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On the Schrödinger–Maxwell system involving sublinear terms

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ABSTRACT

In this paper, we study the coupled Schrödinger–Maxwell system

$$\begin{cases} -\Delta u + u + e\phi u = \lambda\alpha(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \qquad (SM_\lambda)$$

where $e > 0$, $\alpha \in L^\infty(\mathbb{R}^3) \cap L^{6/(5-q)}(\mathbb{R}^3)$ for some $q \in (0, 1)$, and the continuous function $f : \mathbb{R} \to \mathbb{R}$ is superlinear at zero and sublinear at infinity, e.g., $f(s) = \min(|s|^r, |s|^p)$ with $0 < r < 1 < p$. First, for small values of $\lambda > 0$, we prove a non-existence result for (SM_λ), while for $\lambda > 0$ large enough, a recent Ricceri-type result guarantees the existence of at least two non-trivial solutions for (SM_λ) as well as the 'stability' of system (SM_λ) with respect to an arbitrary subcritical perturbation of the Schrödinger equation.

1. Introduction

The problem of coupled Schrödinger–Maxwell equations

$$\begin{cases} -\dfrac{\hbar^2}{2m}\Delta u + \omega u + e\phi u = g(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \qquad (SM)$$

has been widely studied in the recent years, describing the interaction of a charged particle with a given electrostatic field. The quantities m, e, ω and $\hbar$ are the mass, the charge, the phase, and the Planck's constant, respectively. The unknown terms $u : \mathbb{R}^3 \to \mathbb{R}$ and $\phi : \mathbb{R}^3 \to \mathbb{R}$ are the fields associated to the particle and the electric potential, respectively, while the nonlinear term $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ describes the interaction between the particles or an external nonlinear perturbation of the 'linearly' charged fields in the presence of the electrostatic field.

System (SM) is well-understood for the model nonlinearity $g(x, s) = \alpha(x)|s|^{p-1}s$, where $p > 0, \alpha : \mathbb{R}^3 \to \mathbb{R}$ is measurable; various existence and multiplicity results are available for (SM) in the case $1 < p < 5$; see [1–19] (for bounded domains). Via a Pohožaev-type argument, D'Aprile and Mugnai [20] proved the non-existence of the solutions (u, ϕ) in (SM) for every $p \in (0, 1] \cup [5, \infty)$ when $\alpha = 1$. Further non-existence results can be found in the papers of Ruiz [21], and Wang and Zhou [22].

Besides of the model nonlinearity $g(x, s) = \alpha(x)|s|^{p-1}s$, important contributions can be found in the theory of the Schrödinger–Maxwell system when the right-hand side nonlinearity is more general, verifying various growth assumptions near the origin and at infinity. We recall two such classes of nonlinearities (for simplicity, we consider only the autonomous case $g = g(x, \cdot)$):

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(AR) $g \in C(\mathbb{R}, \mathbb{R})$ verifies the global *Ambrosetti–Rabinowitz growth assumption*, i.e., there exists $\mu > 2$ such that

$$0 < \mu G(s) \le sg(s) \quad \text{for all } s \in \mathbb{R} \setminus \{0\}, \tag{1.1}$$

where $G(s) = \int_0^s g(t)\mathrm{d}t$. Note that (1.1) implies the superlinearity at infinity of g, i.e., there exist $c, s_0 > 0$ such that $|g(s)| \ge c|s|^{\mu-1}$ for all $|s| \ge s_0$. Up to some further technicalities, by standard mountain pass arguments one can prove that (SM) has at least a nontrivial solution $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$; see [6] for the pure-power case $g(s) = |s|^{p-1}s$, $3 < p < 5$.

(BL) $g \in C(\mathbb{R}, \mathbb{R})$ verifies the *Berestycki–Lions growth assumptions*, i.e.,
- $-\infty \le \limsup_{s\to\infty} \frac{g(s)}{s^5} \le 0$;
- $-\infty < \liminf_{s\to 0^+} \frac{g(s)}{s} \le \limsup_{s\to 0^+} \frac{g(s)}{s} = -m < 0$;
- There exists $s_0 \in \mathbb{R}$ such that $G(s_0) > 0$.

In the case when $\omega = 0$ and e is small enough, Azzollini et al. [23] proved the existence of at least a nontrivial solution $(u_e, \phi_e) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ for the system (SM) via suitable truncation and monotonicity arguments.

The purpose of the present paper is to describe a new phenomenon for Schrödinger–Maxwell systems (rescaling the mass, the phase and the Planck's constant as $2m = \omega = \hbar = 1$), by considering the non-autonomous eigenvalue problem

$$\begin{cases} -\Delta u + u + e\phi u = \lambda\alpha(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{SM_λ}$$

where $\lambda > 0$ is a parameter, $\alpha \in L^\infty(\mathbb{R}^3)$, and the continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ verifies the assumptions

(f1) $\lim_{|s|\to\infty} \frac{f(s)}{s} = 0$;
(f2) $\lim_{s\to 0} \frac{f(s)}{s} = 0$;
(f3) There exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$.

Remark 1.1. (a) Property (f1) is a *sublinearity growth assumption at infinity* on f which complements the Ambrosetti–Rabinowitz-type assumption (1.1).

(b) If (f1)–(f3) hold for f, then the function $g(s) = -s + f(s)$ verifies all the assumptions in (BL) whenever $1 < \max_{s\ne 0} \frac{2F(s)}{s^2}$. Consequently, the results of Azzollini et al. [23] can be applied also for (SM_λ), guaranteeing the existence of at least one nontrivial pair of solutions when $\lambda = \alpha(x) = 1$, and $e > 0$ is sufficiently small.

On account of Remark 1.1(b), we could expect a much stronger conclusion when (f1)–(f3) hold. Indeed, the real effect of the sublinear nonlinear term $f : \mathbb{R} \to \mathbb{R}$ will be reflected in the following two results.

Let $e > 0$ be arbitrarily fixed. According to hypotheses (f1)–(f3), one can define the number

$$c_f = \max_{s\ne 0} \frac{|f(s)|}{|s| + 4\sqrt{\pi} e s^2} > 0. \tag{1.2}$$

In view of the papers of Ruiz [21], and Wang and Zhou [22], the following non-existence result for the system (SM_λ) is expected whenever $\lambda > 0$ is small enough. More precisely, we have

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (f1)–(f3), *and $\alpha \in L^\infty(\mathbb{R}^3)$. Then for every $\lambda \in [0, \|\alpha\|_\infty^{-1} c_f^{-1})$ (with convention $1/0 = +\infty$), problem (SM_λ) has only the solution $(u, \phi) = (0, 0)$.*

In spite of the above non-existence result, the situation changes significantly for larger values of $\lambda > 0$. In order to state our main theorem, we consider a *perturbed* form of the system (SM_λ) as follows:

$$\begin{cases} -\Delta u + u + e\phi u = \lambda\alpha(x)f(u) + \theta\beta(x)g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$SM_{\lambda,\theta}$}$$

where $\theta \in \mathbb{R}$, $\beta \in L^\infty(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, while $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for some $c > 0$ and $1 < p < 5$, one has

(g1) $|g(s)| \le c(|s| + |s|^p)$ for all $s \in \mathbb{R}$.

The main result reads as follows:

Theorem 1.2. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions which satisfy* (f1)–(f3) *and* (g1), *respectively, $\alpha \in L^\infty(\mathbb{R}^3) \cap L^{6/(5-q)}(\mathbb{R}^3)$ be a non-negative, non-zero, radially symmetric function for some $q \in (0, 1)$, and $\beta \in L^\infty(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ be a radially symmetric function. Then there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, there is $\delta > 0$ with the property that for every $\theta \in [-\delta, \delta]$, system $(SM_{\lambda,\theta})$ has at least two distinct, radially symmetric, nontrivial pairs of solutions $(u^i_{\lambda,\theta}, \phi^i_{\lambda,\theta}) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$, $i \in \{1, 2\}$.*

Some remarks are in order.

Remark 1.2. To prove Theorem 1.2 we use a recent abstract three critical point theorem of Ricceri [24]. Note that for the unperturbed system $(SM_{\lambda,0}) = (SM_\lambda)$, the conclusion follows from standard variational arguments. Indeed, due to (f1)–(f2), the energy functional associated to system (SM_λ) on the space of radially symmetric functions is coercive, weakly lower semicontinuous, and satisfies the standard Palais–Smale condition. By combining the principle of symmetric criticality with a global minimization and the mountain pass argument, one can guarantee the existence of $\lambda^* > 0$ such that for $\lambda > \lambda^*$ system (SM_λ) has at least two non-zero solutions. The power of Theorem 1.2 relies on the fact that a precise information on the *stability* of system (SM_λ) is given with respect to an arbitrary subcritical perturbation of the Schrödinger equation.

Remark 1.3. The proof of Theorem 1.2 gives an exact, but quite involved form for λ^*; see (3.8). It is clear from the conclusions of Theorems 1.1 and 1.2 that we should have

$$\|\alpha\|_\infty^{-1} c_f^{-1} \le \lambda^*. \tag{1.3}$$

Although the constructions of the numbers c_f and λ^* are *independent* (compare relations (1.2) and (3.8), respectively), sharp estimates are used in Proposition 3.1 to prove the inequality (1.3) which tacitly implies that the two values $\|\alpha\|_\infty^{-1} c_f^{-1}$ and λ^* are close to each other. However, no information is available concerning the number of solutions of the system $(\mathrm{SM}_{\lambda,0}) = (SM_\lambda)$ when $\lambda \in [\|\alpha\|_\infty^{-1} c_f^{-1}, \lambda^*]$.

Remark 1.4. The proof of Theorem 1.2 shows that for every compact interval $[a, b] \subset (\lambda^*, \infty)$, there exists a number $\nu > 0$ such that for every $\lambda \in [a, b]$, the solutions $(u^i_{\lambda,\theta}, \phi^i_{\lambda,\theta}) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3),\ i \in \{1, 2\}$ of $(SM_{\lambda,\theta})$ verify

$$\|u^i_{\lambda,\theta}\|_{H^1} \le \nu \quad \text{and} \quad \|\phi^i_{\lambda,\theta}\|_{\mathcal{D}^{1,2}} \le \nu. \tag{1.4}$$

Remark 1.5. A Strauss-type argument shows that the solutions in Theorem 1.2 are homoclinic, i.e., for every $\lambda > \lambda^*, \theta \in [-\delta, \delta]$, and $i \in \{1, 2\}$, we have

$$u^i_{\lambda,\theta}(x) \to 0 \quad \text{and} \quad \phi^i_{\lambda,\theta}(x) \to 0 \quad \text{as } |x| \to \infty.$$

Example 1.1. Typical nonlinearities which fulfil hypotheses (f1)–(f3) are:

(a) $f(s) = \min(|s|^r, |s|^p)$ with $0 < r < 1 < p$.
(b) $f(s) = \min(s_+^r, s_+^p)$ with $0 < r < 1 < p$, where $s_+ = \max(0, s)$;
(c) $f(s) = \ln(1 + s^2)$.

The proof of Theorem 1.1 is based on a direct calculation. Theorem 1.2 is proved by means of a very recent three critical point result of Ricceri [24], by deeply exploiting some important properties of the Maxwell equation $-\Delta\phi = 4\pi e u^2$. In Section 3, we provide additional information about the number λ^* which appears in Theorem 1.2.

Notations and embeddings

- For every $p \in [1, \infty]$, $\|\cdot\|_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^3)$.
- The standard Sobolev space $H^1(\mathbb{R}^3)$ is endowed with the norm $\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2\right)^{1/2}$. Note that the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for every $p \in [2, 6]$; let $s_p > 0$ be the best Sobolev constant in the above embedding. $H^1_{\mathrm{rad}}(\mathbb{R}^3)$ denotes the radially symmetric functions of $H^1(\mathbb{R}^3)$. The embedding $H^1_{\mathrm{rad}}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is compact for every $p \in (2, 6)$.
- The space $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\phi\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla\phi|^2\right)^{1/2}$. Note that the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous; let $d^* > 0$ be the best constant in this embedding. $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^3)$ denotes the radially symmetric functions of $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

2. Preliminaries

2.1. The Maxwell equation

Let $e > 0$ be fixed. By the Lax–Milgram theorem it follows that for every $u \in H^1(\mathbb{R}^3)$, the Maxwell equation

$$-\Delta\phi = 4\pi e u^2 \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

has a unique solution $\Phi[u] = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. In the sequel, we recall/prove some important properties of the function $u \mapsto \phi_u$ which are interesting in their own right as well.

Proposition 2.1. *The map $u \mapsto \phi_u$ has the following properties:*

(a) $\|\phi_u\|^2_{\mathcal{D}^{1,2}} = 4\pi e \int_{\mathbb{R}^3} \phi_u u^2$ *and* $\phi_u \ge 0$;
(b) $\|\phi_u\|_{\mathcal{D}^{1,2}} \le 4\pi e d^* \|u\|^2_{12/5}$ *and* $\int_{\mathbb{R}^3} \phi_u u^2 \le 4\pi e d^{*2} \|u\|^4_{12/5}$;
(c) *If the sequence* $\{u_n\} \subset H^1_{\mathrm{rad}}(\mathbb{R}^3)$ *weakly converges to* $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$, *then* $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2$ *converges to* $\int_{\mathbb{R}^3} \phi_u u^2$.
(d) *The map* $u \mapsto \int_{\mathbb{R}^3} \phi_u u^2$ *is convex;*
(e) $\int_{\mathbb{R}^3} (\phi_u u - \phi_v v)(u - v) \ge 0$ *for all* $u, v \in H^1(\mathbb{R}^3)$.

Proof. A straightforward adaptation of [23, Lemma 2.1] and [21, Lemma 2.1] give the properties (a)–(c). It remains to prove (d) and (e).

(d) Let us fix $u, v \in H^1(\mathbb{R}^3)$, and $t, s \in [0, 1]$ with $t + s = 1$. First, we have

$$\begin{aligned} -\Delta\phi_{tu+sv} &= 4\pi e(tu + sv)^2 \\ &\le 4\pi e(tu^2 + sv^2) \\ &= -t\Delta\phi_u - s\Delta\phi_v \\ &= -\Delta(t\phi_u + s\phi_v). \end{aligned}$$

Thus, the comparison principle implies that

$$\phi_{tu+sv} \le t\phi_u + s\phi_v. \tag{2.2}$$

Multiplying the equation $-\Delta\phi_u = 4\pi eu^2$ by ϕ_v and $-\Delta\phi_v = 4\pi ev^2$ by ϕ_u, after integrations, we obtain that

$$\int_{\mathbb{R}^3} \nabla\phi_u\nabla\phi_v = 4\pi e\int_{\mathbb{R}^3} \phi_v u^2 = 4\pi e\int_{\mathbb{R}^3} \phi_u v^2. \tag{2.3}$$

By combining relations (2.2), (2.3) and property (a), we have

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{tu+sv}(tu+sv)^2 &\le \int_{\mathbb{R}^3} (t\phi_u + s\phi_v)(tu^2 + sv^2) \\ &= t^2\int_{\mathbb{R}^3} \phi_u u^2 + ts\int_{\mathbb{R}^3} (\phi_u v^2 + \phi_v u^2) + s^2\int_{\mathbb{R}^3} \phi_v v^2 \\ &= \frac{1}{4\pi e}\left(t^2\int_{\mathbb{R}^3} |\nabla\phi_u|^2 + 2ts\int_{\mathbb{R}^3} \nabla\phi_u\nabla\phi_v + s^2\int_{\mathbb{R}^3} |\nabla\phi_v|^2\right) \\ &\le \frac{1}{4\pi e}\left(t\int_{\mathbb{R}^3} |\nabla\phi_u|^2 + s\int_{\mathbb{R}^3} |\nabla\phi_v|^2\right) \\ &= t\int_{\mathbb{R}^3} \phi_u u^2 + s\int_{\mathbb{R}^3} \phi_v v^2. \end{aligned}$$

(e) We recall that for all $x, y \ge 0$, we have

$$(xy)^{1/2}(x+y) \le x^2 + y^2.$$

This inequality, relation (2.3), (a), and the Hölder inequality imply that

$$\begin{aligned} \int_{\mathbb{R}^3} (\phi_u uv + \phi_v uv) &\le \left(\int_{\mathbb{R}^3} \phi_u u^2\right)^{1/2}\left(\int_{\mathbb{R}^3} \phi_u v^2\right)^{1/2} + \left(\int_{\mathbb{R}^3} \phi_v v^2\right)^{1/2}\left(\int_{\mathbb{R}^3} \phi_v u^2\right)^{1/2} \\ &= \frac{1}{4\pi e}\left(\int_{\mathbb{R}^3} \nabla\phi_u\nabla\phi_v\right)^{1/2}\left(\|\phi_u\|_{\mathcal{D}^{1,2}} + \|\phi_v\|_{\mathcal{D}^{1,2}}\right) \\ &\le \frac{1}{4\pi e}\left(\int_{\mathbb{R}^3} |\nabla\phi_u|^2\right)^{1/4}\left(\int_{\mathbb{R}^3} |\nabla\phi_v|^2\right)^{1/4}\left(\|\phi_u\|_{\mathcal{D}^{1,2}} + \|\phi_v\|_{\mathcal{D}^{1,2}}\right) \\ &= \frac{1}{4\pi e}\|\phi_u\|_{\mathcal{D}^{1,2}}^{1/2}\|\phi_v\|_{\mathcal{D}^{1,2}}^{1/2}\left(\|\phi_u\|_{\mathcal{D}^{1,2}} + \|\phi_v\|_{\mathcal{D}^{1,2}}\right) \\ &\le \frac{1}{4\pi e}\left(\|\phi_u\|_{\mathcal{D}^{1,2}}^2 + \|\phi_v\|_{\mathcal{D}^{1,2}}^2\right) \\ &= \int_{\mathbb{R}^3} (\phi_u u^2 + \phi_v v^2), \end{aligned}$$

which gives exactly the required relation. □

Remark 2.1. One can prove alternatively properties (d) and (e) from the previous proposition by using the representation formula

$$\phi_u(x) = 4\pi e\int_{\mathbb{R}^3} u^2(y)G(x,y)\mathrm{d}y,$$

where $G(x, y)$ is the Green function of the Laplacian in $\mathbb{R}^3$. In particular, we have the Coulomb energy in the form

$$\int_{\mathbb{R}^3} \phi_u u^2 = e\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y.$$

2.2. Variational framework

We are interested in the existence of weak solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ for the system $(SM_{\lambda,\theta})$, i.e.,

$$\int_{\mathbb{R}^3} (\nabla u \nabla v + uv + e\phi uv) = \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) v + \theta \int_{\mathbb{R}^3} \beta(x) g(u) v, \quad \forall v \in H^1(\mathbb{R}^3), \tag{2.4}$$

$$\int_{\mathbb{R}^3} \nabla\phi \nabla\psi = 4\pi e \int_{\mathbb{R}^3} u^2 \psi, \quad \forall \psi \in \mathcal{D}^{1,2}(\mathbb{R}^3), \tag{2.5}$$

whenever (f1)–(f3) and (g1) hold, $\alpha \in L^\infty(\mathbb{R}^3)$ and $\beta \in L^\infty(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$. Note that all terms in (2.4)–(2.5) are finite; we will check only the right hand sides in both expressions, the rest being straightforward. First, (f1) and (f2) imply in particular that one can find a number $n_f > 0$ such that

$$|f(s)| \le n_f |s| \quad \text{for all } s \in \mathbb{R}. \tag{2.6}$$

By using (g1), one can easily prove that the last term of (2.4) is also well-defined. Moreover, for every $(u, \psi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} u^2 |\psi| &\le \left(\int_{\mathbb{R}^3} |u|^{12/5} \right)^{5/6} \left(\int_{\mathbb{R}^3} \psi^6 \right)^{1/6} \\ &= \|u\|_{12/5}^2 \|\psi\|_6 \\ &\le s_{12/5}^2 d^* \|u\|_{H^1}^2 \|\psi\|_{\mathcal{D}^{1,2}} < \infty. \end{aligned}$$

For every $\lambda > 0$ and $\theta \in \mathbb{R}$, we define the functional $J_{\lambda,\theta} : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ by

$$J_{\lambda,\theta}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2 + \frac{e}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{1}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi|^2 - \lambda \mathcal{F}(u) - \theta \mathcal{G}(u),$$

where

$$\mathcal{F}(u) = \int_{\mathbb{R}^3} \alpha(x) F(u), \qquad \mathcal{G}(u) = \int_{\mathbb{R}^3} \beta(x) G(u).$$

It is clear that $J_{\lambda,\theta}$ is well-defined and is of class C^1 on $H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$. Moreover, a simple calculation shows that its critical points are precisely the weak solutions for $(SM_{\lambda,\theta})$, i.e., the relations

$$\left\langle \frac{\partial J_{\lambda,\theta}}{\partial u}(u, \phi), v \right\rangle = 0 \quad \text{and} \quad \left\langle \frac{\partial J_{\lambda,\theta}}{\partial \phi}(u, \phi), \psi \right\rangle = 0,$$

give (2.4) and (2.5), respectively. Consequently, to prove the existence of solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ for the system $(SM_{\lambda,\theta})$, it is enough to seek critical points of the functional $J_{\lambda,\theta}$.

Note that $J_{\lambda,\theta}$ is a strongly indefinite functional; thus, the location of its critical points is a challenging problem in itself. However, the standard trick is to introduce a 'one-variable' energy functional instead of $J_{\lambda,\theta}$ via the map $u \mapsto \phi_u$; see relation (2.1). More precisely, we define the functional $I_{\lambda,\theta} : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$I_{\lambda,\theta}(u) = J_{\lambda,\theta}(u, \phi_u).$$

On account of Proposition 2.1(a), we have

$$I_{\lambda,\theta}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) + \frac{e}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \lambda \mathcal{F}(u) - \theta \mathcal{G}(u), \tag{2.7}$$

which is of class C^1 on $H^1(\mathbb{R}^3)$. By using standard variational arguments for functionals of two variables, we can state the following result.

Proposition 2.2. *A pair $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a critical point of $J_{\lambda,\theta}$ if and only if u is a critical point of $I_{\lambda,\theta}$ and $\phi = \Phi[u] = \phi_u$.*

Furthermore, since Eq. (2.1) is solved throughout the relation (2.5), we clearly have that $\frac{\partial J_{\lambda,\theta}}{\partial \phi}(u, \phi_u) = 0$. Thus, the derivative of $I_{\lambda,\theta}$ is given by

$$\begin{aligned} \langle I'_{\lambda,\theta}(u), v \rangle &= \left\langle \frac{\partial J_{\lambda,\theta}}{\partial u}(u, \phi_u), v \right\rangle + \left\langle \frac{\partial J_{\lambda,\theta}}{\partial \phi}(u, \phi_u) \circ \phi'_u, v \right\rangle \\ &= \left\langle \frac{\partial J_{\lambda,\theta}}{\partial u}(u, \phi_u), v \right\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u \nabla v + uv + e\phi_u uv) - \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) v - \theta \int_{\mathbb{R}^3} \beta(x) g(u) v. \end{aligned}$$

We conclude this section by recalling the following Ricceri-type three critical point theorem which plays a crucial role in the proof of Theorem 1.2 together with the principle of symmetric criticality restricting the functional $I_{\lambda,\theta}$ to the space $H^1_{\rm rad}(\mathbb{R}^3)$. Before doing that, we recall the following notion: if X is a Banach space, we denote by $\mathcal{W}_X$ the class of those functionals $E : X \to \mathbb{R}$ having the property that if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_n E(u_n) \le E(u)$ then $\{u_n\}$ has a subsequence converging strongly to u.

Theorem 2.1 (*[24, Theorem 2]*)**.** *Let X be a separable and reflexive real Banach space, let $E_1 : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; and $E_2 : X \to \mathbb{R}$ a C^1 functional with a compact derivative. Assume that E_1 has a strict local minimum u_0 with $E_1(u_0) = E_2(u_0) = 0$. Setting the numbers*

$$\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u\to u_0} \frac{E_2(u)}{E_1(u)}\right\}, \tag{2.8}$$

$$\chi = \sup_{E_1(u)>0} \frac{E_2(u)}{E_1(u)}, \tag{2.9}$$

assume that $\tau < \chi$.

Then, for each compact interval $[a, b] \subset (1/\chi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $E_3 : X \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that for each $\theta \in [0, \delta]$, the equation

$$E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0$$

admits at least three solutions in X having norm less than κ.

3. Proofs

Proof of Theorem 1.1. Let us fix $0 \le \lambda < \|\alpha\|_\infty^{-1} c_f^{-1}$ (when $\alpha = 0$, we choose simply $\lambda \ge 0$), and assume that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution for (SM_λ). By choosing $v := u$ and $\psi := \phi$ in relations (2.4) and (2.5), respectively, we obtain that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2 + e\phi u^2) = \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) u,$$

and

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 = 4\pi e \int_{\mathbb{R}^3} \phi u^2. \tag{3.1}$$

Moreover, let us choose also $\psi := |u| \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ in (2.5); we obtain that

$$4\pi e \int_{\mathbb{R}^3} |u|^3 = \int_{\mathbb{R}^3} \nabla\phi \nabla|u|,$$

thus,

$$4\sqrt{\pi} e \int_{\mathbb{R}^3} |u|^3 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^3} \nabla\phi\nabla|u| \le \int_{\mathbb{R}^3} \left(\frac{1}{4\pi}|\nabla\phi|^2 + |\nabla u|^2\right).$$

Combining the above three relations and the definition of c_f from (1.2), this yields

$$\begin{aligned}\int_{\mathbb{R}^3} (u^2 + 4\sqrt{\pi} e|u|^3) &\le \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 + \frac{1}{4\pi}|\nabla\phi|^2\right) \\ &= \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) u \\ &\le \lambda \int_{\mathbb{R}^3} |\alpha(x)|\, |f(u)|\, |u| \\ &\le \lambda \|\alpha\|_\infty c_f \int_{\mathbb{R}^3} (u^2 + 4\sqrt{\pi} e|u|^3).\end{aligned}$$

If $\alpha = 0$, then $u = 0$. If $\alpha \ne 0$, and $0 \le \lambda < \|\alpha\|_\infty^{-1} c_f^{-1}$, the last estimates give that $u = 0$. Moreover, (3.1) implies that $\phi = 0$ as well, which concludes the proof. □

Remark 3.1. (a) The last estimates in the proof of Theorem 1.1 show that if f is a globally Lipschitz function with Lipschitz constant $L_f > 0$ and $f(0) = 0$, then (SM_λ) has only the solution $(u, \phi) = (0, 0)$ for every $0 \le \lambda < \|\alpha\|_\infty^{-1} L_f^{-1}$, no matter if the assumptions (f1)–(f3) hold or not. In addition, if f fulfils (f1)–(f3) then $c_f \le L_f$, and as expected, the range of those values of λ's where non-existence occurs for (SM_λ) is larger than in the previous statement.

(b) If $f(s) = \min(s_+^r, s_+^p)$ with $0 < r < 1 < p$, then $L_f = p$ and $c_f = \max_{s\neq 0} \frac{\min(s_+^r, s_+^p)}{|s|+4\sqrt{\pi e}s^2} \le \max_{s>0} \min(s^{r-1}, s^{p-1}) = 1$ for every $e > 0$.

(c) If $f(s) = \ln(1+s^2)$, then $L_f = 1$ and $c_f = \max_{s\neq 0} \frac{\ln(1+s^2)}{|s|+4\sqrt{\pi e}s^2} \le \max_{s\neq 0} \frac{\ln(1+s^2)}{|s|} \approx 0.804$ for every $e > 0$.

Proof of Theorem 1.2. In the rest of this section, we assume that the assumptions of Theorem 1.2 are fulfilled. For every $\lambda \ge 0$ and $\theta \in \mathbb{R}$, let $\mathcal{R}_{\lambda,\theta} = I_{\lambda,\theta}|_{H^1_{\text{rad}}(\mathbb{R}^3)} : H^1_{\text{rad}}(\mathbb{R}^3) \to \mathbb{R}$ be the functional defined by

$$\mathcal{R}_\lambda(u) = E_1(u) - \lambda E_2(u) - \theta E_3(u),$$

where

$$E_1(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{e}{4}\int_{\mathbb{R}^3} \phi_u u^2, \qquad E_2(u) = \mathcal{F}(u), \quad \text{and} \quad E_3(u) = \mathcal{G}(u), \quad u \in H^1_{\text{rad}}(\mathbb{R}^3). \tag{3.2}$$

It is clear that E_i are C^1 functionals, $i \in \{1, 2, 3\}$. To complete the proof of Theorem 1.2, some lemmas need to be proven. □

Lemma 3.1. *The functional E_1 is coercive, sequentially weakly lower semicontinuous which belongs to $\mathcal{W}_{H^1_{\text{rad}}(\mathbb{R}^3)}$, bounded on each bounded subset of $H^1_{\text{rad}}(\mathbb{R}^3)$, and its derivative admits a continuous inverse on $H^1_{\text{rad}}(\mathbb{R}^3)^*$.*

Proof. It is clear that E_1 is coercive on $H^1_{\text{rad}}(\mathbb{R}^3)$. On account of Brézis [25, Corollaire III.8] and Proposition 2.1(c), the functional E_1 is sequentially weakly lower semicontinuous on $H^1_{\text{rad}}(\mathbb{R}^3)$. Now, let $\{u_n\} \subset H^1_{\text{rad}}(\mathbb{R}^3)$ which converges weakly to $u \in H^1_{\text{rad}}(\mathbb{R}^3)$ and $\liminf_n E_1(u_n) \le E_1(u)$. On account of Proposition 2.1(c), we obtain $\liminf_n \|u_n\|_{H^1}^2 \le \|u\|_{H^1}^2$. Thus, standard arguments show that $u_n \to u$ strongly in $H^1_{\text{rad}}(\mathbb{R}^3)$, i.e., E_1 belongs to $\mathcal{W}_{H^1_{\text{rad}}(\mathbb{R}^3)}$. Proposition 2.1(a)–(b) implies that E_1 sends bounded sets of $H^1_{\text{rad}}(\mathbb{R}^3)$ to bounded sets. It remains to prove that the derivative of E_1 has a continuous inverse on $H^1_{\text{rad}}(\mathbb{R}^3)^*$.

We first show that E_1' is invertible. To do this, let us fix $h \in H^1_{\text{rad}}(\mathbb{R}^3)^*$ arbitrarily. We prove that equation

$$E_1'(u) = h$$

has a unique solution u. Note that the solution of the above equation is precisely the critical point of the functional $\mathcal{H} : H^1_{\text{rad}}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{H}(u) = E_1(u) - \langle h, u\rangle.$$

The functional $\mathcal{H}$ is clearly coercive and bounded from below; moreover, on account of Proposition 2.1(d), E_1 is strictly convex. Therefore, E_1 has a unique critical point which is its unique minimizer.

Now, let $\{h_n\} \subset H^1_{\text{rad}}(\mathbb{R}^3)^*$ and $h \in H^1_{\text{rad}}(\mathbb{R}^3)^*$ such that $h_n \to h$ in $H^1_{\text{rad}}(\mathbb{R}^3)^*$. Consequently, there exist a unique sequence $\{u_n\} \subset H^1_{\text{rad}}(\mathbb{R}^3)$ and $u \in H^1_{\text{rad}}(\mathbb{R}^3)$ such that

$$E_1'(u) = h, \quad \text{and} \quad E_1'(u_n) = h_n \quad \text{for all } n \in \mathbb{N}.$$

In particular, from these relations we obtain that

$$\langle E_1'(u) - E_1'(u_n), u - u_n\rangle = \langle h - h_n, u - u_n\rangle \quad \text{for all } n \in \mathbb{N}.$$

Now, Proposition 2.1(e) gives that

$$\begin{aligned} \|u - u_n\|_{H^1}^2 &= \langle E_1'(u) - E_1'(u_n), u - u_n\rangle - e\int_{\mathbb{R}^3} (\phi_u u - \phi_{u_n} u_n)(u - u_n) \\ &\le \langle E_1'(u) - E_1'(u_n), u - u_n\rangle \\ &= \langle h - h_n, u - u_n\rangle \\ &\le \|h - h_n\|_{(H^1)^*}\|u - u_n\|_{H^1}, \end{aligned}$$

i.e., $\|u - u_n\|_{H^1} \le \|h - h_n\|_{(H^1)^*}$. This fact shows that $u_n \to u$ strongly in $H^1_{\text{rad}}(\mathbb{R}^3)$, that is, the inverse of E_1' is continuous. □

Lemma 3.2. *E_2 and E_3 have compact derivatives.*

Proof. We prove the statement only for E_2; the argument for E_3 is similar. Let $\{u_n\} \subset H^1_{\rm rad}(\mathbb{R}^3)$ be a bounded sequence. In particular, for some $c > 0$, one has that $\sup_n \|u_n\|_2 \leq c$ for some $c > 0$. First, we prove that the sequence $\{E_2'(u_n)\} \subset H^1_{\rm rad}(\mathbb{R}^3)^*$ is bounded; the latter fact follows from the uniform boundedness principle, i.e., the sequence $\{|\langle E_2'(u_n), v\rangle|\}$ is uniformly bounded for every $v \in H^1_{\rm rad}(\mathbb{R}^3)$. Indeed, due to (2.6), for every $v \in H^1_{\rm rad}(\mathbb{R}^3)$ one has

$$\begin{aligned} |\langle E_2'(u_n), v\rangle| &\leq \int_{\mathbb{R}^3} \alpha(x)|f(u_n)||v|{\rm d}x \leq n_f\|\alpha\|_\infty \int_{\mathbb{R}^3} |u_n||v|{\rm d}x, \\ &\leq n_f\|\alpha\|_\infty\|u_n\|_2\|v\|_2 \leq n_f\|\alpha\|_\infty c\|v\|_2 < \infty. \end{aligned}$$

Up to a subsequence, $\{E_2'(u_n)\}$ weakly converges to some $h \in H^1_{\rm rad}(\mathbb{R}^3)^*$. Arguing by contradiction, we assume that there exists $\delta > 0$ such that

$$\|E_2'(u_n) - h\|_{(H^1)^*} > \delta \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

In particular, for every $n \in \mathbb{N}$, there exists $v_n \in H^1_{\rm rad}(\mathbb{R}^3)$ such that $\|v_n\|_{H^1} = 1$ and

$$\langle E_2'(u_n) - h, v_n\rangle > \delta.$$

Up to a subsequence, we may assume that $\{v_n\}$ weakly converges to some $v \in H^1_{\rm rad}(\mathbb{R}^3)$, and $\{v_n\}$ strongly converges to v in $L^3(\mathbb{R}^3)$, since the embedding $H^1_{\rm rad}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ is compact. Therefore, we obtain

$$\begin{aligned} \langle E_2'(u_n) - h, v_n\rangle &= \langle E_2'(u_n) - h, v\rangle + \langle E_2'(u_n), v_n - v\rangle + \langle h, v - v_n\rangle \\ &\leq \langle E_2'(u_n) - h, v\rangle + \int_{\mathbb{R}^3} \alpha(x)|f(u_n)||v_n - v|{\rm d}x + \langle h, v - v_n\rangle, \end{aligned}$$

and each term in the above expression tends to 0. Indeed, the case of the first and last expressions is immediate, while from (f1) and (f2), it follows in particular that for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + c_\varepsilon s^2 \quad \text{for all } s \in \mathbb{R}. \tag{3.4}$$

Therefore,

$$\int_{\mathbb{R}^3} \alpha(x)|f(u_n)||v_n - v|{\rm d}x \leq \|\alpha\|_\infty(\varepsilon\|u_n\|_{H^1}\|v_n - v\|_{H^1} + c_\varepsilon\|u_n\|_3^2\|v_n - v\|_3).$$

The arbitrariness of ε and the fact that $\{v_n\}$ strongly converges to v in $L^3(\mathbb{R}^3)$ imply that the right-hand side of the above inequality tends to 0. Combining these facts, we arrive to a contradiction with (3.3), which concludes the proof. □

Lemma 3.3. $\limsup_{\|u\|_{H^1}\to\infty} \frac{E_2(u)}{E_1(u)} \leq 0$.

Proof. According to (f1) and (f2), for every $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, 1)$ such that

$$|f(s)| < \frac{\varepsilon}{2(1 + \|\alpha\|_\infty)}|s| \quad \text{for all } |s| \leq \delta_\varepsilon \text{ and } |s| \geq \delta_\varepsilon^{-1}.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$, there also exists a number $M_\varepsilon > 0$ such that

$$\frac{|f(s)|}{|s|^q} \leq M_\varepsilon \quad \text{for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $q \in (0, 1)$ is from the hypothesis for $\alpha \in L^{6/(5-q)}(\mathbb{R}^3)$. Combining the above two relations, we obtain that

$$|f(s)| \leq \frac{\varepsilon}{2(1 + \|\alpha\|_\infty)}|s| + M_\varepsilon|s|^q \quad \text{for all } s \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} E_2(u) &\leq \int_{\mathbb{R}^3} \alpha(x)|F(u)| \\ &\leq \int_{\mathbb{R}^3} \alpha(x)\left[\frac{\varepsilon}{4(1 + \|\alpha\|_\infty)}u^2 + \frac{M_\varepsilon}{q + 1}|u|^{q+1}\right] \\ &\leq \frac{\varepsilon}{4}\|u\|_{H^1}^2 + \frac{M_\varepsilon}{q + 1}\|\alpha\|_{6/(5-q)}S_6^{q+1}\|u\|_{H^1}^{q+1}. \end{aligned}$$

For every $u \neq 0$, we have that

$$\frac{E_2(u)}{E_1(u)} \leq \frac{\frac{\varepsilon}{4}\|u\|_{H^1}^2 + \frac{M_\varepsilon}{q+1}\|\alpha\|_{6/(5-q)} s_6^{q+1}\|u\|_{H^1}^{q+1}}{\frac{1}{2}\|u\|_{H^1}^2 + \frac{e}{4}\int_{\mathbb{R}^3}\phi_u u^2} \leq \frac{\varepsilon}{2} + 2\frac{M_\varepsilon}{q+1}\|\alpha\|_{6/(5-q)} s_6^{q+1}\|u\|_{H^1}^{q-1}.$$

Taking the 'limsup' of the above estimation when $\|u\|_{H^1} \to \infty$, the arbitrariness of $\varepsilon > 0$ gives the required inequality. □

Lemma 3.4. $\limsup_{u\to 0} \frac{E_2(u)}{E_1(u)} \leq 0$.

Proof. A similar argument as in (3.4) shows that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{4(1+\|\alpha\|_\infty)} s^2 + c_\varepsilon |s|^3 \quad \text{for all } s \in \mathbb{R}.$$

This inequality implies that for every $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$, we have

$$E_2(u) \leq \int_{\mathbb{R}^3} \alpha(x)|F(u)| \leq \int_{\mathbb{R}^3} \alpha(x)\left[\frac{\varepsilon}{4(1+\|\alpha\|_\infty)}u^2 + c_\varepsilon|u|^3\right] \leq \frac{\varepsilon}{4}\|u\|_{H^1}^2 + c_\varepsilon s_3^3\|\alpha\|_\infty\|u\|_{H^1}^3.$$

Thus, for every $u \neq 0$,

$$\frac{E_2(u)}{E_1(u)} \leq \frac{\frac{\varepsilon}{4}\|u\|_{H^1}^2 + c_\varepsilon s_3^3\|\alpha\|_\infty\|u\|_{H^1}^3}{\frac{1}{2}\|u\|_{H^1}^2 + \frac{e}{4}\int_{\mathbb{R}^3}\phi_u u^2} \leq \frac{\varepsilon}{2} + 2c_\varepsilon s_3^3\|\alpha\|_\infty\|u\|_{H^1},$$

and the argument is similar as in the previous lemma. □

For any $0 \leq r_1 \leq r_2$, let $A[r_1, r_2] = \{x \in \mathbb{R}^3 : r_1 \leq |x| \leq r_2\}$ be the closed annulus (perhaps degenerate) with radii r_1 and r_2.

By assumption, since $\alpha \in L^\infty(\mathbb{R}^3)$ is a radially symmetric function with $\alpha \geq 0$ and $\alpha \not\equiv 0$, there are real numbers $R > r \geq 0$ and $\alpha_0 > 0$ such that

$$\operatorname{essinf}_{x\in A[r,R]}\alpha(x) \geq \alpha_0. \tag{3.5}$$

Let $s_0 \in \mathbb{R}$ from (f3). For a fixed element $\sigma \in (0, 1)$, define the function $u_\sigma \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ such that

(a) $\operatorname{supp} u_\sigma \subseteq A[(r-(1-\sigma)(R-r))_+, R]$;
(b) $u_\sigma(x) = s_0$ for every $x \in A[r, r+\sigma(R-r)]$;
(c) $\|u_\sigma\|_\infty \leq |s_0|$,

where we use the notation $t_+ = \max(0, t)$ for $t \in \mathbb{R}$. A simple calculation shows that

$$E_1(u_\sigma) \geq \frac{1}{2}\|u_\sigma\|_{H^1}^2 \geq \frac{2\pi s_0^2}{3}\left[(r+\sigma(R-r))^3 - r^3\right], \tag{3.6}$$

and

$$E_2(u_\sigma) \geq \frac{4\pi}{3}[\alpha_0 F(s_0)((r+\sigma(R-r))^3 - r^3) - \|\alpha\|_\infty \max_{|t|\leq|s_0|}|F(t)| \times \left(r^3 - (r-(1-\sigma)(R-r))_+^3 + R^3 - (r+\sigma(R-r))^3\right)] \stackrel{\text{not.}}{=:} M(\alpha_0, s_0, \sigma, R, r). \tag{3.7}$$

We observe that for σ close enough to 1, the right-hand sides of both inequalities become strictly positive; therefore, we can define the number

$$\lambda^* = \inf_{E_2(u)>0} \frac{E_1(u)}{E_2(u)}. \tag{3.8}$$

Keeping the notations from (1.2) and (2.9), we state the following relations between χ, λ^* and c_f.

Proposition 3.1. $\lambda^* = \chi^{-1} \geq c_f^{-1} \|\alpha\|_\infty^{-1}$.

Proof. First of all, the estimates (3.6) and (3.7) for the expressions $E_1(u_\sigma)$ and $E_2(u_\sigma)$ (for σ close to 1 and s_0 from (f3)) clearly show that $\chi > 0$; see (2.9). Moreover, by (3.8), we clearly have $\lambda^* = \chi^{-1}$. By (1.2), we have that

$$|f(s)| \leq c_f(|s| + 4\sqrt{\pi}es^2) \quad \text{for all } s \in \mathbb{R}.$$

Therefore, for every $u \in H^1_{\text{rad}}(\mathbb{R}^3)$, one has

$$E_2(u) \leq \int_{\mathbb{R}^3} \alpha(x)|F(u)| \leq c_f \|\alpha\|_\infty \int_{\mathbb{R}^3} \left(\frac{u^2}{2} + \frac{4\sqrt{\pi}e}{3}|u|^3 \right).$$

The Maxwell equation (2.1) and the Hölder inequality give that

$$4\pi e \int_{\mathbb{R}^3} |u|^3 = \int_{\mathbb{R}^3} \nabla\phi_u \nabla|u| \leq 2\sqrt{2\pi} \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{16\pi}|\nabla\phi_u|^2 \right).$$

Combining the above inequalities, we obtain that

$$E_2(u) \leq c_f \|\alpha\|_\infty \int_{\mathbb{R}^3} \left(\frac{1}{2}u^2 + \frac{2\sqrt{2}}{3} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{16\pi}|\nabla\phi_u|^2 \right) \right).$$

Since $\frac{2\sqrt{2}}{3} < 1$, and

$$E_1(u) = \int_{\mathbb{R}^3} \left(\frac{1}{2}u^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{16\pi}|\nabla\phi_u|^2 \right),$$

we have

$$\chi = \sup_{E_1(u)>0} \frac{E_2(u)}{E_1(u)} \leq c_f \|\alpha\|_\infty,$$

which ends the proof. □

Proof of Theorem 1.2 (*concluded*)**.** We apply Theorem 2.1, by choosing $X = H^1_{\text{rad}}(\mathbb{R}^3)$, as well as E_1, E_2 and E_3 from (3.2). On account of Lemmas 3.1 and 3.2, the functionals E_1 and E_2 fulfil the hypotheses of Theorem 2.1. Moreover, E_1 has a strict global minimum $u_0 = 0$, and $E_1(0) = E_2(0) = 0$. The definition of the number τ in Theorem 2.1, see (2.8), and Lemmas 3.3 and 3.4 give that $\tau = 0$. On account of Proposition 3.1, we also have that $0 = \tau < \chi = (\lambda^*)^{-1}$. Therefore, we may apply Theorem 2.1: for every compact interval $[a, b] \subset (\lambda^*, \infty)$ there exists $\kappa > 0$ such that for each $\lambda \in [a, b]$ there exists $\delta > 0$ with the property that for every $\theta \in [0, \delta]$, the equation $\mathcal{R}'_{\lambda,\theta}(u) \equiv E'_1(u) - \lambda E'_2(u) - \theta E'_3(u) = 0$ admits at least three solutions $u^i_{\lambda,\theta} \in H^1_{\text{rad}}(\mathbb{R}^3)$, $i \in \{1, 2, 3\}$, having H^1-norms less than κ. Note that we may repeat the above argument with $-E_3$ instead of the function E_3, by obtaining an interval of the form $[-\delta, \delta]$ for the parameter θ.

A similar argument as in [6, p. 416] shows that

$$\phi_{\gamma u} = \gamma\phi_u \quad \text{for all } \gamma \in O(3),\ u \in H^1(\mathbb{R}^3),$$

where the compact group $O(3)$ acts linearly and isometrically on $H^1(\mathbb{R}^3)$ in the standard way. Consequently, the functional $I_{\lambda,\theta}$ from (2.7) is $O(3)$-invariant. Moreover, since

$$H^1_{\text{rad}}(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : \gamma u = u \text{ for all } \gamma \in O(3)\},$$

the principle of symmetric criticality of Palais implies that the critical points $u^i_{\lambda,\theta} \in H^1_{\text{rad}}(\mathbb{R}^3)$ ($i \in \{1, 2, 3\}$) of the functional $\mathcal{R}_{\lambda,\theta} = I_{\lambda,\theta}|_{H^1_{\text{rad}}(\mathbb{R}^3)}$ are also critical points of $I_{\lambda,\theta}$. Now, by Proposition 2.2 it follows that $(u^i_{\lambda,\theta}, \phi^i_{\lambda,\theta}) \in H^1_{\text{rad}}(\mathbb{R}^3) \times \mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^3)$ are critical points of $J_{\lambda,\theta}$, and thus are weak solutions for the system $(SM_{\lambda,\theta})$, where $\phi^i_{\lambda,\theta} = \phi_{u^i_{\lambda,\theta}}$. On account of (f2) and (g1), one has $f(0) = g(0) = 0$, thus the pair $(0, 0)$ is a solution to $(SM_{\lambda,\theta})$; consequently, there exist at least two nontrivial pairs of solutions $(u^i_{\lambda,\theta}, \phi^i_{\lambda,\theta}) \in H^1_{\text{rad}}(\mathbb{R}^3) \times \mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^3)$ to problem $(SM_{\lambda,\theta})$, ($i \in \{1, 2\}$) with the required properties, which concludes the proof. □

Remark 3.2. (a) The norm-estimates in Remark 1.4 (see (1.4)) follow by Theorem 2.1 and Proposition 2.1(a), by choosing $\nu = \max(\kappa, 4\pi ed^{*2}s^2_{12/5}\kappa^2)$.

(b) Since the expression of λ^* is involved (see (3.8)), we give in the sequel an upper estimate of it which can be easily calculated. This fact can be done in terms of α_0, s_0, σ_0, R and r, see (3.5), where $\sigma_0 \in (0, 1)$ is such a number for which the right hand side of (3.7) becomes positive, i.e., $M(\alpha_0, s_0, \sigma_0, R, r) > 0$. In order to avoid technicalities, we assume that $r = 0$

which slightly restricts our arguments, imposing that α does not vanish near the origin; see (3.5). The truncation function $u_{\sigma_0} \in H^1_{\rm rad}(\mathbb{R}^3)$ defined by

$$u_{\sigma_0}(x) = \begin{cases} 0 & \text{if } |x| > R, \\ s_0 & \text{if } |x| \leq \sigma_0 R, \\ \dfrac{s_0}{R(1-\sigma_0)}(R - |x|) & \text{if } \sigma_0 R < |x| \leq R, \end{cases}$$

verifies the properties (a)–(c) from above. Moreover, by using Proposition 2.1(b), we have

$$E_1(u_{\sigma_0}) \leq \frac{t}{2} + \pi e d^{*2} s^4_{12/5} t^2 \overset{\text{not.}}{=:} N(s_0, \sigma_0, R),$$

where

$$t = \frac{4\pi}{3} R s_0^2 \left[R^2 + \frac{1 + \sigma_0 + \sigma_0^2}{1 - \sigma_0} \right].$$

Combining the above estimation with relation (3.7), we obtain

$$\lambda^* \leq \frac{N(s_0, \sigma_0, R)}{M(\alpha_0, s_0, \sigma_0, R, 0)} = \lambda_0.$$

Now, the conclusions of Theorem 1.2 are valid for every $\lambda \geq \lambda_0$.

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