Received: 8 April 2021 | Revised: 26 May 2021 | Accepted: 26 May 2021

DOI: 10.1002/mma.7606

RESEARCH ARTICLE

WILEY

On weak solutions for fourth-order problems involving the Leray–Lions type operators

Khaled Kefi[1] | **Dušan D. Repovš**[2] | **Kamel Saoudi**[3]

[1]Faculty of Computer Science and Information Technology, Northern Border University, Rafha, Saudi Arabia

[2]Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia

[3]College of Sciences, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia

Correspondence
Dušan D. Repovš, Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia.
Email: dusan.repovs@guest.arnes.si

Communicated by: M. A. Ragusa

Funding information
Slovenian Research Agency, Grant/Award Number: P1-0292, N1-0114, N1-0083, N1-0064 and J1-8131

We investigate existence and multiplicity of weak solutions for fourth-order problems involving the Leray–Lions type operators in variable exponent spaces and improve a result of Bonanno and Chinnì (2011). We use variational methods and apply a multiplicity theorem of Bonanno and Marano (2010).

KEYWORDS

fourth-order PDE, generalized Sobolev space, Leray–Lions type operator, variable exponent, weak solution, variational method

MSC CLASSIFICATION

35J20; 35J60; 35G30; 35J35; 46E35

1 | INTRODUCTION

The objective of this work is to study the existence of solutions for the following problems involving the Leray–Lions type operators in variable exponent spaces

$$\begin{cases} \Delta\left(a(x,\Delta u)\right)=\lambda f(x,u) \text{ in } \Omega, \\ u=\Delta u=0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^{N\geq 2}$ with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, f is a Carathéodory function, $p \in C(\overline{\Omega})$ satisfies $\inf_{x\in\Omega} p(x) > N/2$, for all $x\in\Omega$, and the potential a satisfies a set of conditions (see Section 2.2). The operators include the $p(x)$-biharmonic operator and other important cases. We point out that the extension from the p-biharmonic problem to the $p(x)$-biharmonic problem is nontrivial since the $p(x)$-biharmonic problem possesses a more complicated structure; for example, it is nonhomogeneous, and it usually does not have the so-called first eigenvalue. Here, $\Delta(a(x,\Delta u))$ is the Leray–Lions operator, where a is a Carathéodory function satisfying some suitable supplementary conditions.

Investigations of this type of operators have been going on in various fields, for example, in electrorheological fluids (see Ružička[1]), elasticity theory (see Zhikov[2]), stationary thermorheological viscous flows of non-Newtonian fluids (see Antontsev and Rodrigues[3]), image processing (see Chen et al.[4]), and mathematical description of the processes filtration of barotropic gas through a porous medium (see Antontsev and Shmarev[5]). For more details about this kind of operators, the reader is referred to Leray and Lions[6] (see also Papageorgiou et al.[7] and the references therein).

We briefly recall the literature concerning related problems involving the Leray–Lions type operators. The existence of three solutions for a problem involving the $p(x)$-Laplacian in a variable space was established by Bonanno and Chinnì.[8] In particular, in the absence of small perturbations of the nonlinear term, they proved that one of the solutions is the trivial solution. Bonanno and Chinnì[9] also proved the existence of three solutions for a problem without small perturbations of the nonlinear term, whenever $p(x) > N$. Motivated by these results, we shall establish in this paper the existence of three nontrivial solutions for more general problems involving the Leray–Lions type operators in variable exponent spaces (see Theorem 2 in Section 3).

This paper is organized as follows: in Section 2, we present preliminary definitions and results, Section 3 is devoted to the statement and the proof proof of the main result, and Section 4 presents an application.

2 | PRELIMINARIES

In this section, we first review key definitions and basic properties of variable exponent Sobolev spaces and then introduce all conditions for parameters and nonlinearities which we need for the statement and proof of our main result (Theorem 2).

2.1 | Variable exponent Sobolev spaces

For a comprehensive treatment, we refer the reader to Rădulescu and Repovš.[10] Let

$$C_+(\bar{\Omega}) := \{h | h \in C(\bar{\Omega}), h(x) > 1, \text{ for all } \ x \in \bar{\Omega}\},$$

and let $p \in C_+(\bar{\Omega}), p \in C_+(\bar{\Omega})$ be such that

$$1 < p^- := \min_{x\in\bar{\Omega}} p(x) \le p^+ := \max_{x\in\bar{\Omega}} p(x) < +\infty. \tag{2}$$

We define the Lebesgue space with variable exponent as follows:

$$L^{p(x)}(\Omega) := \left\{ u | u : \Omega \to \mathbb{R} \ \text{ is measurable}, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

and we equip it with the Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \mu > 0 | \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

Variable exponent Lebesgue spaces are like classical Lebesgue spaces in many respects: they are Banach spaces, and they are reflexive if and only if $1 < p^- \le q^+ < \infty$. Moreover, the inclusion between Lebesgue spaces is generalized naturally, that is, if q_1, q_2 are such that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$, then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

For every $u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$, the Hölder inequality

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)} \tag{3}$$

holds, where $p'(x)$ satisfies $1/p(x) + 1/p'(x) = 1$. The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $k \in \mathbb{N}$, we define the Sobolev space with variable exponents as follows:

$$W^{k,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) | D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $\alpha := (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, and

$$D^{\alpha}u := \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1 \ \dots \ .\partial^{\alpha_N}x_N}.$$

Then $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(x)} := \sum_{|\alpha| \leq k} |D^{\alpha}u|_{p(x)}.$$

The space $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$. It is well-known that both $W^{2,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces (for more details, see Rădulescu and Repovš[10]).

It follows that $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is also a separable and reflexive Banach space, equipped with the norm

$$\|u\|_X := \|u\|_{W^{2,p(x)}(\Omega)} + \|u\|_{W_0^{1,p(x)}(\Omega)}.$$

Let

$$\|u\| := \inf \left\{ \mu > 0 | \int_{\Omega} \left| \frac{\Delta u}{\mu} \right|^{p(x)} dx \leq 1 \right\}$$

represent a norm, which is equivalent to $\|.\|_X$ on X (see Remark 2.1 in Amrouss and Ourraoui[11]). Therefore, in what follows, we shall consider $(X, \|.\|)$. The modular on the space X is the map $\rho_{p(x)} : X \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) := \int_{\Omega} |\Delta u|^{p(x)} dx.$$

This mapping satisfies the following properties.

Lemma 1 *(El Amrous et al.[12]). For every $u, u_n \in W^{2,p(.)}(\Omega)$,*

(1) $\|u\| < 1$ (*resp.* $= 1, > 1$) $\iff \rho_{p(x)}(u) < 1$ (*resp.* $= 1, > 1$)*;*
(2) $\min\{\|u\|^{p^-}, \|u\|^{p^+}\} \leq \rho_{p(x)} \leq \max\{\|u\|^{p^-}, \|u\|^{p^+}\}$*; and*
(3) $\|u_n\| \to 0$ (*resp.* $\to \infty$) $\iff \rho_{p(x)}(u_n) \to 0$ (*resp.* $\to \infty$).

We recall that the critical Sobolev exponent is defined as follows:

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-2p(x)}, & \text{if } p(x) < \frac{N}{2}, \\ \infty, & \text{if } p(x) \geq \frac{N}{2}. \end{cases}$$

Remark 1 (Rădulescu and Repovš[10]). Assume that $p \in C^+(\bar{\Omega})$ satisfies $p^- > N/2$. Then there exist a continuous embedding $X \hookrightarrow W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega)$ and a compact embedding $W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, such that X is compactly embedded in $C^0(\bar{\Omega})$ and $\|u\|_{\infty} \leq c_0\|u\|$, where c_0 is a positive constant and $\|u\|_{\infty} := \sup_{x\in\Omega}|u(x)|$.

Proposition 1 *(Gasiński and Papageorgiou[13]). If X is a reflexive Banach space, Y is a Banach space, $Z \subset X$ is nonempty, closed and convex, and $J : Z \to Y$ is completely continuous, then J is compact.*

Our key tool will be Theorem 3.6 in Bonanno and Marano,[14] which we restate here in a more convenient form.

Theorem 1 *(Bonanno and Marano[14]). Let X be a reflexive real Banach space and let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable, and sequentially weakly lower semicontinuous functional, whose Gâteaux derivative admits* *a continuous inverse on X. Let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and such that $\inf_{x\in X}\Phi(X) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0, \bar{x} \in X$, such that $r < \Phi(\bar{x})$, $r^{-1}\sup_{\Phi(x)\le r}\Psi(x) < \Psi(\bar{x})/\Phi(\bar{x})$, and for each $\lambda \in \Lambda_r := \left(\Phi(\bar{x})/\Psi(\bar{x}), r(\sup_{\Phi(x)\le r}\Psi(x))^{-1}\right)$, functional $\Phi - \lambda\Psi$ is coercive. Then for each $\lambda \in \Lambda_r$, functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.*

2.2 | Conditions for parameters and nonlinearities

In problem (1), parameters and nonlinearities are assumed to satisfy the following conditions:

(H_1) $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $a(x,0) = 0$, for a.e. $x \in \Omega$.

(H_2) There exist $c_1 > 0$ and a nonnegative function $d \in L^{\frac{p(x)}{p(x)-1}}(\Omega)$, such that

$$|a(x,t)| \le c_1(d(x) + |t|^{p(x)-1}), \text{ for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

(H_3) For all $s, t \in \mathbb{R}$, the inequality $|a(x,t) - a(x,s)|(t-s) \ge 0$ holds, for a.e. $x \in \Omega$, with equality if and only if $s = t$.

(H_4) There exists $1 \le c_2$ such that

$$c_2|t|^{p(x)} \le \min\{a(x,t)t, p(x)A(x,t)\}, \text{ for a.e.} x \in \Omega \text{ and all } s,t \in \mathbb{R},$$

where c_1 is the constant from condition (H_2) and $A : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ represents the antiderivative of a,

$$A(x,t) = \int_0^t a(x,s)ds.$$

(H_5) $|f(x,t)| \le \xi(x) + \zeta|t|^{q(x)-1}$, for all $(x,t) \in \Omega \times \mathbb{R}$, where $\xi \in L^1(\Omega)$, ζ is a positive constant, and $1 < q^- \le q^+ < p^-$.

Remark 2. (Boureanu[15]) Concerning conditions $(H_1) - (H_5)$, the following can be observed:

(i) $A(x,t)$ is a C^1-Carathéodory function; that is, for every $t \in \mathbb{R}$, $A(.,t) : \Omega \to \mathbb{R}$ is measurable, and for a.e. $x \in \Omega$, $A(x,.)$ is $C^1(\mathbb{R})$.

(ii) There exists a constant c_3 such that $|A(x,t)| \le c_3(d(x)|t| + |t|^{p(x)})$, for a.e.$x \in \Omega$ and all $t \in \mathbb{R}$.

In order to formulate the variational approach to problem (1), recall that a weak solution for our problem satisfies the following definition.

Definition 1. We say that $u \in X\setminus\{0\}$ is a weak solution of problem (1) if $\Delta u = 0$ on $\partial\Omega$ and

$$\int_\Omega a(x,\Delta u)\Delta v dx - \lambda \int_\Omega f(x,u)v dx = 0, \text{ for all } v \in X.$$

Denote

$$\Phi(u) := \int_\Omega F(x,u)dx.$$

The Euler–Lagrange functional corresponding to problem (1) is defined by $\Psi_\lambda : X \to \mathbb{R}$, where $\Psi_\lambda(u) = J(u) - \lambda\Phi(u)$, for all $u \in X$, such that

$$J(u) = \int_\Omega A(x,\Delta u)dx.$$

By condition (H_5), we have

$$|F(x,u)| \le \xi(x)|u| + \frac{\zeta}{q(x)}|u|^{q(x)};$$

hence,

$$\Phi(u) \le |\xi(x)|_{L^1(\Omega)}\|u\|_\infty + \frac{\zeta}{q^-}\int_\Omega (|u|^{q^+} + |u|^{q^-})dx,$$

so we get

$$\Phi(u) \leq |\xi(x)|_{L^1(\Omega)}\|u\|_\infty + \frac{\zeta}{q^-}(c_0^{q^+}\|u\|^{q^+} + c_0^{q^-}\|u\|^{q^-})|\Omega|.$$

This shows that Φ is well defined. In the sequel, we shall need the following lemma.

Lemma 2 *(Boureanu[15]). Functional J is coercive on X, and $J' : X \to X^*$ is a strictly monotone homeomorphism.*

Proof. It is clear from Lemma 1 and hypothesis (H_4) that for $u \in X$ with $\|u\|>1$,

$$J(u) \geq \int_\Omega \frac{c_2}{p(x)}|\Delta u|^{p(x)}dx \geq \frac{1}{p^+}\rho_{p(x)}(u) \geq \frac{1}{p^+}\|u\|^{p^-}, \tag{4}$$

and thus, J is coercive. For the rest of the proof of Lemma 2, see Boureanu.[15] □

Next, we shall show that $\Phi'(u)$ is compact. Let $v_n \rightharpoonup v$ in X. Remark 1 asserts that $v_n \to v$ in $C_0(\overline{\Omega})$, so for any $u \in X$,

$$| < \Phi'(u), v_n > | - | < \Phi'(u), v > | \leq \|v_n - v\|_\infty \int_\Omega f(x,u)dx.$$

As a consequence of condition (H_5), we have

$$| < \Phi'(u), v_n > | \to | < \Phi'(u), v > |, \text{ as } n \to +\infty.$$

This means that $\Phi'(u)$ is completely continuous, hence compact by Proposition 1.

3 | THE MAIN RESULT

Setting $\delta(x) := \sup\{\delta > 0 | B(x,\delta) \subseteq \Omega\}$, for all $x \in \Omega$, one can prove that there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where $D := \sup_{x\in\Omega}\delta(x)$. For each $r > 0$, let $\gamma_r := \max\left\{(p^+ r)^{1/p^+}, (p^+ r)^{1/p^-}\right\}$. Let

$$L := w\left(D^N - \left(\frac{D}{2}\right)^N\right), \text{where } w := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)},$$

and Γ denotes the Euler function. We can now state the main result of this paper.

Theorem 2. *Let $a : \Omega \times \mathbb{R}^N \to R^N$ be potential satisfying conditions $(H_2) - (H_4)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function satisfying condition (H_5) and the following requirement*

$$ess\ inf_{x\in\Omega}F(x,t) := ess \inf_{x\in\Omega}\int_0^t f(x,s)ds \geq 0, \quad for\ all\ \ t \in [0,h],$$

where h is a nonnegative constant. Suppose that there exist $r > 0$, $h > 0$ such that

$$r < \frac{L}{p^+}\min\left\{\left(\frac{8hN}{3D^2}\right)^{p^-}, \left(\frac{8hN}{3D^2}\right)^{p^+}\right\},$$

$$\beta_h := \frac{w\left(\frac{D}{2}\right)^N ess\ inf_{x\in\Omega}F(x,h)}{c_3 L^{\frac{1}{p^+}}\left[N^{\frac{1}{p^+}}\frac{8h}{3D^2}|d(x)|_{\frac{p(x)}{p(x)-1}} + L^{\frac{p^+-1}{p^+}}\max\left\{\left(\frac{8hN}{3D^2}\right)^{p^-}, \left(\frac{8hN}{3D^2}\right)^{p^+}\right\}\right]} \tag{5}$$

$$> \alpha_r := \frac{1}{r}\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx.$$

Then for every $\lambda \in \bar{\Lambda} := (1/\beta_h, 1/\alpha_r)$, problem (1) admits at least three weak solutions.

Proof. We shall apply Theorem 1. Let $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and $\Psi_\lambda(u) := J(u) - \lambda\Phi(u)$, for all $u \in X$, where

$$J(u) = \int_\Omega A(x, \Delta u)dx \text{ and } \phi(u) = \int_\Omega F(x,u)dx.$$

As we have seen before, functionals J and Φ satisfy the regularity assumptions of Theorem 1. Now let $\bar{v} \in X$ be defined by

$$\bar{v} = \begin{cases} 0, & \text{if } x \in \Omega\backslash B(x_0, D) \\ h, & \text{if } x \in B\left(x_0, \frac{D}{2}\right) \\ \frac{4h}{3D^2}(D^2 - |x-x_0|^2), & \text{if } x \in B(x_0,D)\backslash B\left(x_0, \frac{D}{2}\right), \end{cases}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^N$, and

$$\triangle\bar{v} = \begin{cases} 0, & \text{if } x \in \Omega\backslash B(x_0, D) \cup B\left(x_0, \frac{D}{2}\right) \\ \frac{-8h}{3D^2}N, & \text{if } x \in B(x_0,D)\backslash B\left(x_0, \frac{D}{2}\right). \end{cases}$$

Using the above information, Remark 1, Lemma 1, and the continuity of embedding $L^{p^+}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, we obtain

$$\begin{aligned} &\frac{L}{p^+}\min\left\{\left(\frac{8hN}{3D^2}\right)^{p^-}, \left(\frac{8hN}{3D^2}\right)^{p^+}\right\} \le J(\bar{v}) \\ &\le c_3 L^{\frac{1}{p^+}}\left[N^{\frac{1}{p^+}}\frac{8h}{3D^2}|d(x)|_{\frac{p(x)}{p(x)-1}} + L^{\frac{p^+-1}{p^+}}\max\left\{\left(\frac{8hN}{3D^2}\right)^{p^-}, \left(\frac{8hN}{3D^2}\right)^{p^+}\right\}\right], \end{aligned}$$

and

$$\Phi(\bar{v}) \ge \int_{B\left(x_0,\frac{D}{2}\right)} F(x, \bar{v}(x)dx \ge w\left(\frac{D}{2}\right)^N \operatorname*{essinf}_{x\in\Omega} F(x,h).$$

It follows from (5) that $r < J(\bar{v})$. Moreover, since the embedding $X \hookrightarrow C_0(\bar{\Omega})$ is continuous, we have

$$\max_{x\in\Omega}|u(x)| \le c_0 \max\left\{(p^+r)^{\frac{1}{p^+}}, (p^+r)^{\frac{1}{p^-}}\right\} = c_0\gamma_r, \text{ for all } u \in X, \text{ such that } J(u) \le r.$$

Therefore,

$$\sup_{J(u)\le r}\Phi(u) \le \int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx.$$

By the definitions of α_r and β_h in Theorem 2, we obtain

$$\frac{\sup_{J(u)\le r}\Phi(u)}{r} \le \alpha_r < \beta_h \le \frac{\Phi(\bar{v})}{J(\bar{v})}.$$

Hence, the first condition of Theorem 1 has been verified.

Next, we shall prove that for each $\lambda > 0$, the energy functional $J - \lambda\Phi$ is coercive. For any $q \in C_+(\bar{\Omega})$, we have $|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}$. Condition (H_5), Remark 1, and the above inequality imply

$$\begin{aligned}\Phi(u) &\leq |\xi(x)|_{L^1(\Omega)}\|u\|_\infty + \frac{\alpha}{q^-}\int_\Omega (|u|^{q^+} + |u|^{q^-})dx\\ &\leq |\xi(x)|_{L^1(\Omega)}\|u\|_\infty + \frac{\alpha}{q^-}\left(\|u\|_{+\infty}^{q^-} + (\|u\|_{+\infty}^{q^+}\right)|\Omega|\\ &\leq |\xi(x)|_{L^1(\Omega)}\|u\|_\infty (c_0^{q^+}\|u\|^{q^+} + c_0^{q^-}\|u\|^{q^-})|\Omega|.\end{aligned}$$

The above inequality and (4) give

$$J(u) - \lambda\Phi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - c_0|\xi(x)|_{L^1(\Omega)}\|u\| + \frac{\alpha}{q^-}(c_0^{q^+}\|u\|^{q^+} + c_0^{q^-}\|u\|^{q^-})|\Omega|.$$

Since $1 \leq q^- \leq q^+ < p^-$, it follows that $J(u) - \lambda\Phi(u)$ is coercive.

Finally, we use the fact that

$$\bar{\Lambda} := \left(\frac{1}{\beta_h}, \frac{1}{\alpha_r}\right) \subseteq \left(\frac{J(\bar{v})}{\Phi(\bar{v})}, \frac{r}{\sup\limits_{J(u)\leq r}\Phi(u)}\right).$$

Theorem 1 now ensures that for each $\lambda \in \bar{\Lambda}$, functional $J(u) - \lambda\Phi(u)$ admits at least three critical points in X which are weak solutions for problem (1). This completes the proof of Theorem 2. □

Remark 3. Setting $r = 1$ in Theorem 2, we get $\gamma_r = (p^+)^{1/p^-}$, and inequalities (5) become

$$p^+ < L\min\left\{\left(\frac{8hN}{3D^2}\right)^{p^-}, \left(\frac{8hN}{3D^2}\right)^{p^+}\right\},$$

and

$$\beta_h := \frac{w\left(\frac{D}{2}\right)^N \operatorname*{essinf}_{x\in\Omega} F(x,h)}{c_3 L^{\frac{1}{p^+}}\left[N^{\frac{1}{p^+}}\frac{8h}{3D^2}|d(x)|_{\frac{p(x)}{p(x)-1}} + L^{\frac{p^+-1}{p^+}}\max\left\{\left(\frac{8hN}{3D^2}\right)^{p^-}, \left(\frac{8hN}{3D^2}\right)^{p^+}\right\}\right]}$$

$$> \alpha := \int_\Omega \sup_{|t|\leq c_0(p^+)^{\frac{1}{p^-}}} F(x,t)dx.$$

Remark 4. We note that, if $f(x,0) \neq 0$, then by Theorem 2, there exist at least three nonzero solutions.

Remark 5. We are interested in the Leray–Lions type operators because they are quite general. Indeed, consider

$$a(x,t) = \theta(x)|t|^{p(x)-2}t, \tag{6}$$

where $p(x)$ satisfies condition (2), and for $\theta \in L^\infty(\Omega)$, there exists $\theta_0 > 0$ with $\theta(x) \geq \theta_0 > 0$, for a.e. $x \in \Omega$. One can see that Equation (6) satisfies conditions $(H_1) - (H_4)$, and we arrive at operator $\theta(.)\Delta^2\left(|\Delta|^{p(.)-2}u\right)$.

Note that when $\theta \equiv 1$, we get the well-known $p(x)$-biharmonic operator $\Delta^2_{p(.)}(u)$. Moreover, we can make the following choice $a(x,t) = \theta(x)(1+|t|^2)^{p(x)/p(x)-2}t$ and obtain operator $\theta(.)\Delta\left((1+|\Delta u|^2)^{p(.)/p(.)-2}\Delta u\right)$, where p and θ are as above.

4 | AN APPLICATION

We present an interesting application of Theorem 2. Let $\alpha : [0,1] \to \mathbb{R}$, be a positive, bounded, and measurable function. Put $\alpha_0 = \operatorname*{ess\,inf}_{x\in[0,1]} \alpha(x)$ and $\|\alpha\|_1 = \|\alpha\|_{L^1([0,1])}$. Moreover, set

$$k = \frac{1}{p^+ c_3}\left(\frac{3}{8}\right)^{p^+} \frac{\alpha_0}{\|\alpha\|_1},$$

where c_3 is the constant from Remark 2.

Theorem 3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that $\lim_{|t|\to+\infty} g(t)|t|^{-\nu} = 0$, for some $0 \le \nu < p^- - 1$ and $g(0) \neq 0$. Let $G(\xi) = \int_0^\xi g(t)dt$, for all $\xi \in \mathbb{R}$, and assume that there exist positive constants h and l, with $l \le 1 \le (8/3h)^{p^-/p^+}(1/4)^{1/p^+}$, such that $G(l)/l^{p^+} < kG(h)/h^{p^+}$. Then for every $\lambda \in \left(\frac{\left(\frac{8}{3}\right)^{p^+} h^{p^+} c_3}{\alpha_0 G(h)}, \frac{l^{p^+}}{p^+\|\alpha\|_1 G(l)}\right)$, the following problem*

$$\begin{cases} (|u''|^{p(x)-2}u'')'' = \lambda\alpha(x)g(u) \quad in \quad (0,1), \\ u(0) = u(1) = 0, \\ u''(1) = u''(0) = 0 \end{cases} \tag{7}$$

has at least three nontrivial solutions.

Proof. For each $u \in X$ and $x \in [0,1]$, we have

$$u(x) = \int_0^x u'(t)dt = \int_1^x u'(t)dt$$

and

$$|u(x)| = \frac{1}{2}\left(\left|\int_0^x u'(t)dt\right| + \left|\int_1^x u'(t)dt\right|\right) = \frac{1}{2}\left[\int_0^x |u'(t)|dt + \int_1^x |u'(t)dt|\right] = \frac{1}{2}\int_0^1 u'(t)dt. \tag{8}$$

For each $u \in C^2([0,1])$, there exists $\eta \in (0,1)$ such that $u'(\eta) = 0$. Hence, one has

$$u'(t) = \int_\eta^t u''(s)ds, \text{ for all } t \in (0,1).$$

By the Hölder inequality (3), we can conclude that

$$|u'(t)| \le \int_0^1 |u''(s)|ds \le \frac{1}{2}|u''|_{L^{p(x)}([0,1])}|1|_{L^{q(x)}([0,1])}, \ 0 < \eta \le t \le 1,$$

where $1/p(x) + 1/q(x) = 1$. Since $|1|_{L^{q(x)}([0,1])} \le 1$, we see that

$$|u(x)| \le \frac{1}{4}\|u\|, \text{ for all } t \in [0,1], u \in W^{2,p(x)}([0,1]) \cap W_0^{1,p(x)}([0,1]),$$

so $c_0 \le 1$, where c_0 is the constant from Remark 1. If we take $r = l^{p^+}/p^+$, we get $\gamma_r = l$, and a simple computation shows that all hypotheses of Theorem 2 hold, and thus, it can be applied. This completes the proof of Theorem 3. □

Remark 6. Our Theorem 3 generalizes Theorem 3.3 in Bonanno and Chinnì.[9]

ACKNOWLEDGEMENT

D. Repovš was supported by the Slovenian Research Agency program P1-0292 and grants N1-0114, N1-0083, N1-0064, and J1-8131.

ORCID

Khaled Kefi https://orcid.org/0000-0001-9277-5820
Dušan D. Repovš https://orcid.org/0000-0002-6643-1271
Kamel Saoudi https://orcid.org/0000-0002-7647-4814

REFERENCES

1. Ružička M. *Electrorheological Fluids: Modeling and Mathematical Theory*. Lecture Notes in Mathematics Berlin: Springer; 2000:1748.
2. Zhikov VV. Lavrentiev phenomenon and homogenization for some variational problems. *C R Acad Sci Paris Sér I Math*. 1993;316(5):435-439. Zbl 0783.35005.
3. Antontsev SN, Rodrigues JF. On stationary thermo-rheological viscous flows. *Ann Univ Ferrara*. 2006;52:19-36. https://doi.org/10.1007/s11565-006-0002-9
4. Chen Y, Levine S, Rao M. Variable exponent, linear growth functionals in image processing. *SIAM J Appl Math*. 2006;66:1383-1406. https://doi.org/10.1137/050624522
5. Antontsev SN, Shmarev SI. A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions. *Nonlinear Anal*. 2005;60:515-545. https://doi.org/10.1016/j.na.2004.09.026
6. Leray J, Lions JL. Quelques ré,sultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. *Bull Soc Math France*. 1965;93:97-107. https://doi.org/10.24033/bsmf.1617
7. Papageorgiou NS, Rădulescu VD, Repovš DD. *Nonlinear Analysis - Theory and Methods*. Cham: Springer; 2019.
8. Bonanno G, Chinnì A. Discontinuous elliptic problems involving the $p(x)$-Laplacian. *Math Nachr*. 2011;284(5-6):639-652. https://doi.org/10.1002/mana.200810232
9. Bonanno G, Chinnì A. Multiple solutions for elliptic problems involving the $p(x)$-Laplacian. *Matematiche (Catania)*. 2011;66:105-113. https://doi.org/10.4418/2011.66.1.10
10. Rădulescu VD, Repovš DD. *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*. Boca Raton, FL: Chapman and Hall/CRC Press; 2015.
11. El Amrouss AR, Ourraoui A. Existence of solutions for a boundary problem involving $p(x)$-biharmonic operator. *Bol Soc Parana Mat*. 2013;31:179-192. https://doi.org/10.5269/bspm.v31i1.15148
12. El Amrous A, Moradi F, Moussaoui M. Existence of solutions for fourth-order PDEs with variable exponents. *Electron J Differ Equ*. 2009;153:1-13. http://ejde.math.txstate.edu
13. Gasiński L, Papageorgiou NS. *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*. Boca Raton: Chapman and Hall/CRC Press; 2005.
14. Bonanno G, Marano SA. On the structure of the critical set of non-differentiable functions with a weak compactness condition. *Appl Anal*. 2010;89(1):1-10. https://doi.org/10.1080/00036810903397438
15. Boureanu MM. Fourth-order problems with Leray-Lions type operators in variable exponent spaces. *Discrete Contin Dyn Syst Ser S*. 2019;12(2):231-243. https://doi.org/10.3934/dcdss.2019016

How to cite this article: Kefi K, Repovš DD, Saoudi K. On weak solutions for fourth-order problems involving the Leray-Lions type operators. *Math Meth Appl Sci*. 2021;44:13060-13068. https://doi.org/10.1002/mma.7606