseteq \mathbb{R}^N$ be an open subset and let $\{u_n\}_n$ be a sequence in $H^s(\mathbb{R}^N)$, weakly converging to u as $n \to \infty$ and such that $|u_n|^{2_s^*} \rightharpoonup \nu$ and $|(-\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \mu$ in the sense of measures. Then either $u_n \to u$ in $L_{loc}^{2_s^*}(\mathbb{R}^N)$ or there exist a (at most countable) set of distinct points $\{x_j\}_{j\in I} \subseteq \overline{\Omega}$ and positive numbers $\{\nu_j\}_{j\in I}$ such that*

$$\nu = |u|^{2_s^*} + \sum_{j \in I} \delta_{x_j} \nu_j, \quad \nu_j > 0.$$

If, in addition, Ω is bounded, then there exist a positive measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^N)$ with $supp\tilde{\mu} \subseteq \overline{\Omega}$ and positive numbers $\{\mu_j\}_{j\in I}$ such that

$$\mu = |(-\Delta)^{\frac{s}{2}} u|^2 + \tilde{\mu} + \sum_{j\in I} \delta_{x_j}\mu_j, \quad \mu_j > 0$$

and

$$\nu_j \leq (S^{-1}\mu(\{x_j\}))^{\frac{2_s^*}{2}},$$

where S is the best Sobolev constant, ie,

$$S = \inf_{u\in H^s(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\int_{\mathbb{R}^N} |u|^{2_s^*} dx},$$

$x_j \in \mathbb{R}^N$, δ_{x_j} are Dirac measures at x_j and μ_j, ν_j are constants.

In the case $\Omega = \mathbb{R}^N$, the above principle of concentration compactness does not provide any information about the possible loss of mass at infinity. The following result expresses this fact in quantitative terms.

Lemma 4. *(see Zhang et al[40], lemma 3.5) Let $\{u_n\}_n \subset H^s(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ weakly converges in $H^s(\mathbb{R}^N)$, $|u_n|^{2_s^*} \rightharpoonup \nu$ and $|(-\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \mu$ weakly-$*$ converges in $\mathcal{M}(\mathbb{R}^N)$ and define*

(i) $\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N : |x|>R\}} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx,$

(ii) $\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N : |x|>R\}} |u_n|^{2_s^*} dx.$

Then the quantities ν_∞ and μ_∞ exist and satisfy the following

(iii) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty,$

(iv) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty,$

(v) $\nu_\infty \leq (S^{-1}\nu_\infty)^{\frac{2_s^*}{2}}.$

The main result of this section is the following compactness result:

Lemma 5. *Suppose that conditions (V), (M) and (F) hold. Let $\{u_n\}_n \subset E$ be a $(PS)_c$ sequence of functional J_ε, ie,*

$$J_\varepsilon(u_n) \to c \ \text{ and } \ J_\varepsilon'(u_n) \to 0 \ \text{ in } \ E'$$

as $n \to \infty$, where E' is the dual of E. Then for any $0 < \varepsilon < 1$, J_ε satisfies $(PS)_c$ condition, for all $c \in \left(0,\ \sigma_0 \varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}}\right)$, where $\sigma_0 := \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right) \left(m_1 S^{2\sigma}\right)^{\frac{2_s^*}{2_s^*-4\sigma}}$, i.e. any $(PS)_c$-sequence $\{u_n\}_n \subset E$ has a strongly convergent subsequence in E.

Proof. If $\inf_{n\geq 1} \|u\|_\varepsilon = 0$, then there exists a subsequence of $\{u_n\}_n$ (still denoted by $\{u_n\}_n$) such that $u_n \to 0$ in E as $n \to \infty$. Thus, we assume that $d := \inf_{n\geq 1} \|u\|_\varepsilon > 0$ in the sequel. By $J_\varepsilon(u_n) \to c$ and $J_\varepsilon'(u_n) \to 0$ in E', there exists $C > 0$ such that

$$\begin{aligned} c + o(1)\|u_n\|_\varepsilon &= J_\varepsilon(u_n) - \frac{1}{\mu}\langle J_\varepsilon'(u_n), u_n\rangle = \frac{1}{2}\tilde{M}\left([u_n]_{s,A}^2\right) - \frac{1}{\mu} M\left([u_n]_{s,A}^2\right)[u_n]_{s,A}^2 \\ &\quad + \left(\frac{1}{2} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_n|^2 dx + \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \\ &\quad + \varepsilon^{-2s}\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2))\left(\frac{1}{\mu} f(|u_n|^2)|u_n|^2 - \frac{1}{4}F(|u_n|^2)\right) dx. \end{aligned} \tag{3.1}$$

It follows by (M_2) and (f_3) that

$$C + C\|u_n\|_\varepsilon \geq \left(\frac{1}{2\sigma} - \frac{1}{\mu}\right) M\left([u_n]_{s,A}^2\right)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_n|^2 dx$$
$$\geq \left(\frac{1}{2\sigma} - \frac{1}{\mu}\right) m_1[u_n]_{s,A}^{2\sigma} + \left(\frac{1}{2} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_n|^2 dx.$$

This, together with $2 < 2\sigma < 2_s^*$, implies that $\{u_n\}_n$ is bounded in E. Furthermore, we can obtain $c \geq 0$ by passing to the limit in (3.1). Hence, by diamagnetic inequality, $\{|u_n|\}_n$ is bounded in $H^s(\mathbb{R}^N)$. Therefore for some subsequence, there is $u \in E$ such that $u_n \rightharpoonup u$ in E. Since $2 < p < \frac{2N-\alpha}{N-2s} < 2_s^*$ and $2 < \frac{4N}{2N-\alpha} < 2_s^*$, by Proposition 1 we get that $|u_n| \to |u|$ strongly in $L^{\frac{2Np}{2N-\alpha}}(\mathbb{R}^N) \cap L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)$. Hence the Brézis-Lieb Lemma implies that $u_n \to u$ strongly in $L^{\frac{2Np}{2N-\alpha}}(\mathbb{R}^N, \mathbb{C}) \cap L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N, \mathbb{C})$. By (f_2), we have

$$\int_{\mathbb{R}^N}\left|F(|u_n|^2 - F(|u|^2))\right|^{\frac{2N}{2N-\alpha}} dx \leq \int_{\mathbb{R}^N}\left|f(|u|^2 + \varrho(|u_n|^2 - |u|^2))\right|^{\frac{2N}{2N-\alpha}}\left||u_n|^2 - |u|^2\right|^{\frac{2N}{2N-\alpha}} dx$$
$$\leq \int_{\mathbb{R}^N}\left[C(1 + (|u_n| + |u|)^{p-2})\right]^{\frac{2N}{2N-\alpha}}(|u_n| + |u|)^{\frac{2N}{2N-\alpha}}|u_n - u|^{\frac{2N}{2N-\alpha}} dx$$
$$\leq C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}}\int_{\mathbb{R}^N}(|u_n| + |u|)^{\frac{2N}{2N-\alpha}}|u_n - u|^{\frac{2N}{2N-\alpha}} dx$$
$$+ C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}}\int_{\mathbb{R}^N}(|u_n| + |u|)^{(p-1)\frac{2N}{2N-\alpha}}|u_n - u|^{\frac{2N}{2N-\alpha}} dx.$$

Using the Hölder inequality, we can deduce

$$\int_{\mathbb{R}^N}\left|F(|u_n|^2 - F(|u|^2))\right|^{\frac{2N}{2N-\alpha}} dx \leq C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}}\|(|u_n| + |u|)^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)}\||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)}$$
$$+ C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}}\|(|u_n| + |u|)^{(p-1)\frac{2N}{2N-\alpha}}\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)}\||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^p(\mathbb{R}^N)}$$
$$\leq C\||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)} + C\||u_n - u|^{\frac{2N}{2N-\alpha}}\|_{L^p(\mathbb{R}^N)}$$
$$\to 0,$$

as $n \to \infty$, where $C > 0$ is independent of n. Thus, we obtain that $F(|u_n|^2 \to F(|u|^2)$ in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$. Note that by the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map form $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$. Then

$$\mathcal{K}_\alpha * F(|u_n|^2) \to \mathcal{K}_\alpha * F(|u|^2) \quad \text{in} \quad L^{\frac{2N}{\alpha}}(\mathbb{R}^N) \tag{3.2}$$

as $n \to \infty$.

For $\varphi \in E$ fixed, by (f_2) with $\varepsilon = 1$ we have

$$\int_{\mathbb{R}^N}\left|f(|u_n|^2)u_n\overline{\varphi}\right|^{\frac{2N}{2N-\alpha}} dx \leq C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}}\left(\int_{\mathbb{R}^N}(|u_n||\varphi|)^{\frac{2N}{2N-\alpha}} dx + \int_{\mathbb{R}^N}|u_n|^{(p-1)\frac{2N}{2N-\alpha}}|\varphi|^{\frac{2N}{2N-\alpha}} dx\right)$$
$$\leq C^{\frac{2N}{2N-\alpha}} 2^{\frac{\alpha}{2N-\alpha}}\left(\||u_n|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)}\||\varphi|^{\frac{2N}{2N-\alpha}}\|_{L^2(\mathbb{R}^N)}\right.$$
$$\left.+\||u_n|^{(p-1)\frac{2N}{2N-\alpha}}\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)}\||\varphi|^{\frac{2N}{2N-\alpha}}\|_{L^p(\mathbb{R}^N)}\right)$$
$$\leq C,$$

thanks to $2 < \frac{4N}{2N-\alpha} < 2_s^*$ and $2 < \frac{2pN}{2N-\alpha} < 2_s^*$, where $C > 0$ denotes various constants.

Clearly, $f(|u_n|^2)u_n\overline{\varphi} \to f(|u|^2)u\overline{\varphi}$ ae in $\mathbb{R}^N$. Hence, up to a subsequence, $\mathrm{Re}\left\{f(|u_n|^2)u_n\overline{\varphi}\right\}$ weakly converges to $\mathrm{Re}\left\{f(|u|^2)u\overline{\varphi}\right\}$ in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$. This together with (3.2) yields that

$$\lim_{n\to\infty} \mathrm{Re}\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)u_n\overline{\varphi}dx = \lim_{n\to\infty} \mathrm{Re}\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u|^2))f(|u|^2)u\overline{\varphi}dx \tag{3.3}$$

for each $\varphi \in E$.

We claim that as $n \to \infty$,

$$\int_{\mathbb{R}^N} |u_n|^{2_s^*}dx \to \int_{\mathbb{R}^N} |u|^{2_s^*}dx. \tag{3.4}$$

In order to prove this claim, we invoke Prokhorov's Theorem (see Bogachev[41], theorem 8.6.2) to conclude that there exist $\mu, \nu \in \mathcal{M}(\mathbb{R}^N)$ such that

$$\begin{aligned}
&|(-\Delta)^{\frac{s}{2}}u_n|^2 \rightharpoonup \mu \quad \text{(weak*-sense of measures)},\\
&|u_n|^{2_s^*} \rightharpoonup \nu \quad \text{(weak*-sense of measures)},
\end{aligned}$$

where μ and ν are a nonnegative bounded measures on $\mathbb{R}^N$. It follows by Lemma 3 that either $u_n \to u$ in $L^{2_s^*}_{loc}(\mathbb{R}^N)$ or $\nu = |u|^{2_s^*} + \sum_{j\in I}\delta_{x_j}\nu_j$, as $n \to \infty$, where I is a countable set, $\{\nu_j\}_j \subset [0,\infty)$, $\{x_j\}_j \subset \mathbb{R}^N$.

Take $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \phi \le 1$; $\phi \equiv 1$ in $B(x_j,\rho)$, $\phi(x) = 0$ in $\mathbb{R}^N \backslash B(x_j, 2\rho)$. For any $\rho > 0$, define $\phi_\rho = \phi\left(\frac{x-x_j}{\rho}\right)$, where $j \in I$. It follows that

$$\begin{aligned}
&\int\int_{\mathbb{R}^{2N}} \frac{|u_n(x)\phi_\rho(x) - u_n(y)\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy\\
&\le 2\int\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2\phi_\rho^2(y)}{|x-y|^{N+2s}}dxdy + 2\int\int_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy\\
&\le 2\int\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}dxdy + 2\int\int_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy.
\end{aligned} \tag{3.5}$$

Similar to the proof of Zhang and Zhang,[42], lemma 3.4 we can show that

$$\int\int_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy \le C\rho^{-2s}\int_{B(x_j,K\rho)} |u_n(x)|^2dx + CK^{-N}, \tag{3.6}$$

where $K > 4$. Since $\{u_n\}_n$ is bounded in E, it follows from (3.5) and (3.6) that $\{u_n\phi_\rho\}_n$ is bounded in E. Then $\langle J'_\varepsilon(u_n), u_n\phi_\rho\rangle \to 0$, which implies

$$\begin{aligned}
&M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2\phi_\rho(y)}{|x-y|^{N+2s}}dxdy + \varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_n|^2\phi_\rho(x)dx\\
&= -\mathrm{Re}\left\{M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}} \frac{(u_n(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_\rho(x) - \phi_\rho(y))}}{|x-y|^{N+2s}}dxdy\right\}\\
&+ \varepsilon^{-2s}\int_{\mathbb{R}^N} |u_n|^{2_s^*}\phi_\rho dx + \varepsilon^{-2s}\int_{\mathbb{R}^N} \mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2\phi_\rho dx + o_n(1).
\end{aligned} \tag{3.7}$$

Note that by (M_2) and diamagnetic inequality, the following holds

$$
\begin{aligned}
&M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2\phi_\rho(y)}{|x-y|^{N+2s}}dxdy\\
&\geq m_1\left(\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2\phi_\rho(y)}{|x-y|^{N+2s}}dxdy\right)^{2\sigma}\\
&\geq m_1\left(\int\int_{\mathbb{R}^{2N}}\frac{||u_n(x)|-|u_n(y)||^2\phi_\rho(y)}{|x-y|^{N+2s}}dxdy\right)^{2\sigma}.
\end{aligned}
$$

It is easy to verify that

$$
\int\int_{\mathbb{R}^{2N}}\frac{||u_n(x)|-|u_n(y)||^2\phi_\rho(y)}{|x-y|^{N+2s}}dxdy\to\int_{\mathbb{R}^N}\phi_\rho d\mu,
$$

as $n\to\infty$ and

$$
\int_{\mathbb{R}^N}\phi_\rho d\mu\to\mu(\{x_i\})
$$

as $\rho\to 0$. Note that the Hölder inequality implies

$$
\begin{aligned}
&\left|\mathrm{Re}\left\{M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_\rho(x)-\phi_\rho(y))}}{|x-y|^{N+2s}}dxdy\right\}\right|\\
&\leq C\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|\cdot|\phi_\rho(x)-\phi_\rho(y)|\cdot|u_n(x)|}{|x-y|^{N+2s}}dxdy\\
&\leq C\left(\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}.
\end{aligned}
\tag{3.8}
$$

Similar to the proof Zhang and Zhang,[42], lemma 3.4 we can show that

$$
\lim_{\rho\to 0}\lim_{n\to\infty}\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy=0. \tag{3.9}
$$

It follows from

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathcal{K}_\alpha*F(|u_n|^2))f(|u_n|^2)|u_n|^2\phi_\rho dx=\int_{\mathbb{R}^N}\mathcal{K}_\alpha*F(|u|^2))f(|u|^2)|u|^2\phi_\rho dx
$$

and

$$
\lim_{\rho\to 0}\int_{\mathbb{R}^N}\mathcal{K}_\alpha*F(|u|^2))f(|u|^2)|u|^2\phi_\rho dx=0
$$

that

$$
\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathcal{K}_\alpha*F(|u_n|^2))f(|u_n|^2)|u_n|^2\phi_\rho dx=0.
$$

Since ϕ_ρ has compact support, letting $n\to\infty$ in (3.7), we can deduce from (3.8)-(3.9) and the diamagnetic inequality that

$$
m_1\left(\mu(\{x_j\})\right)^{2\sigma}\leq\varepsilon^{-2s}\nu_j.
$$

Combining this fact with Lemma 3, we obtain

$$
\nu_j\geq m_1\varepsilon^{2s}S^{2\sigma}\nu_j^{\frac{4\sigma}{2_s^*}}.
$$

This result implies that

$$
\text{(I)}\quad \nu_j=0\quad\text{or}\quad\text{(II)}\quad \nu_j\geq\left(m_1S^{2\sigma}\right)^{\frac{2_s^*}{2_s^*-4\sigma}}\varepsilon^{\frac{2s2_s^*}{2_s^*-4\sigma}}.
$$

To obtain the possible concentration of mass at infinity, we similarly define a cut off function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x) = 0$ on $|x| < R$ and $\phi_R(x) = 1$ on $|x| > R+1$. We can verify that $\{u_n\phi_R\}_n$ is bounded in E, hence $\langle J_\varepsilon'(u_n), u_n\phi_R\rangle \to 0$, as $n \to \infty$, which implies

$$\begin{aligned} &M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2\phi_R(y)}{|x-y|^{N+2s}}dxdy+\varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u_n|^2\phi_R(x)dx\\ &=-\mathrm{Re}\left\{M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{N+2s}}dxdy\right\}\\ &\quad+\varepsilon^{-2s}\int_{\mathbb{R}^N}|u_n|^{2_s^*}\phi_R dx+\varepsilon^{-2s}\int_{\mathbb{R}^N}\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2\phi_R(x)dx+o_n(1). \end{aligned} \tag{3.10}$$

It is easy to verify that

$$\limsup_{R\to\infty}\ \limsup_{n\to\infty}\int\int_{\mathbb{R}^{2N}}\frac{||u_n(x)|-|u_n(y)||^2\phi_R(y)}{|x-y|^{N+2s}}dxdy=\mu_\infty$$

and

$$\begin{aligned} &\left|\mathrm{Re}\left\{M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{N+2s}}dxdy\right\}\right|\\ &\le C\left(\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}. \end{aligned}$$

Note that

$$\begin{aligned} &\limsup_{R\to\infty}\ \limsup_{n\to\infty}\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}dxdy\\ &=\limsup_{R\to\infty}\ \limsup_{n\to\infty}\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{N+2s}}dxdy. \end{aligned}$$

Similar to the proof of Zhang and Zhang,[42], lemma 3.4 we can show that

$$\limsup_{R\to\infty}\ \limsup_{n\to\infty}\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{N+2s}}dxdy=0.$$

It follows from the fact that (M_2), Lemma 1 and Lemma 4 that

$$\begin{aligned} &\limsup_{R\to\infty}\ \limsup_{n\to\infty} M\left([u_n]_{s,A}^2\right)\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y)|^2\phi_R(y)}{|x-y|^{N+2s}}dxdy\\ &\ge\limsup_{R\to\infty}\ \limsup_{n\to\infty} m_1\left(\int\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y)|^2\phi_R(y)}{|x-y|^{N+2s}}dxdy\right)^{2\sigma}\\ &\ge\limsup_{R\to\infty}\ \limsup_{n\to\infty} m_1\left(\int\int_{\mathbb{R}^{2N}}\frac{||u_n(x)|-|u_n(y)||^2\phi_R(y)}{|x-y|^{N+2s}}dxdy\right)^{2\sigma}=m_1\mu_\infty^{2\sigma}. \end{aligned}$$

It is easy to see that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathcal{K}_\alpha * F(|u_n|^2))f(|u_n|^2)|u_n|^2\phi_R(x)dx=0.$$

By Lemma 4 and letting $R\to\infty$ in (3.10), we obtain

$$\nu_\infty\ge m_1\varepsilon^{2s}S^{2\sigma}\nu_\infty^{\frac{4\sigma}{2_s^*}}.$$

This result implies that

$$\text{(III)} \quad \nu_\infty = 0 \quad \text{or} \qquad \text{(IV)} \quad \nu_\infty \geq \left(m_1 S^{2\sigma}\right)^{\frac{2_s^*}{2_s^*-4\sigma}} \varepsilon^{\frac{2s2_s^*}{2_s^*-4\sigma}}.$$

Next, we claim that (II) and (IV) cannot occur. If the case (IV) holds for some $j \in I$, then by Lemma 4, (M) and (H), we have

$$\begin{aligned}
c &= \lim_{n\to\infty} \left(J_\varepsilon(u_n) - \frac{1}{\mu}\langle J_\varepsilon'(u_n), u_n\rangle \right) \\
&\geq \left(\frac{1}{2\sigma} - \frac{1}{\mu}\right) M\left([u_n]_{s,A}^2\right)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_n|^2 dx \\
&\quad + \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx + \varepsilon^{-2s}\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * F(|u_n|^2)) \left(\frac{1}{\mu} f(|u_n|^2)|u_n|^2 - \frac{1}{4}F(|u_n|^2)\right) dx \\
&\geq \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \geq \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\nu_\infty \\
&\geq \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)(m_1 S^\sigma)^{\frac{2_s^*}{2_s^*-4\sigma}} \varepsilon^{\frac{4s\sigma}{2_s^*-4\sigma}} = \sigma_0 \varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}},
\end{aligned}$$

where $\sigma_0 = \left(\frac{1}{\mu} - \frac{1}{2_s^*}\right)(m_1 S^\sigma)^{\frac{2_s^*}{2_s^*-2\sigma}}$, which is impossible.

Consequently, $\nu_j = 0$ for all $j \in I$. Similarly, we can prove that (II) cannot occur for any j. Thus,

$$\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx \to \int_{\mathbb{R}^N} |u|^{2_s^*} dx. \tag{3.11}$$

The Brézis-Lieb Lemma implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^{2_s^*} dx = 0.$$

Therefore, we get

$$u_n \to u \quad \text{in} \quad L^{2_s^*}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty.$$

By the weak lower semicontinuity of the norm, conditon (m_1) and the Brézis-Lieb lemma, we have

$$\begin{aligned}
o(1)\|u_n\|_\varepsilon &= \langle J_\varepsilon'(u_n), u_n\rangle = M\left([u_n]_{s,A}^2\right)[u_n]_{s,A}^2 + \varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_n|^2 dx \\
&\quad - \varepsilon^{-2s}\int_{\mathbb{R}^N} |u_n|^{2_s^*} dx - \varepsilon^{-2s}\int_{\mathbb{R}^N} \mathcal{K}_\alpha * F(|u_n|^2) f(|u_n|^2)|u_n|^2 dx \\
&\geq m_1\left([u_n]_{s,A}^{2\sigma} - [u]_{s,A}^{2\sigma}\right) + \varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)(|u_n|^2 - |u|^2)dx + M\left([u]_{s,A}^2\right)[u]_{s,A}^2 \\
&\quad + \varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \varepsilon^{-2s}\int_{\mathbb{R}^N} |u|^{2_s^*} dx - \varepsilon^{-2s}\int_{\mathbb{R}^N} \mathcal{K}_\alpha * F(|u|^2) f(|u|^2)|u|^2 dx \\
&\geq \min\{m_1, 1\}\min\{\|u_n - u\|_\varepsilon^{2\sigma}, \|u_n - u\|_\varepsilon^2\} + o(1)\|u\|_\varepsilon.
\end{aligned}$$

Here, we use the fact that $J_\varepsilon'(u) = 0$. Thanks to $2 < 2\sigma$, we have proved that $\{u_n\}_n$ strongly converges to u in E. Hence the proof is complete. □

4 | PROOFS OF MAIN THEOREMS

In this section, we will prove our main results. We shall first establish Theorem 1.

Note that $J_\varepsilon(u)$ does not satisfy $(PS)_c$ condition for any $c > 0$. Thus, in the sequel we will find a special finite-dimensional subspace by which we construct sufficiently small minimax levels.

Recall that the assumption (V) implies that there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality we can assume from now on that $x_0 = 0$.

Proposition 2. *(see Binlin et al[34], theorem 3.2) For any $q \in (2, 2_s^*)$, we have*

$$\inf\left\{\int\int_{\mathbb{R}^{2N}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}dxdy \,:\, \phi \in C_0^\infty(\mathbb{R}^N), |\phi|_q = 1\right\} = 0.$$

By Proposition 2, one can choose $\phi_\zeta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\zeta|_q = 1$ and $\operatorname{supp}\phi_\zeta \subset B_{r_\zeta}(0)$ so that

$$\int\int_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(x)-\phi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy \le C\zeta^{\frac{2N-(N-2s)q}{q}},$$

for any $1 > \zeta > 0$.

Set

$$\psi_\zeta(x) = e^{iA(0)x}\phi_\zeta(x), \quad \psi_{\varepsilon,\zeta}(x) = \psi_\zeta(\varepsilon^{-\tau}x) \tag{4.1}$$

and

$$\tau := \frac{2s2_s^*}{N(2_s^*-4\sigma)}. \tag{4.2}$$

By (f_3), for any $t > 0$ we get

$$\begin{aligned}
J_\varepsilon(t\psi_{\varepsilon,\zeta}) &\le \frac{C_0}{2}t^{2\sigma}\left(\int\int_{\mathbb{R}^{2N}} \frac{|\psi_{\varepsilon,\zeta}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}\psi_{\varepsilon,\zeta}(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{2\sigma} \\
&\quad + \frac{t^2}{2}\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|\psi_{\varepsilon,\zeta}|^2dx - t^{2_s^*}\frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx \\
&\le \varepsilon^{N\tau-2s}\left[\frac{C_0}{2}t^{2\sigma}\left(\int\int_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x)-e^{i(\varepsilon^\tau x-\varepsilon^\tau y)\cdot A(\frac{\varepsilon^\tau x+\varepsilon^\tau y}{2})}\psi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{2\sigma}\right. \\
&\quad \left. + \frac{t^2}{2}\int_{\mathbb{R}^N} V(\varepsilon^\tau x)|\psi_\zeta|^2dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}|\psi_\zeta|^{2_s^*}dx\right] \\
&= \varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}}I_\varepsilon(t\psi_\zeta),
\end{aligned}$$

where $I_\varepsilon \in C^1(E,\mathbb{R})$ is defined by

$$\begin{aligned}
I_\varepsilon(u) &:= \frac{C_0}{2}\left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-e^{i(\varepsilon^\tau x-\varepsilon^\tau y)\cdot A(\frac{\varepsilon^\tau x+\varepsilon^\tau y}{2})}u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{2\sigma} \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon^\tau x)|u|^2dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx.
\end{aligned}$$

Since $2_s^* > 2\sigma$, there exists a finite number $t_0 \in [0,+\infty)$ such that

$$\begin{aligned}
\max_{t\ge 0} I_\varepsilon(t\psi_\zeta) &= \frac{C_0}{2}t_0^{2\sigma}\left(\int\int_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x)-e^{i(\varepsilon^\tau x-\varepsilon^\tau y)\cdot A(\frac{\varepsilon^\tau x+\varepsilon^\tau y}{2})}\psi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{2\sigma} \\
&\quad + \frac{t_0^2}{2}\int_{\mathbb{R}^N} V(\varepsilon^\tau x)|\psi_\zeta|^2dx - \frac{t_0^{2_s^*}}{2_s^*}\int_{\mathbb{R}^N}|\psi_\zeta|^{2_s^*}dx \\
&\le \frac{C_0}{2}t_0^{2\sigma}\left(\int\int_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x)-e^{i(\varepsilon^\tau x-\varepsilon^\tau y)\cdot A(\frac{\varepsilon^\tau x+\varepsilon^\tau y}{2})}\psi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{2\sigma} \\
&\quad + \frac{t_0^2}{2}\int_{\mathbb{R}^N} V(\varepsilon^\tau x)|\psi_\zeta|^2dx.
\end{aligned}$$

Let $\psi_\zeta(x) = e^{iA(0)x}\phi_\zeta(x)$, where $\phi_\zeta(x)$ is as defined above. Then, we have the following lemma.

Lemma 6. *(see Binlin et al[34], lemma 3.6) (Norm estimate) For any $\zeta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\zeta) > 0$ such that*

$$\int\int_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x) - e^{i(\varepsilon^\tau x - \varepsilon^\tau y)\cdot A(\frac{\varepsilon^\tau x + \varepsilon^\tau y}{2})}\psi_\zeta(y)|^2}{|x-y|^{N+2s}} dxdy \leq C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s},$$

for all $0 < \varepsilon < \varepsilon_0$ and some constant $C > 0$ depending only on $[\phi]_{s,0}$.

On the one hand, since $V(0) = 0$ and note that supp $\phi_\zeta \subset B_{r_\zeta}(0)$, there is $\varepsilon^* > 0$ such that

$$V(\varepsilon^\tau x) \leq \frac{\zeta}{|\phi_\zeta|_2^2} \quad \text{for all} \quad |x| \leq r_\zeta \quad \text{and} \quad 0 < \varepsilon < \varepsilon^*.$$

This implies that

$$\max_{t \geq 0} I_\varepsilon(t\phi_\delta) \leq \frac{C_0}{2} t_0^{2\sigma} \left(C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s} \right)^{2\sigma} + \frac{t_0^2}{2}\zeta. \tag{4.3}$$

Therefore, for all $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon^*\}$, we have

$$\max_{t \geq 0} J_\varepsilon(t\psi_{\lambda,\zeta}) \leq \left[\frac{C_0}{2} t_0^{2\sigma} \left(C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s} \right)^{2\sigma} + \frac{t_0^2}{2}\zeta \right] \varepsilon^{\frac{8s\sigma}{2_s^* - 4\sigma}}. \tag{4.4}$$

Thus, we have the following result.

Lemma 7. *Under the assumptions of Lemma 2, for any $\kappa > 0$ there exists $\mathcal{E}_\kappa > 0$ such that for each $0 < \varepsilon < \mathcal{E}_\kappa$, there is $\hat{e}_\varepsilon \in E$ with $\|\hat{e}_\varepsilon\| > \varrho_\varepsilon$, $J_\varepsilon(\hat{e}_\varepsilon) \leq 0$ and*

$$\max_{t \in [0,1]} J_\varepsilon(t\hat{e}_\varepsilon) \leq \kappa \varepsilon^{\frac{8s\sigma}{2_s^* - 4\sigma}}. \tag{4.5}$$

Proof. Choose $\zeta > 0$ so small that

$$\frac{C_0}{2} t_0^{2\sigma} \left(C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s} \right)^{2\sigma} + \frac{t_0^2}{2}\zeta \leq \kappa.$$

Let $\psi_{\varepsilon,\zeta} \in E$ be the function defined by (4.2). Set $\mathcal{E}_\kappa = \min\{\varepsilon_0, \varepsilon^*\}$. Let $\hat{t}_\varepsilon > 0$ be such that $\hat{t}_\varepsilon \|\psi_{\varepsilon,\zeta}\|_\varepsilon > \varrho_\varepsilon$ and $J_\varepsilon(t\psi_{\varepsilon,\zeta}) \leq 0$ for all $t \geq \hat{t}_\varepsilon$. By (4.4), let $\hat{e}_\varepsilon = \hat{t}_\varepsilon \psi_{\varepsilon,\zeta}$ we know that the conclusion of Lemma 7 holds. □

Proof. For any $0 < \kappa < \sigma_0$, by Lemma 5, we choose $\mathcal{E}_\kappa > 0$ and define for $0 < \varepsilon < \mathcal{E}_\kappa$, the minimax value

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(t\hat{e}_\varepsilon),$$

where

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = \hat{e}_\varepsilon\}.$$

By Lemma 2, we have $\alpha_\varepsilon \leq c_\varepsilon \leq \kappa \varepsilon^{\frac{8s\sigma}{2_s^* - 4\sigma}}$. By virtue of Lemma 5, we know that J_ε satisfies the $(PS)_{c_\lambda}$ condition, there is $u_\varepsilon \in E$ such that $J\varepsilon'(u_\varepsilon) = 0$ and $J_\varepsilon(u_\varepsilon) = c_\varepsilon$, Then u_ε is a nontrivial mountain pass solution of problem (2.2).

Since u_ε is a critical point of J_ε, by (M) and (H), we have for $\tau \in [2\sigma, 2_s^*]$

$$\begin{aligned}
\kappa\varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}} &\geq J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon) - \frac{1}{\tau}\langle J_\varepsilon'(u_\varepsilon), u_\varepsilon\rangle \\
&= \frac{1}{2}\tilde{M}\left([u_\varepsilon]_{s,A_\varepsilon}^2\right) - \frac{1}{\tau}M\left([u_\varepsilon]_{s,A_\varepsilon}^2\right)[u_\varepsilon]_{s,A_\varepsilon}^2 + \left(\frac{1}{2}-\frac{1}{\tau}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2dx \\
&\quad + \left(\frac{1}{\tau}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}|u_\varepsilon|^{2_s^*}dx + \varepsilon^{-2s}\int_{\mathbb{R}^N}(\mathcal{K}_\alpha * F(|u_\varepsilon|^2))\left(\frac{1}{\tau}f(|u_\varepsilon|^2)|u_\varepsilon|^2 - \frac{1}{4}F(|u_\varepsilon|^2)\right)dx \\
&\geq \left(\frac{1}{2\sigma}-\frac{1}{\tau}\right)m_1[u_\varepsilon]_{s,A_\varepsilon}^{2\sigma} + \left(\frac{1}{2}-\frac{1}{\tau}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2dx \\
&\quad + \left(\frac{1}{\tau}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}|u_\varepsilon|^{2_s^*}dx + \left(\frac{\mu}{\tau}-\frac{1}{4}\right)\varepsilon^{-2s}\int\int_{\mathbb{R}^{2N}}\frac{F(|u_\varepsilon(x)|^2)F(|u_\varepsilon(y)|^2)}{|x-y|^\alpha}dxdy.
\end{aligned} \tag{4.6}$$

Taking $\tau = 2/\sigma$, we obtain the estimate (1.7) and taking $\tau = \mu$ we obtain the estimate (1.8). This completes the proof of Theorem 1. □

Next, we shall establish Theorem 2. Again, we first need to prove a lemma.

For any $m^* \in \mathbb{N}$, one can choose m^* functions $\phi_\zeta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\zeta^i \cap \operatorname{supp}\phi_\zeta^k = \emptyset$, $i \neq k$, $|\phi_\zeta^i|_s = 1$ and

$$\int\int_{\mathbb{R}^{2N}}\frac{|\phi_\zeta^i(x)-\phi_\zeta^i(y)|^2}{|x-y|^{N+2s}}dxdy \leq C\zeta^{\frac{2N-(N-2s)q}{q}}.$$

Let $r_\zeta^{m^*} > 0$ be such that $\operatorname{supp}\phi_\zeta^i \subset B_{r_\zeta}^i(0)$ for $i = 1, 2, \ldots, m^*$. Set

$$\psi_\zeta^i(x) = e^{iA(0)x}\phi_\zeta^i(x) \tag{4.7}$$

and

$$\psi_{\varepsilon,\zeta}^i(x) = \psi_\zeta^i(\varepsilon^{-1}x). \tag{4.8}$$

Denote

$$\mathcal{H}_{\varepsilon\zeta}^{m^*} = \operatorname{span}\{\psi_{\varepsilon,\zeta}^1, \psi_{\varepsilon,\zeta}^2, \ldots, \psi_{\varepsilon,\zeta}^{m^*}\}.$$

Observe that for each $u = \sum_{i=1}^{m^*} c_i\psi_{\varepsilon,\zeta}^i \in \mathcal{H}_{\varepsilon\zeta}^{m^*}$, we have

$$[u]_{s,A_\varepsilon}^2 \leq C\sum_{i=1}^{m^*}|c_i|^2[\psi_{\varepsilon,\zeta}^i]_{s,A_\varepsilon}^2,$$

for some constant $C > 0$. Therefore,

$$J_\varepsilon(u) \leq C\sum_{i=1}^{m^*}J_\varepsilon(c_i\psi_{\varepsilon,\zeta}^i)$$

for some constant $C > 0$. Based on a similar argument as before, we see that

$$J_\varepsilon(c_i\psi_{\varepsilon,\zeta}^i) \leq \varepsilon^{N-2s}\Psi(|c_i|\psi_\zeta^i).$$

As before, we can obtain the following estimate:

$$\max_{u\in\mathcal{H}_{\varepsilon\delta}^{m^*}} J_\varepsilon(u) \leq Cm^*\left[\frac{C_0}{2}t_0^{2\sigma}\left(C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s}\right)^{2\sigma} + \frac{t_0^2}{2}\zeta\right]\varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}} \tag{4.9}$$

for all small enough ζ and some constant $C > 0$. From the estimate (4.9) we have the following:

Lemma 8. *Under the assumptions of Lemma 2, for any $m^* \in \mathbb{N}$ and $\kappa > 0$ there exists $\mathcal{E}_{m^*\kappa} > 0$ such that for each $0 < \varepsilon < \mathcal{E}_{m^*\kappa}$, there exists an m^*-dimensional subspace $\mathcal{F}_{\lambda m^*}$ satisfying*

$$\max_{u \in \mathcal{F}_{\lambda m^*}} J_\varepsilon(u) \le \kappa \varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}}.$$

□

Proof. Choose $\zeta > 0$ so small that

$$Cm^* \left[\frac{C_0}{2} t_0^{2\sigma} \left(C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s} \right)^{2\sigma} + \frac{t_0^2}{2}\zeta \right] \le \kappa.$$

Set $\mathcal{F}_{\varepsilon m^*} = \mathcal{H}_{\varepsilon\zeta}^{m^*} = \text{span}\{\psi_{\varepsilon,\zeta}^1, \psi_{\varepsilon,\zeta}^2, \dots, \psi_{\varepsilon,\zeta}^{m^*}\}$. Now, the conclusion of Lemma 8 follows from (4.9). □

Proof. Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ, for each $Z \in \Sigma$. Let gen(Z) be the Krasnoselski genus and

$$j(Z) := \min_{\eta \in \Phi_{m^*}} \text{gen}(\eta(Z) \cap \partial B_{\varrho_\varepsilon}),$$

where Φ_{m^*} is the set of all odd homeomorphisms $\eta \in C(E, E)$ and ϱ_ε is the number from Lemma 2. Then j is a version of Benci's pseudoindex[43]. Let

$$c_{\varepsilon i} := \inf_{j(Z) \ge i} \sup_{u \in Z} J_\varepsilon(u), \quad 1 \le i \le m^*.$$

Since $J_\varepsilon(u) \ge \alpha_\varepsilon$ for all $u \in \partial B_{\varrho_\varepsilon}^+$ and since $j(\mathcal{F}_{\varepsilon m^*}) = \dim \mathcal{F}_{\varepsilon m^*} = m^*$, we obtain

$$\alpha_\varepsilon \le c_{\varepsilon 1} \le \dots \le c_{\varepsilon m^*} \le \sup_{u \in H_{\varepsilon m^*}} J_\varepsilon(u) \le \kappa \varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}}.$$

It follows from Lemma 5 that J_ε satisfies the $(PS)_{c_\varepsilon}$ condition at all levels $c < \sigma_0 \varepsilon^{N-2s}$. By the usual critical point theory, all $c_{\varepsilon i}$ are critical levels and J_ε has at least m^* pairs of nontrivial critical points satisfying

$$\alpha_\varepsilon \le J_\varepsilon(u_\varepsilon) \le \kappa \varepsilon^{\frac{8s\sigma}{2_s^*-4\sigma}}.$$

Hence, problem (2.2) has at least m^* pairs of solutions. Finally, as in the proof of Theorem 1, we see that these solutions satisfy the estimates (1.7) and (1.8). □

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CONFLICT OF INTEREST

The authors declare that this work does not represent any conflict of interest.

ORCID

Dušan D. Repovš https://orcid.org/0000-0002-6643-1271

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